Quantizing a Free Scalar Field

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Contents

Suggested Use

Full screen single-slide manual advance

(Under macOS 12, SlidePilot transitions are cleaner than Preview's)

Special Notation

$$f(\lambda) = \int_0^{\pi} d\theta \left(1 + e^{i\lambda\theta}\right)$$

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \sim \frac{1}{137}$$

Lorentz Invariants

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

Lorentz transformation

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

$$x'^{0} = \Lambda^{0}{}_{0} x^{0} + \Lambda^{0}{}_{1} x^{1} + \Lambda^{0}{}_{2} x^{2} + \Lambda^{0}{}_{3} x^{3}$$
$$x'^{1} = \Lambda^{1}{}_{0} x^{0} + \Lambda^{1}{}_{1} x^{1} + \Lambda^{1}{}_{2} x^{2} + \Lambda^{1}{}_{3} x^{3}$$
$$x'^{2} = \Lambda^{2}{}_{0} x^{0} + \Lambda^{2}{}_{1} x^{1} + \Lambda^{2}{}_{2} x^{2} + \Lambda^{2}{}_{3} x^{3}$$
$$x'^{3} = \Lambda^{3}{}_{0} x^{0} + \Lambda^{3}{}_{1} x^{1} + \Lambda^{3}{}_{2} x^{2} + \Lambda^{3}{}_{3} x^{3}$$

Expand implied sum

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

$$\gamma = 1/\sqrt{1 - v^2}$$

$$x'^{0} = \Lambda^{0}{}_{0} x^{0} + \Lambda^{0}{}_{1} x^{1} + \Lambda^{0}{}_{2} x^{2} + \Lambda^{0}{}_{3} x^{3}$$
$$x'^{1} = \Lambda^{1}{}_{0} x^{0} + \Lambda^{1}{}_{1} x^{1} + \Lambda^{1}{}_{2} x^{2} + \Lambda^{1}{}_{3} x^{3}$$
$$x'^{2} = \Lambda^{2}{}_{0} x^{0} + \Lambda^{2}{}_{1} x^{1} + \Lambda^{2}{}_{2} x^{2} + \Lambda^{2}{}_{3} x^{3}$$
$$x'^{3} = \Lambda^{3}{}_{0} x^{0} + \Lambda^{3}{}_{1} x^{1} + \Lambda^{3}{}_{2} x^{2} + \Lambda^{3}{}_{3} x^{3}$$

$$t' = \gamma(t + vx)$$
$$x' = \gamma(x + vt)$$
$$y' = y$$
$$z' = z$$

Label relative motion direction x & x'

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$
\begin{aligned}
x'^0 &= \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 & t' &= \gamma(t + vx) & E' &= \gamma(E + vp_x) \\
x'^1 &= \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 & x' &= \gamma(x + vt) & p'_x &= \gamma(p_x + vE) \\
x'^2 &= \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 & y' &= y & p'_y &= p_y \\
x'^3 &= \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 & z' &= z & p'_z &= p_z
\end{aligned}
$$

Energy-momentum Lorentz transformation

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$\begin{aligned}
x'^0 &= \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 & t' &= \gamma(t + vx) & E' &= \gamma(E + vp_x) & dE' &= \gamma(dE + v\,dp_x)\\
x'^1 &= \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 & x' &= \gamma(x + vt) & p'_x &= \gamma(p_x + vE) & dp'_x &= \gamma(dp_x + v\,dE)\\
x'^2 &= \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 & y' &= y & p'_y &= p_y & dp'_y &= dp_y\\
x'^3 &= \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 & z' &= z & p'_z &= p_z & dp'_z &= dp_z
\end{aligned}$$

Differential form

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1 - v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2$$

Mass is spacetime momentum length

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2}$$

Spacetime split

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1 - v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

Rectangular components

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

Differential, assuming relative motion in the x direction

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$\begin{aligned}
x'^0 &= \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 & t' &= \gamma(t+vx) & E' &= \gamma(E + v p_x) & dE' &= \gamma(dE + v\, dp_x)\\
x'^1 &= \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 & x' &= \gamma(x+vt) & p'_x &= \gamma(p_x + vE) & dp'_x &= \gamma(dp_x + v\, dE)\\
x'^2 &= \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 & y' &= y & p'_y &= p_y & dp'_y &= dp_y\\
x'^3 &= \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 & z' &= z & p'_z &= p_z & dp'_z &= dp_z
\end{aligned}$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\, dE - 2p_x dp_x - 0 - 0$$

$$E\, dE = p_x dp_x$$

Rearrange

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^{0} = \Lambda^{0}{}_{0}x^{0} + \Lambda^{0}{}_{1}x^{1} + \Lambda^{0}{}_{2}x^{2} + \Lambda^{0}{}_{3}x^{3} \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^{1} = \Lambda^{1}{}_{0}x^{0} + \Lambda^{1}{}_{1}x^{1} + \Lambda^{1}{}_{2}x^{2} + \Lambda^{1}{}_{3}x^{3} \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^{2} = \Lambda^{2}{}_{0}x^{0} + \Lambda^{2}{}_{1}x^{1} + \Lambda^{2}{}_{2}x^{2} + \Lambda^{2}{}_{3}x^{3} \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^{3} = \Lambda^{3}{}_{0}x^{0} + \Lambda^{3}{}_{1}x^{1} + \Lambda^{3}{}_{2}x^{2} + \Lambda^{3}{}_{3}x^{3} \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$dp'_x = \gamma\,(dp_x + v\,dE)$$

Lorentz transformation

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\, dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\, dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\, dE - 2p_x dp_x - 0 - 0$$

$$E\, dE = p_x dp_x$$

$$E\, dp'_x = E\gamma\,(dp_x + v\, dE)$$

Multiply

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\, dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\, dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\, dE - 2p_x dp_x - 0 - 0$$

$$E\, dE = p_x dp_x$$

$$E\, dp'_x = \gamma\left(E\, dp_x + vE\, dE\right)$$

Distribute

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$E\,dp'_x = \gamma\left(E\,dp_x + vp_x dp_x\right)$$

Substitute

$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$ $\gamma = 1/\sqrt{1-v^2}$

$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3$ $t' = \gamma(t + vx)$ $E' = \gamma(E + vp_x)$ $dE' = \gamma(dE + v\, dp_x)$

$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3$ $x' = \gamma(x + vt)$ $p'_x = \gamma(p_x + vE)$ $dp'_x = \gamma(dp_x + v\, dE)$

$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3$ $y' = y$ $p'_y = p_y$ $dp'_y = dp_y$

$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3$ $z' = z$ $p'_z = p_z$ $dp'_z = dp_z$

$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$

$0 = 2E\, dE - 2p_x dp_x - 0 - 0$

$E\, dE = p_x dp_x$

$E\, dp'_x = \gamma\,(E + vp_x)\, dp_x$

Factor

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$E\,dp'_x = E' dp_x$$

Lorentz transformation

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^2 = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$\frac{dp'_x}{E'} = \frac{dp_x}{E}$$

Rearrange

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t+vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x+vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^2 = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$\frac{dp'_x dp'_y dp'_z}{E'} = \frac{dp_x dp_y dp_z}{E}$$

No change perpendicular to motion

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$\frac{d^3p'}{E'} = \frac{d^3p}{E}$$

Consolidate

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1 - v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\, dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\, dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\, dE - 2p_x dp_x - 0 - 0$$

$$E\, dE = p_x dp_x$$

$$\frac{d^3 p'}{E'} = \frac{d^3 p}{E}$$

d^3p/E is invariant, where $E^2 = \vec{p}^{\,2} + m^2$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t+vx) \qquad E' = \gamma(E+vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x+vt) \qquad p'_x = \gamma(p_x+vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^2 = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$\frac{d^3p'}{E'} = \frac{d^3p}{E}$$

$$\frac{d^3k'}{\omega'} = \frac{d^3k}{\omega}$$

$$E = \hbar\omega,\ p = \hbar k$$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$\frac{d^3p'}{E'} = \frac{d^3p}{E}$$

$$\frac{d^3k'}{\omega'} = \frac{d^3k}{\omega}$$

$E = \omega$, $p = k$ in natural units

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^2 = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$\frac{d^3p'}{E'} = \frac{d^3p}{E}$$

$$\frac{d^3k'}{\omega'} = \frac{d^3k}{\omega}$$

d^3k/ω is invariant, where $\omega^2 = \vec{k}^2 + m^2$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$\begin{aligned}
x'^0 &= \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 & t' &= \gamma(t+vx) & E' &= \gamma(E+vp_x) & dE' &= \gamma(dE + v\,dp_x)\\
x'^1 &= \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 & x' &= \gamma(x+vt) & p'_x &= \gamma(p_x+vE) & dp'_x &= \gamma(dp_x + v\,dE)\\
x'^2 &= \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 & y' &= y & p'_y &= p_y & dp'_y &= dp_y\\
x'^3 &= \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 & z' &= z & p'_z &= p_z & dp'_z &= dp_z
\end{aligned}$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^2 = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$\frac{d^3p'}{E'} = \frac{d^3p}{E}$$

$$\frac{d^3k'}{\omega'} = \frac{d^3k}{\omega} \;\blacksquare$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\, dp_x)$$
$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\, dE)$$
$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$
$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\, dE - 2p_x dp_x - 0 - 0$$

$$E\, dE = p_x dp_x$$

$$\frac{d^3p'}{E'} = \frac{d^3p}{E} \qquad 1 = \int dp'_x \delta(p'_x) = \int dp_x \delta(p_x)$$

Dirac delta unit normalization

$$\frac{d^3k'}{\omega'} = \frac{d^3k}{\omega} \blacksquare$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^2 = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$\frac{d^3p'}{E'} = \frac{d^3p}{E} \qquad dp'_x \delta(p'_x) = dp_x \delta(p_x)$$

$$\frac{d^3k'}{\omega'} = \frac{d^3k}{\omega} \quad \blacksquare$$

1D variable change

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$\frac{d^3p'}{E'} = \frac{d^3p}{E} \qquad E\,dp'_x\,\delta(p'_x) = E\,dp_x\,\delta(p_x)$$

Multiply

$$\frac{d^3k'}{\omega'} = \frac{d^3k}{\omega} \quad \blacksquare$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\, dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\, dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\, dE - 2p_x dp_x - 0 - 0$$

$$E\, dE = p_x dp_x$$

$$\frac{d^3p'}{E'} = \frac{d^3p}{E} \qquad E' dp_x \delta(p'_x) = E dp_x \delta(p_x)$$

$$\frac{d^3k'}{\omega'} = \frac{d^3k}{\omega} \;\blacksquare$$

Substitute

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1 - v^2}$$

$$\begin{aligned}
x'^0 &= \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 & t' &= \gamma(t + vx) & E' &= \gamma(E + v p_x) & dE' &= \gamma(dE + v\, dp_x) \\
x'^1 &= \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 & x' &= \gamma(x + vt) & p'_x &= \gamma(p_x + vE) & dp'_x &= \gamma(dp_x + v\, dE) \\
x'^2 &= \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 & y' &= y & p'_y &= p_y & dp'_y &= dp_y \\
x'^3 &= \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 & z' &= z & p'_z &= p_z & dp'_z &= dp_z
\end{aligned}$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\, dE - 2p_x dp_x - 0 - 0$$

$$E\, dE = p_x dp_x$$

$$\frac{d^3p'}{E'} = \frac{d^3p}{E} \qquad E'\,\delta(p'_x) = E\,\delta(p_x)$$

Cancel

$$\frac{d^3k'}{\omega'} = \frac{d^3k}{\omega} \ \blacksquare$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1 - v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + v p_x) \qquad dE' = \gamma(dE + v\, dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\, dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\, dE - 2p_x dp_x - 0 - 0$$

$$E\, dE = p_x dp_x$$

$$\frac{d^3p'}{E'} = \frac{d^3p}{E} \qquad E'\,\delta(p'_x)\delta(p'_y)\delta(p'_z) = E\,\delta(p_x)\delta(p_y)\delta(p_z)$$

No change perpendicular to motion

$$\frac{d^3k'}{\omega'} = \frac{d^3k}{\omega} \blacksquare$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$\frac{d^3p'}{E'} = \frac{d^3p}{E} \qquad E'\,\delta^3(\vec{p}\,') = E\,\delta^3(\vec{p})$$

Consolidate

$$\frac{d^3k'}{\omega'} = \frac{d^3k}{\omega} \quad \blacksquare$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t+vx) \qquad E' = \gamma(E+vp_x) \qquad dE' = \gamma(dE + v\,dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x+vt) \qquad p'_x = \gamma(p_x+vE) \qquad dp'_x = \gamma(dp_x + v\,dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$\frac{d^3p'}{E'} = \frac{d^3p}{E} \qquad E'\,\delta^3(\vec{p}\,') = E\,\delta^3(\vec{p})$$

$$\frac{d^3k'}{\omega'} = \frac{d^3k}{\omega} \;\blacksquare \qquad E\,\delta^3(\vec{p})\ \text{is invariant, where}\ E^2 = \vec{p}^{\,2} + m^2$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$\begin{aligned}
x'^0 &= \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 & t' &= \gamma(t+vx) & E' &= \gamma(E+vp_x) & dE' &= \gamma(dE + v\,dp_x)\\
x'^1 &= \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 & x' &= \gamma(x+vt) & p'_x &= \gamma(p_x+vE) & dp'_x &= \gamma(dp_x + v\,dE)\\
x'^2 &= \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 & y' &= y & p'_y &= p_y & dp'_y &= dp_y\\
x'^3 &= \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 & z' &= z & p'_z &= p_z & dp'_z &= dp_z
\end{aligned}$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$\frac{d^3p'}{E'} = \frac{d^3p}{E} \qquad E'\,\delta^3(\vec{p}\,') = E\,\delta^3(\vec{p})$$

$$\frac{d^3k'}{\omega'} = \frac{d^3k}{\omega} \;\blacksquare \qquad \omega'\,\delta^3(\vec{k}') = \omega\,\delta^3(\vec{k})$$

$$E = \hbar\omega,\; p = \hbar k$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$\begin{aligned}
x'^0 &= \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 & t' &= \gamma(t+vx) & E' &= \gamma(E+vp_x) & dE' &= \gamma(dE + v\,dp_x)\\
x'^1 &= \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 & x' &= \gamma(x+vt) & p'_x &= \gamma(p_x+vE) & dp'_x &= \gamma(dp_x + v\,dE)\\
x'^2 &= \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 & y' &= y & p'_y &= p_y & dp'_y &= dp_y\\
x'^3 &= \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 & z' &= z & p'_z &= p_z & dp'_z &= dp_z
\end{aligned}$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$\frac{d^3p'}{E'} = \frac{d^3p}{E} \qquad E'\,\delta^3(\vec{p}\,') = E\,\delta^3(\vec{p})$$

$$\frac{d^3k'}{\omega'} = \frac{d^3k}{\omega} \;\blacksquare \qquad \omega'\,\delta^3(\vec{k}') = \omega\,\delta^3(\vec{k})$$

$E = \omega$, $p = k$ in natural units

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 \qquad t' = \gamma(t + vx) \qquad E' = \gamma(E + v p_x) \qquad dE' = \gamma(dE + v\, dp_x)$$

$$x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 \qquad x' = \gamma(x + vt) \qquad p'_x = \gamma(p_x + vE) \qquad dp'_x = \gamma(dp_x + v\, dE)$$

$$x'^2 = \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 \qquad y' = y \qquad p'_y = p_y \qquad dp'_y = dp_y$$

$$x'^3 = \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 \qquad z' = z \qquad p'_z = p_z \qquad dp'_z = dp_z$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\, dE - 2p_x dp_x - 0 - 0$$

$$E\, dE = p_x dp_x$$

$$\frac{d^3 p'}{E'} = \frac{d^3 p}{E} \qquad E'\,\delta^3(\vec{p}\,') = E\,\delta^3(\vec{p})$$

$$\frac{d^3 k'}{\omega'} = \frac{d^3 k}{\omega} \;\blacksquare \qquad \omega'\,\delta^3(\vec{k}') = \omega\,\delta^3(\vec{k})$$

$\omega\,\delta^3(\vec{k})$ is invariant, where $\omega^2 = \vec{k}^2 + m^2$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \gamma = 1/\sqrt{1-v^2}$$

$$\begin{aligned}
x'^0 &= \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3 & t' &= \gamma(t+vx) & E' &= \gamma(E+vp_x) & dE' &= \gamma(dE + v\,dp_x)\\
x'^1 &= \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 + \Lambda^1{}_2 x^2 + \Lambda^1{}_3 x^3 & x' &= \gamma(x+vt) & p'_x &= \gamma(p_x+vE) & dp'_x &= \gamma(dp_x + v\,dE)\\
x'^2 &= \Lambda^2{}_0 x^0 + \Lambda^2{}_1 x^1 + \Lambda^2{}_2 x^2 + \Lambda^2{}_3 x^3 & y' &= y & p'_y &= p_y & dp'_y &= dp_y\\
x'^3 &= \Lambda^3{}_0 x^0 + \Lambda^3{}_1 x^1 + \Lambda^3{}_2 x^2 + \Lambda^3{}_3 x^3 & z' &= z & p'_z &= p_z & dp'_z &= dp_z
\end{aligned}$$

$$m^2 = p^2 = (p^0)^2 - \vec{p}^{\,2} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$0 = 2E\,dE - 2p_x dp_x - 0 - 0$$

$$E\,dE = p_x dp_x$$

$$\frac{d^3p'}{E'} = \frac{d^3p}{E} \qquad E'\,\delta^3(\vec{p}\,') = E\,\delta^3(\vec{p})$$

$$\frac{d^3k'}{\omega'} = \frac{d^3k}{\omega}\ \blacksquare \qquad \omega'\,\delta^3(\vec{k}') = \omega\,\delta^3(\vec{k})\ \blacksquare$$

$$1 = \int d^3p \, \delta^3(\vec{p} - \vec{p}\,')$$

Dirac delta normalization is Lorentz invariant

$$1 = \int d^3p\, \delta^3(\vec{p} - \vec{p}\,') = \int \frac{d^3p}{E} E\, \delta^3(\vec{p} - \vec{p}\,')$$

Component invariants, with $E^2 = \vec{p}^{\,2} + m^2$

$$1 = \int d^3p\,\delta^3(\vec{p} - \vec{p}\,') = \int \frac{d^3p}{E} E\,\delta^3(\vec{p} - \vec{p}\,') = \int \frac{d^3p}{(2\pi)^3 2E}\ (2\pi)^3 2E\,\delta^3(\vec{p} - \vec{p}\,')$$

Convenient normalization

$$1 = \int d^3p\, \delta^3(\vec{p} - \vec{p}\,') = \int \frac{d^3p}{E} E\, \delta^3(\vec{p} - \vec{p}\,') = \int \frac{d^3p}{(2\pi)^3 2E}\ (2\pi)^3 2E\, \delta^3(\vec{p} - \vec{p}\,')$$

$$1 = \int d^3k\, \delta^3(\vec{k} - \vec{k}') = \int \frac{d^3k}{\omega} \omega\, \delta^3(\vec{k} - \vec{k}') = \int \frac{d^3k}{(2\pi)^3 2\omega}\ (2\pi)^3 2\omega\, \delta^3(\vec{k} - \vec{k}')$$

$E = \hbar\omega,\ p = \hbar k$

$$1 = \int d^3p\,\delta^3(\vec{p} - \vec{p}\,') = \int \frac{d^3p}{E} E\,\delta^3(\vec{p} - \vec{p}\,') = \int \frac{d^3p}{(2\pi)^3 2E}\ (2\pi)^3 2E\,\delta^3(\vec{p} - \vec{p}\,')$$

$$1 = \int d^3k\,\delta^3(\vec{k} - \vec{k}') = \int \frac{d^3k}{\omega} \omega\,\delta^3(\vec{k} - \vec{k}') = \int \frac{d^3k}{(2\pi)^3 2\omega}\ (2\pi)^3 2\omega\,\delta^3(\vec{k} - \vec{k}')$$

$E = \omega$, $p = k$ in natural units

$$1 = \int d^3p\,\delta^3(\vec{p} - \vec{p}\,') = \int \frac{d^3p}{E} E\,\delta^3(\vec{p} - \vec{p}\,') = \int \frac{d^3p}{(2\pi)^3 2E}\;(2\pi)^3 2E\,\delta^3(\vec{p} - \vec{p}\,')$$

$$1 = \int d^3k\,\delta^3(\vec{k} - \vec{k}') = \int \frac{d^3k}{\omega} \omega\,\delta^3(\vec{k} - \vec{k}') = \int \frac{d^3k}{(2\pi)^3 2\omega}\;(2\pi)^3 2\omega\,\delta^3(\vec{k} - \vec{k}') \quad \blacksquare$$

Euler-Lagrange Equations

$$S = \int dt\, L$$

Action is the temporal integral of the Lagrangian

$$S = \int dt\, L \;= \int_T dt \int_V d^3x\, \mathcal{L}$$

Lagrangian density

$$S = \int dt\, L \;\; = \int_T dt \int_V d^3x\, \mathcal{L} \;\; = \int_T \int_V dt\, d^3x\, \mathcal{L}$$

Iterated integral $\rightarrow$ 4D integral

$$S = \int dt\, L \;= \int_T dt \int_V d^3x\, \mathcal{L} \;= \int_T \int_V dt\, d^3x\, \mathcal{L} \;= \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

4-volume $d^4x = dx^0\, dx^1\, dx^2\, dx^3 = c\, dt\, dx\, dy\, dz$

$$S = \int dt\, L \;=\int_T dt \int_V d^3x\, \mathcal{L} \;=\int_T \int_V dt\, d^3x\, \mathcal{L} \;=\int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

4-volume $d^4x = dx^0\, dx^1\, dx^2\, dx^3 = dt\, dx\, dy\, dz$ in natural units

$$S = \int dt\, L \;= \int_T dt \int_V d^3x\, \mathcal{L} \;= \int_T \int_V dt\, d^3x\, \mathcal{L} \;= \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

Compare mechanics $L(x, \dot{x})$

$$S = \int dt\, L \;= \int_T dt \int_V d^3x\, \mathcal{L} \;= \int_T \int_V dt\, d^3x\, \mathcal{L} \;= \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\phi \to \delta\phi, \quad \delta\phi\Big|_{\partial\Omega} = 0$$

Vary field $\phi(x)$ except on boundary

$$S = \int dt\, L \;\; = \int_T dt \int_V d^3x\, \mathcal{L} \;\; = \int_T \int_V dt\, d^3x\, \mathcal{L} \;\; = \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\phi \to \delta\phi, \quad \delta\phi\Big|_{\partial\Omega} = 0$$

$$0 = \delta S$$

Action is stationary if variation vanishes

$$S = \int dt\, L \;= \int_T dt \int_V d^3x\, \mathcal{L} \;= \int_T \int_V dt\, d^3x\, \mathcal{L} \;= \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\phi \to \delta\phi, \quad \delta\phi\Big|_{\partial\Omega} = 0$$

$$0 = \delta S \;= \int_\Omega d^4x\, \delta\mathcal{L}(\phi, \partial_\mu \phi)$$

Variation and integration commute for fixed $4D$ volume Ω

$$S = \int dt\, L \;= \int_T dt \int_V d^3x\, \mathcal{L} \;= \int_T \int_V dt\, d^3x\, \mathcal{L} \;= \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\phi \to \delta\phi, \quad \delta\phi\Big|_{\partial\Omega} = 0$$

$$\begin{aligned} 0 = \delta S \;&= \int_\Omega d^4x\, \delta\mathcal{L}(\phi, \partial_\mu \phi) \\ &= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) \end{aligned}$$

Variational chain rule

$$S = \int dt\, L \;= \int_T dt \int_V d^3x\, \mathcal{L} \;= \int_T \int_V dt\, d^3x\, \mathcal{L} \;= \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\phi \to \delta\phi, \quad \delta\phi\Big|_{\partial\Omega} = 0$$

$$\begin{aligned}
0 = \delta S \;&= \int_\Omega d^4x\, \delta\mathcal{L}(\phi, \partial_\mu \phi) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) \right)
\end{aligned}$$

Variation and derivative commute

$$\delta q = (q + \delta q) - q \;\; \rightarrow$$

$$\delta(\partial_t q) = \partial_t(q + \delta q) - \partial_t q = \partial_t(\delta q)$$

$$S = \int dt\, L \;= \int_T dt \int_V d^3x\, \mathcal{L} \;= \int_T \int_V dt\, d^3x\, \mathcal{L} \;= \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\phi \to \delta\phi, \quad \delta\phi\Big|_{\partial\Omega} = 0$$

$$\begin{aligned}
0 = \delta S \;&= \int_\Omega d^4x\, \delta\mathcal{L}(\phi, \partial_\mu \phi) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right)
\end{aligned}$$

Partial integration

$$d(uv) = du\, v + u\, dv \rightarrow$$

$$+\int_a^b u\, dv = -\int_a^b du\, v + uv\Big|_a^b$$

$$S = \int dt\, L \;= \int_T dt \int_V d^3x\, \mathcal{L} \;= \int_T \int_V dt\, d^3x\, \mathcal{L} \;= \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\phi \to \delta\phi, \quad \delta\phi\Big|_{\partial\Omega} = 0$$

$$\begin{aligned}
0 = \delta S \;&= \int_\Omega d^4x\, \delta\mathcal{L}(\phi, \partial_\mu \phi) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right)
\end{aligned}$$

Partial integration

$d(uv) = du\, v + u\, dv \rightarrow$

$+ \int_a^b u\, dv = - \int_a^b du\, v$

with vanishing boundary term $\phi(x) = 0$

$$S = \int dt\, L \;= \int_T dt \int_V d^3x\, \mathcal{L} \;= \int_T \int_V dt\, d^3x\, \mathcal{L} \;= \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\phi \to \delta\phi, \quad \delta\phi\Big|_{\partial\Omega} = 0$$

$$\begin{aligned}
0 = \delta S \;&= \int_\Omega d^4x\, \delta\mathcal{L}(\phi, \partial_\mu \phi) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi
\end{aligned}$$

Factor the variation

$$S = \int dt\, L \;= \int_T dt \int_V d^3x\, \mathcal{L} \;= \int_T \int_V dt\, d^3x\, \mathcal{L} \;= \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\phi \to \delta\phi, \quad \delta\phi\Big|_{\partial\Omega} = 0$$

$$\begin{aligned}
0 = \delta S \;&= \int_\Omega d^4x\, \delta\mathcal{L}(\phi, \partial_\mu \phi) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi
\end{aligned}$$

$$\forall\ \delta\phi \to \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$

Since variation is arbitrary, integrand must vanish

$$S = \int dt\, L \;= \int_T dt \int_V d^3x\, \mathcal{L} \;= \int_T \int_V dt\, d^3x\, \mathcal{L} \;= \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\phi \to \delta\phi, \quad \delta\phi\Big|_{\partial\Omega} = 0$$

$$\begin{aligned}
0 = \delta S \;&= \int_\Omega d^4x\, \delta\mathcal{L}(\phi, \partial_\mu \phi) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta \phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi
\end{aligned}$$

$$\forall\ \delta\phi \to \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \qquad \text{Solve}$$

$$S = \int dt\, L \;= \int_T dt \int_V d^3x\, \mathcal{L} \;= \int_T \int_V dt\, d^3x\, \mathcal{L} \;= \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\phi \to \delta\phi, \quad \delta\phi\Big|_{\partial\Omega} = 0$$

$$\begin{aligned}
0 = \delta S \;&= \int_\Omega d^4x\, \delta\mathcal{L}(\phi, \partial_\mu \phi) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi
\end{aligned}$$

$$\forall\ \delta\phi \to \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \qquad \text{Compare 1D mechanics } \frac{\partial L}{\partial q} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}$$

$$S = \int dt\, L \ = \int_T dt \int_V d^3x\, \mathcal{L} \ = \int_T \int_V dt\, d^3x\, \mathcal{L} \ = \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\phi \to \delta\phi, \quad \delta\phi\Big|_{\partial\Omega} = 0$$

$$\begin{aligned}
0 = \delta S \ &= \int_\Omega d^4x\, \delta\mathcal{L}(\phi, \partial_\mu \phi) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi
\end{aligned}$$

$$\forall\ \delta\phi \to \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \ = \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x^\mu)} \qquad \text{Explicit coordinates}$$

$$S = \int dt\, L \;= \int_T dt \int_V d^3x\, \mathcal{L} \;= \int_T \int_V dt\, d^3x\, \mathcal{L} \;= \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\phi \to \delta\phi, \quad \delta\phi\Big|_{\partial\Omega} = 0$$

$$\begin{aligned}
0 = \delta S \;&= \int_\Omega d^4x\, \delta\mathcal{L}(\phi, \partial_\mu \phi) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta \phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right) \\
&= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi
\end{aligned}$$

$$\forall\ \delta\phi \to \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} + \partial_1 \frac{\partial \mathcal{L}}{\partial(\partial_1 \phi)} + \partial_2 \frac{\partial \mathcal{L}}{\partial(\partial_2 \phi)} + \partial_3 \frac{\partial \mathcal{L}}{\partial(\partial_3 \phi)} \quad \text{Einstein summation convention}$$

$$S = \int dt\, L \;= \int_T dt \int_V d^3x\, \mathcal{L} \;= \int_T \int_V dt\, d^3x\, \mathcal{L} \;= \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\phi \to \delta\phi, \quad \delta\phi\Big|_{\partial\Omega} = 0$$

$$\begin{aligned}
0 = \delta S \;&= \int_\Omega d^4x\, \delta\mathcal{L}(\phi, \partial_\mu\phi) \\
&= \int_\Omega d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) \right) \\
&= \int_\Omega d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right) \delta\phi \right) \\
&= \int_\Omega d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right) \delta\phi
\end{aligned}$$

$$\forall\ \delta\phi \to \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \quad \blacksquare$$

Klein-Gordon Equation

$$\mathcal{L} = \frac{1}{2}\partial^\alpha \phi \, \partial_\alpha \phi - \frac{1}{2} m^2 \phi^2$$

Lagrangian density for real fields $\phi(x) \in \mathbb{R}$

$$\mathcal{L} = \frac{1}{2}\partial^\alpha \phi \, \partial_\alpha \phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_\alpha \phi \, \partial_\beta \phi - \frac{1}{2}m^2\phi^2$$

Mostly minuses Minkowski metric $g_{\alpha\beta}$ lowers indices

$$\mathcal{L} = \frac{1}{2}\partial^{\alpha}\phi\,\partial_{\alpha}\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\,\partial_{\beta}\phi - \frac{1}{2}m^2\phi^2$$

$$g_{\alpha\beta} \leftrightarrow \boxed{\begin{array}{cccc} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}} \leftrightarrow g^{\alpha\beta}$$

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - \frac{1}{2}m^2\phi^2 = \mathcal{L}(\phi, \partial_\alpha\phi)$$

Lagrangian density depends on fields and their gradients

$$\mathcal{L} = \frac{1}{2}\partial^{\alpha}\phi\,\partial_{\alpha}\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\,\partial_{\beta}\phi - \frac{1}{2}m^2\phi^2 = \mathcal{L}(\phi, \partial_{\alpha}\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

Rate of change of Lagrangian density with respect to the fields at constant field gradients

$$\mathcal{L} = \frac{1}{2}\partial^{\alpha}\phi\,\partial_{\alpha}\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\,\partial_{\beta}\phi - \frac{1}{2}m^2\phi^2 = \mathcal{L}(\phi, \partial_{\alpha}\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\alpha\beta}\frac{\partial(\partial_{\alpha}\phi)}{\partial(\partial_{\mu}\phi)}\partial_{\beta}\phi + \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\frac{\partial(\partial_{\beta}\phi)}{\partial(\partial_{\mu}\phi)}$$

Rate of change of Lagrangian density with respect to the field gradients at constant fields

$$\mathcal{L} = \frac{1}{2}\partial^{\alpha}\phi\,\partial_{\alpha}\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\,\partial_{\beta}\phi - \frac{1}{2}m^2\phi^2 = \mathcal{L}(\phi, \partial_{\alpha}\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\alpha\beta}\frac{\partial(\partial_{\alpha}\phi)}{\partial(\partial_{\mu}\phi)}\partial_{\beta}\phi + \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\frac{\partial(\partial_{\beta}\phi)}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\mu\beta}\partial_{\beta}\phi + \frac{1}{2}g^{\mu\alpha}\partial_{\alpha}\phi$$

$$\text{Partial derivatives } \frac{\partial(\partial_{\tau}\phi)}{\partial(\partial_{\sigma}\phi)} = \delta_{\sigma}^{\tau} = \begin{cases} 1, & \sigma = \tau \\ 0, & \sigma \neq \tau \end{cases}$$

$$\mathcal{L} = \frac{1}{2}\partial^{\alpha}\phi\,\partial_{\alpha}\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\,\partial_{\beta}\phi - \frac{1}{2}m^2\phi^2 = \mathcal{L}(\phi, \partial_{\alpha}\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\alpha\beta}\frac{\partial(\partial_{\alpha}\phi)}{\partial(\partial_{\mu}\phi)}\partial_{\beta}\phi + \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\frac{\partial(\partial_{\beta}\phi)}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\mu\beta}\partial_{\beta}\phi + \frac{1}{2}g^{\mu\alpha}\partial_{\alpha}\phi = \frac{1}{2}\partial^{\mu}\phi + \frac{1}{2}\partial^{\mu}\phi$$

Metric raises indices of partial derivatives

$$\mathcal{L} = \frac{1}{2}\partial^{\alpha}\phi\,\partial_{\alpha}\phi - \frac{1}{2}m^{2}\phi^{2} = \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\,\partial_{\beta}\phi - \frac{1}{2}m^{2}\phi^{2} = \mathcal{L}(\phi, \partial_{\alpha}\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^{2}\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\alpha\beta}\frac{\partial(\partial_{\alpha}\phi)}{\partial(\partial_{\mu}\phi)}\partial_{\beta}\phi + \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\frac{\partial(\partial_{\beta}\phi)}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\mu\beta}\partial_{\beta}\phi + \frac{1}{2}g^{\mu\alpha}\partial_{\alpha}\phi = \frac{1}{2}\partial^{\mu}\phi + \frac{1}{2}\partial^{\mu}\phi = \partial^{\mu}\phi$$

Addition

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - \frac{1}{2}m^2\phi^2 = \mathcal{L}(\phi, \partial_\alpha\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{1}{2}g^{\alpha\beta}\frac{\partial(\partial_\alpha\phi)}{\partial(\partial_\mu\phi)}\partial_\beta\phi + \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\frac{\partial(\partial_\beta\phi)}{\partial(\partial_\mu\phi)} = \frac{1}{2}g^{\mu\beta}\partial_\beta\phi + \frac{1}{2}g^{\mu\alpha}\partial_\alpha\phi = \frac{1}{2}\partial^\mu\phi + \frac{1}{2}\partial^\mu\phi = \partial^\mu\phi$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial_\mu\partial^\mu\phi$$

Take gradient

$$\mathcal{L} = \frac{1}{2}\partial^{\alpha}\phi\,\partial_{\alpha}\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\,\partial_{\beta}\phi - \frac{1}{2}m^2\phi^2 = \mathcal{L}(\phi, \partial_{\alpha}\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\alpha\beta}\frac{\partial(\partial_{\alpha}\phi)}{\partial(\partial_{\mu}\phi)}\partial_{\beta}\phi + \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\frac{\partial(\partial_{\beta}\phi)}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\mu\beta}\partial_{\beta}\phi + \frac{1}{2}g^{\mu\alpha}\partial_{\alpha}\phi = \frac{1}{2}\partial^{\mu}\phi + \frac{1}{2}\partial^{\mu}\phi = \partial^{\mu}\phi$$

$$\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} = \partial_{\mu}\partial^{\mu}\phi = \partial^2\phi$$

Compact notation

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - \frac{1}{2}m^2\phi^2 = \mathcal{L}(\phi, \partial_\alpha\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{1}{2}g^{\alpha\beta}\frac{\partial(\partial_\alpha\phi)}{\partial(\partial_\mu\phi)}\partial_\beta\phi + \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\frac{\partial(\partial_\beta\phi)}{\partial(\partial_\mu\phi)} = \frac{1}{2}g^{\mu\beta}\partial_\beta\phi + \frac{1}{2}g^{\mu\alpha}\partial_\alpha\phi = \frac{1}{2}\partial^\mu\phi + \frac{1}{2}\partial^\mu\phi = \partial^\mu\phi$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial_\mu\partial^\mu\phi = \partial^2\phi = \Box^2\phi$$

∇^2 is the Laplacian and $\Box^2$ is the d'Alembertian

$$\mathcal{L} = \frac{1}{2}\partial^{\alpha}\phi\,\partial_{\alpha}\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\,\partial_{\beta}\phi - \frac{1}{2}m^2\phi^2 = \mathcal{L}(\phi, \partial_{\alpha}\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\alpha\beta}\frac{\partial(\partial_{\alpha}\phi)}{\partial(\partial_{\mu}\phi)}\partial_{\beta}\phi + \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\frac{\partial(\partial_{\beta}\phi)}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\mu\beta}\partial_{\beta}\phi + \frac{1}{2}g^{\mu\alpha}\partial_{\alpha}\phi = \frac{1}{2}\partial^{\mu}\phi + \frac{1}{2}\partial^{\mu}\phi = \partial^{\mu}\phi$$

$$\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} = \partial_{\mu}\partial^{\mu}\phi = \partial^2\phi = \Box^2\phi$$

(or $\varDelta$ is the Laplacian and $\Box$ is the d'Alembertian)

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - \frac{1}{2}m^2\phi^2 = \mathcal{L}(\phi,\partial_\alpha\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{1}{2}g^{\alpha\beta}\frac{\partial(\partial_\alpha\phi)}{\partial(\partial_\mu\phi)}\partial_\beta\phi + \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\frac{\partial(\partial_\beta\phi)}{\partial(\partial_\mu\phi)} = \frac{1}{2}g^{\mu\beta}\partial_\beta\phi + \frac{1}{2}g^{\mu\alpha}\partial_\alpha\phi = \frac{1}{2}\partial^\mu\phi + \frac{1}{2}\partial^\mu\phi = \partial^\mu\phi$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial_\mu\partial^\mu\phi = \partial^2\phi = \Box^2\phi$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad\rightarrow\quad \partial_\mu\partial^\mu\phi + m^2\phi = 0$$

Euler-Lagrange equations $\rightarrow$ Klein-Gordon equation

$$\mathcal{L} = \frac{1}{2}\partial^{\alpha}\phi\,\partial_{\alpha}\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\,\partial_{\beta}\phi - \frac{1}{2}m^2\phi^2 = \mathcal{L}(\phi, \partial_{\alpha}\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\alpha\beta}\frac{\partial(\partial_{\alpha}\phi)}{\partial(\partial_{\mu}\phi)}\partial_{\beta}\phi + \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\frac{\partial(\partial_{\beta}\phi)}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\mu\beta}\partial_{\beta}\phi + \frac{1}{2}g^{\mu\alpha}\partial_{\alpha}\phi = \frac{1}{2}\partial^{\mu}\phi + \frac{1}{2}\partial^{\mu}\phi = \partial^{\mu}\phi$$

$$\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} = \partial_{\mu}\partial^{\mu}\phi = \partial^2\phi = \Box^2\phi$$

$$\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad \rightarrow \quad \partial_{\mu}\partial^{\mu}\phi + m^2\phi = 0$$

$$(\partial_0\partial^0\phi + \partial_1\partial^1\phi + \partial_2\partial^2\phi + \partial_3\partial^3\phi + m^2)\phi = 0$$

Implied sum

$$\mathcal{L} = \frac{1}{2}\partial^{\alpha}\phi\,\partial_{\alpha}\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\,\partial_{\beta}\phi - \frac{1}{2}m^2\phi^2 = \mathcal{L}(\phi, \partial_{\alpha}\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\alpha\beta}\frac{\partial(\partial_{\alpha}\phi)}{\partial(\partial_{\mu}\phi)}\partial_{\beta}\phi + \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\frac{\partial(\partial_{\beta}\phi)}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\mu\beta}\partial_{\beta}\phi + \frac{1}{2}g^{\mu\alpha}\partial_{\alpha}\phi = \frac{1}{2}\partial^{\mu}\phi + \frac{1}{2}\partial^{\mu}\phi = \partial^{\mu}\phi$$

$$\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} = \partial_{\mu}\partial^{\mu}\phi = \partial^2\phi = \Box^2\phi$$

$$\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad \rightarrow \quad \partial_{\mu}\partial^{\mu}\phi + m^2\phi = 0$$

$$(\partial_0\partial^0\phi + \partial_1\partial^1\phi + \partial_2\partial^2\phi + \partial_3\partial^3\phi + m^2)\phi = 0$$

$$\partial_t^2\phi - \partial_x^2\phi - \partial_y^2\phi - \partial_z^2\phi + m^2\phi = 0$$

Explicit spacetime coordinates

$$\mathcal{L} = \frac{1}{2}\partial^{\alpha}\phi\,\partial_{\alpha}\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\,\partial_{\beta}\phi - \frac{1}{2}m^2\phi^2 = \mathcal{L}(\phi, \partial_{\alpha}\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\alpha\beta}\frac{\partial(\partial_{\alpha}\phi)}{\partial(\partial_{\mu}\phi)}\partial_{\beta}\phi + \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\frac{\partial(\partial_{\beta}\phi)}{\partial(\partial_{\mu}\phi)} = \frac{1}{2}g^{\mu\beta}\partial_{\beta}\phi + \frac{1}{2}g^{\mu\alpha}\partial_{\alpha}\phi = \frac{1}{2}\partial^{\mu}\phi + \frac{1}{2}\partial^{\mu}\phi = \partial^{\mu}\phi$$

$$\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} = \partial_{\mu}\partial^{\mu}\phi = \partial^2\phi = \Box^2\phi$$

$$\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad \rightarrow \quad \partial_{\mu}\partial^{\mu}\phi + m^2\phi = 0$$

$$(\partial_0\partial^0\phi + \partial_1\partial^1\phi + \partial_2\partial^2\phi + \partial_3\partial^3\phi + m^2)\phi = 0$$

$$\partial_t^2\phi - \partial_x^2\phi - \partial_y^2\phi - \partial_z^2\phi + m^2\phi = 0$$

$$(\partial_t^2 - \nabla^2 + m^2)\phi = 0$$

Laplacian form

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - \frac{1}{2}m^2\phi^2 = \mathcal{L}(\phi, \partial_\alpha\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{1}{2}g^{\alpha\beta}\frac{\partial(\partial_\alpha\phi)}{\partial(\partial_\mu\phi)}\partial_\beta\phi + \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\frac{\partial(\partial_\beta\phi)}{\partial(\partial_\mu\phi)} = \frac{1}{2}g^{\mu\beta}\partial_\beta\phi + \frac{1}{2}g^{\mu\alpha}\partial_\alpha\phi = \frac{1}{2}\partial^\mu\phi + \frac{1}{2}\partial^\mu\phi = \partial^\mu\phi$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial_\mu\partial^\mu\phi = \partial^2\phi = \Box^2\phi$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad \rightarrow \quad \partial_\mu\partial^\mu\phi + m^2\phi = 0$$

$$(\partial_0\partial^0\phi + \partial_1\partial^1\phi + \partial_2\partial^2\phi + \partial_3\partial^3\phi + m^2)\phi = 0$$

$$\partial_t^2\phi - \partial_x^2\phi - \partial_y^2\phi - \partial_z^2\phi + m^2\phi = 0$$

$$(\partial_t^2 - \nabla^2 + m^2)\phi = 0$$

$$(\Box^2 + m^2)\phi = 0$$

d'Alembertian form

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - \frac{1}{2}m^2\phi^2 = \mathcal{L}(\phi, \partial_\alpha\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{1}{2}g^{\alpha\beta}\frac{\partial(\partial_\alpha\phi)}{\partial(\partial_\mu\phi)}\partial_\beta\phi + \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\frac{\partial(\partial_\beta\phi)}{\partial(\partial_\mu\phi)} = \frac{1}{2}g^{\mu\beta}\partial_\beta\phi + \frac{1}{2}g^{\mu\alpha}\partial_\alpha\phi = \frac{1}{2}\partial^\mu\phi + \frac{1}{2}\partial^\mu\phi = \partial^\mu\phi$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial_\mu\partial^\mu\phi = \partial^2\phi = \Box^2\phi$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad\rightarrow\quad \partial_\mu\partial^\mu\phi + m^2\phi = 0$$

$$(\partial_0\partial^0\phi + \partial_1\partial^1\phi + \partial_2\partial^2\phi + \partial_3\partial^3\phi + m^2)\phi = 0$$

$$\partial_t^2\phi - \partial_x^2\phi - \partial_y^2\phi - \partial_z^2\phi + m^2\phi = 0$$

$$(\partial_t^2 - \nabla^2 + m^2)\phi = 0$$

$$(\Box^2 + m^2)\phi = 0 \quad \blacksquare$$

Klein-Gordon Solutions

$\phi \propto e^{\pm i k \cdot x}$

Seek sinusoidal solutions

$$\phi \propto e^{\pm i k \cdot x}, \; k \cdot x = k^{\mu} x_{\mu}$$

Spacetime dot product as an implied sum

$$\phi \propto e^{\pm i k\cdot x}, \; k\cdot x = k^{\mu}x_{\mu} = g_{\mu\nu}k^{\mu}x^{\nu}$$

Mostly minus metric raises index for a double sum

$$\phi \propto e^{\pm i k\cdot x}, \;\; k\cdot x = k^{\mu}x_{\mu} = g_{\mu\nu}k^{\mu}x^{\nu} = k^0x^0 - k^1x^1 - k^2x^2 - k^3x^3$$

Expand double sum

$$\phi \propto e^{\pm i k\cdot x},\ \ k\cdot x = k^\mu x_\mu = g_{\mu\nu} k^\mu x^\nu = k^0 x^0 - k^1 x^1 - k^2 x^2 - k^3 x^3 = \omega t - \vec{k}\cdot\vec{x}$$

Spacetime split

$$\phi \propto e^{\pm i k\cdot x}, \;\; k\cdot x = k^\mu x_\mu = g_{\mu\nu} k^\mu x^\nu = k^0 x^0 - k^1 x^1 - k^2 x^2 - k^3 x^3 = \omega t - \vec{k}\cdot\vec{x}$$

$$\partial_\mu \phi = \pm i k_\mu \phi$$

First derivative

$$\phi \propto e^{\pm i k\cdot x}, \;\; k\cdot x = k^{\mu}x_{\mu} = g_{\mu\nu}k^{\mu}x^{\nu} = k^0x^0 - k^1x^1 - k^2x^2 - k^3x^3 = \omega t - \vec{k}\cdot\vec{x}$$

$$\partial_{\mu}\phi = \pm i k_{\mu}\phi$$

$$\partial^{\mu}\partial_{\mu}\phi = -k^{\mu}k_{\mu}\phi$$

Second derivative

$$\phi \propto e^{\pm i k\cdot x}, \;\; k\cdot x = k^\mu x_\mu = g_{\mu\nu} k^\mu x^\nu = k^0 x^0 - k^1 x^1 - k^2 x^2 - k^3 x^3 = \omega t - \vec{k}\cdot\vec{x}$$

$$\partial_\mu \phi = \pm i k_\mu \phi$$

$$\partial^\mu \partial_\mu \phi = -k^\mu k_\mu \phi = -k^2 \phi$$

Compact form

$$\phi \propto e^{\pm i k\cdot x}, \;\; k\cdot x = k^\mu x_\mu = g_{\mu\nu}k^\mu x^\nu = k^0x^0 - k^1x^1 - k^2x^2 - k^3x^3 = \omega t - \vec{k}\cdot\vec{x}$$

$$\partial_\mu \phi = \pm i k_\mu \phi$$

$$\partial^\mu\partial_\mu \phi = -k^\mu k_\mu \phi = -k^2\phi$$

$$(\partial^\mu\partial_\mu + m^2)\phi = 0 \;\; \rightarrow \;\; (-k^2 + m^2)\phi = 0$$

Substitute in the Klein-Gordon equation

$$\phi \propto e^{\pm i k\cdot x},\ \ k\cdot x = k^\mu x_\mu = g_{\mu\nu}k^\mu x^\nu = k^0x^0 - k^1x^1 - k^2x^2 - k^3x^3 = \omega t - \vec{k}\cdot\vec{x}$$

$$\partial_\mu \phi = \pm i k_\mu \phi$$

$$\partial^\mu \partial_\mu \phi = -k^\mu k_\mu \phi = -k^2\phi$$

$$(\partial^\mu\partial_\mu + m^2)\phi = 0 \ \rightarrow\ (-k^2 + m^2)\phi = 0 \ \rightarrow\ k^2 = m^2$$

$$\phi \propto e^{\pm i k\cdot x}, \;\; k\cdot x = k^\mu x_\mu = g_{\mu\nu}k^\mu x^\nu = k^0x^0 - k^1x^1 - k^2x^2 - k^3x^3 = \omega t - \vec{k}\cdot\vec{x}$$

$$\partial_\mu \phi = \pm i k_\mu \phi$$

$$\partial^\mu \partial_\mu \phi = -k^\mu k_\mu \phi = -k^2 \phi$$

$$(\partial^\mu\partial_\mu + m^2)\phi = 0 \;\rightarrow\; (-k^2 + m^2)\phi = 0 \;\rightarrow\; k^2 = m^2$$

Squared 4-momentum is squared mass, where $p = \hbar k$

$$\phi \propto e^{\pm ik\cdot x},\ \ k\cdot x = k^\mu x_\mu = g_{\mu\nu}k^\mu x^\nu = k^0x^0 - k^1x^1 - k^2x^2 - k^3x^3 = \omega t - \vec{k}\cdot\vec{x}$$

$$\partial_\mu \phi = \pm i k_\mu \phi$$

$$\partial^\mu \partial_\mu \phi = -k^\mu k_\mu \phi = -k^2 \phi$$

$$(\partial^\mu \partial_\mu + m^2)\phi = 0 \ \rightarrow\ (-k^2 + m^2)\phi = 0 \ \rightarrow\ k^2 = m^2$$

Squared 4-momentum is squared mass, where $p = k$ in natural units

$$\phi \propto e^{\pm i k\cdot x},\ \ k\cdot x = k^\mu x_\mu = g_{\mu\nu}k^\mu x^\nu = k^0x^0 - k^1x^1 - k^2x^2 - k^3x^3 = \omega t - \vec{k}\cdot\vec{x}$$

$$\partial_\mu \phi = \pm i k_\mu \phi$$

$$\partial^\mu \partial_\mu \phi = -k^\mu k_\mu \phi = -k^2 \phi$$

$$(\partial^\mu\partial_\mu + m^2)\phi = 0 \ \rightarrow\ (-k^2+m^2)\phi = 0 \ \rightarrow\ k^2 = m^2 \ \rightarrow\ k_0^2 = \vec{k}^2 + m^2 = \vec{k}\cdot\vec{k} + m^2 = \omega_{\vec{k}}^2$$

4-momentum time component

$$\phi \propto e^{\pm ik\cdot x}, \;\; k\cdot x = k^\mu x_\mu = g_{\mu\nu}k^\mu x^\nu = k^0x^0 - k^1x^1 - k^2x^2 - k^3x^3 = \omega t - \vec{k}\cdot\vec{x}$$

$$\partial_\mu \phi = \pm i k_\mu \phi$$

$$\partial^\mu \partial_\mu \phi = -k^\mu k_\mu \phi = -k^2 \phi$$

$$(\partial^\mu\partial_\mu + m^2)\phi = 0 \;\; \to \;\; (-k^2 + m^2)\phi = 0 \;\; \to \;\; k^2 = m^2 \;\; \to \;\; k_0^2 = \vec{k}^2 + m^2 = \vec{k}\cdot\vec{k} + m^2 = \omega_{\vec{k}}^2$$

$$\phi(x) \propto \int d^3k\, a(\vec{k}) e^{-ik\cdot x}$$

General solution is the superposition

$$\phi \propto e^{\pm ik\cdot x}, \;\; k\cdot x = k^\mu x_\mu = g_{\mu\nu}k^\mu x^\nu = k^0x^0 - k^1x^1 - k^2x^2 - k^3x^3 = \omega t - \vec{k}\cdot\vec{x}$$

$$\partial_\mu \phi = \pm i k_\mu \phi$$

$$\partial^\mu\partial_\mu\phi = -k^\mu k_\mu \phi = -k^2\phi$$

$$(\partial^\mu\partial_\mu + m^2)\phi = 0 \;\rightarrow\; (-k^2 + m^2)\phi = 0 \;\rightarrow\; k^2 = m^2 \;\rightarrow\; k_0^2 = \vec{k}^2 + m^2 = \vec{k}\cdot\vec{k} + m^2 = \omega_{\vec{k}}^2$$

$$\phi(x) \propto \int d^3k\, a(\vec{k}) e^{-ik\cdot x}$$

$$\phi(x) \propto \int d^3k \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

Add hermitian conjugate so $\phi = \phi^\dagger \in \mathbb{R}$

$$\phi \propto e^{\pm ik\cdot x}, \; k\cdot x = k^\mu x_\mu = g_{\mu\nu} k^\mu x^\nu = k^0 x^0 - k^1 x^1 - k^2 x^2 - k^3 x^3 = \omega t - \vec{k}\cdot\vec{x}$$

$$\partial_\mu \phi = \pm i k_\mu \phi$$

$$\partial^\mu \partial_\mu \phi = -k^\mu k_\mu \phi = -k^2 \phi$$

$$(\partial^\mu \partial_\mu + m^2)\phi = 0 \;\rightarrow\; (-k^2 + m^2)\phi = 0 \;\rightarrow\; k^2 = m^2 \;\rightarrow\; k_0^2 = \vec{k}^2 + m^2 = \vec{k}\cdot\vec{k} + m^2 = \omega_{\vec{k}}^2$$

$$\phi(x) \propto \int d^3k \, a(\vec{k}) e^{-ik\cdot x}$$

$$\phi(x) \propto \int d^3k \left(a(\vec{k}) e^{-ik\cdot x} + a^\dagger(\vec{k}) e^{ik\cdot x} \right)$$

Nationally, $a^\dagger(\vec{k}) = a(\vec{k})^\dagger$, like $\sin^2\theta = \sin^2(\theta) = \sin(\theta)^2 = (\sin\theta)^2$

$$\phi \propto e^{\pm i k \cdot x}, \; k \cdot x = k^\mu x_\mu = g_{\mu\nu} k^\mu x^\nu = k^0 x^0 - k^1 x^1 - k^2 x^2 - k^3 x^3 = \omega t - \vec{k} \cdot \vec{x}$$

$$\partial_\mu \phi = \pm i k_\mu \phi$$

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$$\phi(x) \propto \int d^3k \, a(\vec{k}) e^{-i k \cdot x}$$

$$\phi(x) \propto \int d^3k \left(a(\vec{k}) e^{-i k \cdot x} + a^\dagger(\vec{k}) e^{i k \cdot x} \right)$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \left(a(\vec{k}) e^{-i k \cdot x} + a^\dagger(\vec{k}) e^{i k \cdot x} \right)$$

Include Lorentz invariant measure normalization as $d^3k/\omega_{\vec{k}} = d^3p/E_{\vec{p}}$ is the same for all observers

$$\phi \propto e^{\pm i k\cdot x}, \;\; k\cdot x = k^\mu x_\mu = g_{\mu\nu} k^\mu x^\nu = k^0 x^0 - k^1 x^1 - k^2 x^2 - k^3 x^3 = \omega t - \vec{k}\cdot\vec{x}$$

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$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k}) e^{-ik\cdot x} + a^\dagger(\vec{k}) e^{ik\cdot x} \right)$$

Streamline notation

$$\phi \propto e^{\pm ik\cdot x}, \; k\cdot x = k^\mu x_\mu = g_{\mu\nu} k^\mu x^\nu = k^0 x^0 - k^1 x^1 - k^2 x^2 - k^3 x^3 = \omega t - \vec{k}\cdot\vec{x}$$

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$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k}) e^{-ik\cdot x} + a^\dagger(\vec{k}) e^{ik\cdot x} \right)$$

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}$$

Conjugate momentum density

$$\phi \propto e^{\pm ik\cdot x},\ \ k\cdot x = k^\mu x_\mu = g_{\mu\nu}k^\mu x^\nu = k^0x^0 - k^1x^1 - k^2x^2 - k^3x^3 = \omega t - \vec{k}\cdot\vec{x}$$

$$\partial_\mu \phi = \pm i k_\mu \phi$$

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$$\phi(x) \propto \int d^3k\, a(\vec{k})e^{-ik\cdot x}$$

$$\phi(x) \propto \int d^3k \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi$$

Conjugate momentum density for Lagrangian density $\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2$

$$\phi \propto e^{\pm ik\cdot x},\ \ k\cdot x = k^\mu x_\mu = g_{\mu\nu}k^\mu x^\nu = k^0x^0 - k^1x^1 - k^2x^2 - k^3x^3 = \omega t - \vec{k}\cdot\vec{x}$$

$$\partial_\mu \phi = \pm i k_\mu \phi$$

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$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi = \dot{\phi}$$

Compare mechanics $p = \dfrac{\partial L}{\partial \dot{x}} = m\dot{x}$ for Lagrangian $L = \frac{1}{2}m\dot{x}^2 - V(x)$

$$\phi \propto e^{\pm ik\cdot x},\ \ k\cdot x = k^\mu x_\mu = g_{\mu\nu}k^\mu x^\nu = k^0x^0 - k^1x^1 - k^2x^2 - k^3x^3 = \omega t - \vec{k}\cdot\vec{x}$$

$$\partial_\mu \phi = \pm ik_\mu \phi$$

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$$(\partial^\mu\partial_\mu + m^2)\phi = 0 \ \rightarrow\ (-k^2 + m^2)\phi = 0 \ \rightarrow\ k^2 = m^2 \ \rightarrow\ k_0^2 = \vec{k}^2 + m^2 = \vec{k}\cdot\vec{k} + m^2 = \omega_{\vec{k}}^2$$

$$\phi(x) \propto \int d^3k\, a(\vec{k})e^{-ik\cdot x}$$

$$\phi(x) \propto \int d^3k \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi = \dot{\phi} = \int \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

Time derivative $\partial^0 e^{\pm ik\cdot x} = \pm ik^0 e^{\pm ik\cdot x}$

$$\phi \propto e^{\pm ik\cdot x}, \;\; k\cdot x = k^\mu x_\mu = g_{\mu\nu}k^\mu x^\nu = k^0x^0 - k^1x^1 - k^2x^2 - k^3x^3 = \omega t - \vec{k}\cdot\vec{x}$$

$$\partial_\mu \phi = \pm i k_\mu \phi$$

$$\partial^\mu \partial_\mu \phi = -k^\mu k_\mu \phi = -k^2\phi$$

$$(\partial^\mu\partial_\mu + m^2)\phi = 0 \;\rightarrow\; (-k^2 + m^2)\phi = 0 \;\rightarrow\; k^2 = m^2 \;\rightarrow\; k_0^2 = \vec{k}^2 + m^2 = \vec{k}\cdot\vec{k} + m^2 = \omega_{\vec{k}}^2$$

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$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi = \dot{\phi} = \int \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right) \quad \blacksquare$$

Commutation Relations

$$[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta(\vec{x}-\vec{y})$$

Impose nonzero commutation relation on field operators

$$[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta(\vec{x} - \vec{y})$$

Nonzero only at same place at same time, a Lorentz invariant with which all observers will agree

$$[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta(\vec{x}-\vec{y})$$

Compare quantum mechanics $[x,p] = i\hbar$

$$[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta(\vec{x}-\vec{y})$$

Compare quantum mechanics $[x,p] = i$ in natural units

$$[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta(\vec{x}-\vec{y})$$

Else $[\phi,\phi] = 0 = [\pi,\pi]$

$$[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta(\vec{x}-\vec{y})$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

Recall field operator expansion

$[\phi(\vec{x},t),\pi(\vec{y},t)]=i\delta(\vec{x}-\vec{y})$

$$\phi(x)=\int\frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{-ik\cdot x}+a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\int d^3x\, e^{ik'\cdot x}\phi(x)=\int d^3x\,\frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{i(k'-k)\cdot x}+a^\dagger(\vec{k})e^{i(k'+k)\cdot x}\right)$$

Fourier transform the field operator

$[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta(\vec{x}-\vec{y})$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\int d^3x\, e^{ik'\cdot x}\phi(x) = \int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{i(k'-k)\cdot x} + a^\dagger(\vec{k})e^{i(k'+k)\cdot x}\right)$$

$$= \int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{i(\omega'-\omega)t}e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}} + a^\dagger(\vec{k})e^{i(\omega'+\omega)t}e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}}\right)$$

Spacetime split $k\cdot x = \omega t - \vec{k}\cdot\vec{x}$

$[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta(\vec{x}-\vec{y})$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\int d^3x\, e^{ik'\cdot x}\phi(x) = \int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{i(k'-k)\cdot x} + a^\dagger(\vec{k})e^{i(k'+k)\cdot x}\right)$$

$$= \int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{i(\omega'-\omega)t}e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}} + a^\dagger(\vec{k})e^{i(\omega'+\omega)t}e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}}\right)$$

$$= \int \frac{d^3k}{2\omega}\left(a(\vec{k})e^{i(\omega'-\omega)t}\delta^3(\vec{k}'-\vec{k}) + a^\dagger(\vec{k})e^{i(\omega'+\omega)t}\delta^3(\vec{k}'+\vec{k})\right)$$

Unit Fourier transform is the Dirac delta, $\int \frac{d^3x}{(2\pi)^3} e^{-i\vec{c}\cdot\vec{x}} = \delta^3(\vec{c})$

$[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta(\vec{x}-\vec{y})$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\int d^3x\, e^{ik'\cdot x}\phi(x) = \int d^3x\,\frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{i(k'-k)\cdot x} + a^\dagger(\vec{k})e^{i(k'+k)\cdot x}\right)$$

$$= \int d^3x\,\frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{i(\omega'-\omega)t}e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}} + a^\dagger(\vec{k})e^{i(\omega'+\omega)t}e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}}\right)$$

$$= \int \frac{d^3k}{2\omega}\left(a(\vec{k})e^{i(\omega'-\omega)t}\delta^3(\vec{k}'-\vec{k}) + a^\dagger(\vec{k})e^{i(\omega'+\omega)t}\delta^3(\vec{k}'+\vec{k})\right)$$

$$= \frac{1}{2\omega'}\left(a(\vec{k}') + a^\dagger(-\vec{k}')e^{i2\omega' t}\right)$$

Dirac delta sifting property, $\int d^3k\, f(\vec{k})\delta^3(\vec{k}-\vec{k}') = f(\vec{k}')$

$[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta(\vec{x}-\vec{y})$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\begin{aligned}
\int d^3x\, e^{ik'\cdot x}\phi(x) &= \int d^3x \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{i(k'-k)\cdot x} + a^\dagger(\vec{k})e^{i(k'+k)\cdot x}\right) \\
&= \int d^3x \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{i(\omega'-\omega)t}e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}} + a^\dagger(\vec{k})e^{i(\omega'+\omega)t}e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}}\right) \\
&= \int \frac{d^3k}{2\omega}\left(a(\vec{k})e^{i(\omega'-\omega)t}\delta^3(\vec{k}'-\vec{k}) + a^\dagger(\vec{k})e^{i(\omega'+\omega)t}\delta^3(\vec{k}'+\vec{k})\right) \\
&= \frac{1}{2\omega'}\left(a(\vec{k}') + a^\dagger(-\vec{k}')e^{i2\omega' t}\right)
\end{aligned}$$

Since $\omega^2 = \vec{k}^2 + m^2$, integrating over the Dirac deltas forces $\vec{k} = \pm\vec{k}'$ and hence $\omega = \omega'$

$$[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta(\vec{x}-\vec{y})$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\int d^3x\, e^{ik'\cdot x}\phi(x) = \int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{i(k'-k)\cdot x} + a^\dagger(\vec{k})e^{i(k'+k)\cdot x}\right)$$

$$= \int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{i(\omega'-\omega)t}e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}} + a^\dagger(\vec{k})e^{i(\omega'+\omega)t}e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}}\right)$$

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$$= \frac{1}{2\omega'}\left(a(\vec{k}') + a^\dagger(-\vec{k}')e^{i2\omega' t}\right)$$

$$\int d^3x\, e^{ik\cdot x}\phi(x) = \frac{1}{2\omega}\left(a(\vec{k}) + a^\dagger(-\vec{k})e^{i2\omega t}\right)$$

Omit the prime accents on k

$$\pi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

Recall conjugate momentum operator expansion

$$\pi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\int d^3x\, e^{ik'\cdot x}\pi(x) = \int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{i(k'-k)\cdot x} - a^\dagger(\vec{k})e^{i(k'+k)\cdot x}\right)$$

Fourier transform the conjugate momentum density operator

$$\pi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\int d^3x\, e^{ik'\cdot x}\pi(x) = \int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{i(k'-k)\cdot x} - a^\dagger(\vec{k})e^{i(k'+k)\cdot x}\right)$$

$$= \int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{i(\omega'-\omega)t}e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}} - a^\dagger(\vec{k})e^{i(\omega'+\omega)t}e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}}\right)$$

Spacetime split $k\cdot x = \omega t - \vec{k}\cdot\vec{x}$

$$\pi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

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$$= \int \frac{d^3k}{2\omega}(-i\omega)\left(a(\vec{k})e^{i(\omega'-\omega)t}\delta^3(\vec{k}' - \vec{k}) - a^\dagger(\vec{k})e^{i(\omega'+\omega)t}\delta^3(\vec{k}' + \vec{k})\right)$$

Unit Fourier transform is the Dirac delta, $\int \frac{d^3x}{(2\pi)^3}e^{-i\vec{c}\cdot\vec{x}} = \delta^3(\vec{c})$

$$\pi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\int d^3x\, e^{ik'\cdot x}\pi(x) = \int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{i(k'-k)\cdot x} - a^\dagger(\vec{k})e^{i(k'+k)\cdot x}\right)$$

$$= \int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{i(\omega'-\omega)t}e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}} - a^\dagger(\vec{k})e^{i(\omega'+\omega)t}e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}}\right)$$

$$= \int \frac{d^3k}{2\omega}(-i\omega)\left(a(\vec{k})e^{i(\omega'-\omega)t}\delta^3(\vec{k}'-\vec{k}) - a^\dagger(\vec{k})e^{i(\omega'+\omega)t}\delta^3(\vec{k}'+\vec{k})\right)$$

$$= \frac{1}{2i}\left(a(\vec{k}') - a^\dagger(-\vec{k}')e^{i2\omega' t}\right)$$

Dirac delta sifting property, $\int d^3k\, f(\vec{k})\delta^3(\vec{k}-\vec{k}') = f(\vec{k}')$

$$\pi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\int d^3x\, e^{ik'\cdot x}\pi(x) = \int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{i(k'-k)\cdot x} - a^\dagger(\vec{k})e^{i(k'+k)\cdot x}\right)$$

$$= \int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{i(\omega'-\omega)t}e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}} - a^\dagger(\vec{k})e^{i(\omega'+\omega)t}e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}}\right)$$

$$= \int \frac{d^3k}{2\omega}(-i\omega)\left(a(\vec{k})e^{i(\omega'-\omega)t}\delta^3(\vec{k}'-\vec{k}) - a^\dagger(\vec{k})e^{i(\omega'+\omega)t}\delta^3(\vec{k}'+\vec{k})\right)$$

$$= \frac{1}{2i}\left(a(\vec{k}') - a^\dagger(-\vec{k}')e^{i2\omega' t}\right)$$

Since $\omega^2 = \vec{k}^2 + m^2$, integrating over the Dirac deltas forces $\vec{k} = \pm\vec{k}'$ and hence $\omega = \omega'$

$$\pi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\int d^3x\, e^{ik'\cdot x}\pi(x) = \int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{i(k'-k)\cdot x} - a^\dagger(\vec{k})e^{i(k'+k)\cdot x}\right)$$

$$= \int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}(-i\omega)\left(a(\vec{k})e^{i(\omega'-\omega)t}e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}} - a^\dagger(\vec{k})e^{i(\omega'+\omega)t}e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}}\right)$$

$$= \int \frac{d^3k}{2\omega}(-i\omega)\left(a(\vec{k})e^{i(\omega'-\omega)t}\delta^3(\vec{k}'-\vec{k}) - a^\dagger(\vec{k})e^{i(\omega'+\omega)t}\delta^3(\vec{k}'+\vec{k})\right)$$

$$= \frac{1}{2i}\left(a(\vec{k}') - a^\dagger(-\vec{k}')e^{i2\omega' t}\right)$$

$$\int d^3x\, e^{ik\cdot x}\pi(x) = \frac{1}{2i}\left(a(\vec{k}) - a^\dagger(-\vec{k})e^{i2\omega t}\right)$$

Omit the prime accents on k

$$\int d^3x\, e^{ik\cdot x}\phi(x) = \frac{1}{2\omega}\left(a(\vec{k}) + a^\dagger(-\vec{k})e^{i2\omega t}\right)$$

$$\int d^3x\, e^{ik\cdot x}\pi(x) = \frac{1}{2i}\left(a(\vec{k}) - a^\dagger(-\vec{k})e^{i2\omega t}\right)$$

Recall Fourier transforms of field and momentum density operators

$$\int d^3x\, e^{ik\cdot x}\phi(x) = \frac{1}{2\omega}\left(a(\vec{k}) + a^{\dagger}(-\vec{k})e^{i2\omega t}\right)$$

$$\int d^3x\, e^{ik\cdot x}\pi(x) = \frac{1}{2i}\left(a(\vec{k}) - a^{\dagger}(-\vec{k})e^{i2\omega t}\right)$$

$$\int d^3x\, e^{ik\cdot x}\omega\phi(x) = \frac{1}{2}\left(a(\vec{k}) + a^{\dagger}(-\vec{k})e^{i2\omega t}\right)$$

$$\int d^3x\, e^{ik\cdot x}\, i\,\pi(x) = \frac{1}{2}\left(a(\vec{k}) - a^{\dagger}(-\vec{k})e^{i2\omega t}\right)$$

Multiply by constants ω and i

$$\int d^3x\, e^{ik\cdot x}\phi(x) = \frac{1}{2\omega}\left(a(\vec{k}) + a^\dagger(-\vec{k})e^{i2\omega t}\right)$$

$$\int d^3x\, e^{ik\cdot x}\pi(x) = \frac{1}{2i}\left(a(\vec{k}) - a^\dagger(-\vec{k})e^{i2\omega t}\right)$$

$$\int d^3x\, e^{ik\cdot x}\omega\phi(x) = \frac{1}{2}\left(a(\vec{k}) + a^\dagger(-\vec{k})e^{i2\omega t}\right)$$

$$\int d^3x\, e^{ik\cdot x}\; i\,\pi(x) = \frac{1}{2}\left(a(\vec{k}) - a^\dagger(-\vec{k})e^{i2\omega t}\right)$$

$$a(\vec{k}) = \int d^3x\, e^{+ik\cdot x}\,(\omega\phi(x) + i\pi(x))$$

Add and solve for annihilation operator

$$\int d^3x\, e^{ik\cdot x}\phi(x) = \frac{1}{2\omega}\left(a(\vec{k}) + a^\dagger(-\vec{k})e^{i2\omega t}\right)$$

$$\int d^3x\, e^{ik\cdot x}\pi(x) = \frac{1}{2i}\left(a(\vec{k}) - a^\dagger(-\vec{k})e^{i2\omega t}\right)$$

$$\int d^3x\, e^{ik\cdot x}\omega\phi(x) = \frac{1}{2}\left(a(\vec{k}) + a^\dagger(-\vec{k})e^{i2\omega t}\right)$$

$$\int d^3x\, e^{ik\cdot x}\, i\,\pi(x) = \frac{1}{2}\left(a(\vec{k}) - a^\dagger(-\vec{k})e^{i2\omega t}\right)$$

$$a(\vec{k}) = \int d^3x\, e^{+ik\cdot x}\left(\omega\phi(x) + i\pi(x)\right)$$

$$a^\dagger(\vec{k}) = \int d^3x\, e^{-ik\cdot x}\left(\omega\phi(x) - i\pi(x)\right)$$

Adjoint for creation operator

$$[a(\vec{k}), a^\dagger(\vec{k}')] = a(\vec{k})a^\dagger(\vec{k}') - a^\dagger(\vec{k}')a(\vec{k})$$

Commutator $[a, b] = ab - ba$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = a(\vec{k})a^\dagger(\vec{k}') - a^\dagger(\vec{k}')a(\vec{k})$$

$$= \int d^3x\, e^{ik\cdot x}\, (\omega\phi(\vec{x},t) + i\pi(\vec{x},t)) \int d^3x'\, e^{-ik'\cdot x'}\, \left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right)$$

$$- \int d^3x'\, e^{-ik'\cdot x'}\, \left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right) \int d^3x\, e^{ik\cdot x}\, (\omega\phi(\vec{x},t) + i\pi(\vec{x},t))$$

Substitute expansions in terms of field and momentum density operators, where $\phi(x) \rightarrow \phi(\vec{x},t)$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = a(\vec{k})a^\dagger(\vec{k}') - a^\dagger(\vec{k}')a(\vec{k})$$

$$= \int d^3x\, e^{ik\cdot x}\,(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)) \int d^3x'\, e^{-ik'\cdot x'}\,\big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big)$$

$$- \int d^3x'\, e^{-ik'\cdot x'}\,\big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big) \int d^3x\, e^{ik\cdot x}\,(\omega\phi(\vec{x},t) + i\pi(\vec{x},t))$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'}\Big(\ (\omega\phi(\vec{x},t) + i\pi(\vec{x},t))\,\big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big) - \big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big)\,(\omega\phi(\vec{x},t) + i\pi(\vec{x},t))\ \Big)$$

Combine integrals

$$[a(\vec{k}), a^\dagger(\vec{k}')] = a(\vec{k})a^\dagger(\vec{k}') - a^\dagger(\vec{k}')a(\vec{k})$$

$$= \int d^3x\, e^{ik\cdot x}\, (\omega\phi(\vec{x},t) + i\pi(\vec{x},t)) \int d^3x'\, e^{-ik'\cdot x'}\, \left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right)$$

$$- \int d^3x'\, e^{-ik'\cdot x'}\, \left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right) \int d^3x\, e^{ik\cdot x}\, (\omega\phi(\vec{x},t) + i\pi(\vec{x},t))$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'} \Big(\ (\omega\phi(\vec{x},t) + i\pi(\vec{x},t)) \left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right) \\ - \left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right) (\omega\phi(\vec{x},t) + i\pi(\vec{x},t)) \Big)$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'}\, \left(i\omega[\pi(\vec{x},t), \phi(\vec{x}\,',t)] - i\omega[\phi(\vec{x},t), \pi(\vec{x}\,',t)]\right)$$

Cross terms survive due to nonzero commutators

$[a(\vec{k}), a^\dagger(\vec{k}')] = a(\vec{k})a^\dagger(\vec{k}') - a^\dagger(\vec{k}')a(\vec{k})$

$$= \int d^3x\, e^{ik\cdot x}\, (\omega\phi(\vec{x},t) + i\pi(\vec{x},t)) \int d^3x'\, e^{-ik'\cdot x'}\, \big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big)$$

$$- \int d^3x'\, e^{-ik'\cdot x'}\, \big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big) \int d^3x\, e^{ik\cdot x}\, (\omega\phi(\vec{x},t) + i\pi(\vec{x},t))$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'} \Big(\big(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\big) \big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big) - \big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big) \big(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\big) \Big)$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'} \big(i\omega[\pi(\vec{x},t), \phi(\vec{x}\,',t)] - i\omega[\phi(\vec{x},t), \pi(\vec{x}\,',t)]\big)$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'}\, 2\omega\, \delta^3(\vec{x} - \vec{x}\,')$$

Nonzero commutators are $[\phi(\vec{x},t), \pi(\vec{y},t)] = i\delta(\vec{x} - \vec{y}) = i\delta(\vec{y} - \vec{x})$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = a(\vec{k})a^\dagger(\vec{k}') - a^\dagger(\vec{k}')a(\vec{k})$$

$$= \int d^3x\, e^{ik\cdot x}\,(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)) \int d^3x'\, e^{-ik'\cdot x'}\,\big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big)$$

$$- \int d^3x'\, e^{-ik'\cdot x'}\,\big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big) \int d^3x\, e^{ik\cdot x}\,(\omega\phi(\vec{x},t) + i\pi(\vec{x},t))$$

$$= \int d^3x'\, d^3x\, e^{ik\cdot x - ik'\cdot x'}\Big(\ \big(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\big)\big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big) - \big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big)\big(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\big)\Big)$$

$$= \int d^3x'\, d^3x\, e^{ik\cdot x - ik'\cdot x'}\,\big(i\omega[\pi(\vec{x},t), \phi(\vec{x}\,',t)] - i\omega[\phi(\vec{x},t), \pi(\vec{x}\,',t)]\big)$$

$$= \int d^3x'\, d^3x\, e^{ik\cdot x - ik'\cdot x'}\, 2\omega\, \delta^3(\vec{x} - \vec{x}\,')$$

$$= 2\omega \int d^3x\, e^{i(k-k')\cdot x}$$

Dirac delta sifting property

$$[a(\vec{k}), a^\dagger(\vec{k}')] = a(\vec{k})a^\dagger(\vec{k}') - a^\dagger(\vec{k}')a(\vec{k})$$

$$= \int d^3x\, e^{ik\cdot x}\, (\omega\phi(\vec{x},t) + i\pi(\vec{x},t)) \int d^3x'\, e^{-ik'\cdot x'}\, \left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right)$$

$$- \int d^3x'\, e^{-ik'\cdot x'}\, \left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right) \int d^3x\, e^{ik\cdot x}\, (\omega\phi(\vec{x},t) + i\pi(\vec{x},t))$$

$$= \int d^3x'\, d^3x\, e^{ik\cdot x - ik'\cdot x'} \Big(\left(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\right)\left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right) - \left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right)\left(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\right) \Big)$$

$$= \int d^3x'\, d^3x\, e^{ik\cdot x - ik'\cdot x'} \left(i\omega[\pi(\vec{x},t), \phi(\vec{x}\,',t)] - i\omega[\phi(\vec{x},t), \pi(\vec{x}\,',t)]\right)$$

$$= \int d^3x'\, d^3x\, e^{ik\cdot x - ik'\cdot x'}\, 2\omega\, \delta^3(\vec{x} - \vec{x}\,')$$

$$= 2\omega \int d^3x\, e^{i(k-k')\cdot x}$$

$$t^2 = \vec{x}^2 + \tau^2 \;\&\; \vec{x} = \vec{x}\,' \;\rightarrow\; t = t' \;\rightarrow\; x = x'$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = a(\vec{k})a^\dagger(\vec{k}') - a^\dagger(\vec{k}')a(\vec{k})$$

$$= \int d^3x\, e^{ik\cdot x}\,(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)) \int d^3x'\, e^{-ik'\cdot x'}\,\big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big)$$

$$- \int d^3x'\, e^{-ik'\cdot x'}\,\big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big) \int d^3x\, e^{ik\cdot x}\,(\omega\phi(\vec{x},t) + i\pi(\vec{x},t))$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'} \Big(\big(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\big)\big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big) - \big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big)\big(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\big) \Big)$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'}\,\big(i\omega[\pi(\vec{x},t), \phi(\vec{x}\,',t)] - i\omega[\phi(\vec{x},t), \pi(\vec{x}\,',t)]\big)$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'}\, 2\omega\, \delta^3(\vec{x} - \vec{x}\,')$$

$$= 2\omega \int d^3x\, e^{i(k-k')\cdot x} = 2\omega\, e^{i(\omega-\omega')t} \int d^3x\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

Spacetime split $k \cdot x = \omega t - \vec{k}\cdot\vec{x}$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = a(\vec{k})a^\dagger(\vec{k}') - a^\dagger(\vec{k}')a(\vec{k})$$

$$= \int d^3x\, e^{ik\cdot x}\,(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)) \int d^3x'\, e^{-ik'\cdot x'}\,\big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big)$$

$$- \int d^3x'\, e^{-ik'\cdot x'}\,\big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big) \int d^3x\, e^{ik\cdot x}\,(\omega\phi(\vec{x},t) + i\pi(\vec{x},t))$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'}\Big(\ (\omega\phi(\vec{x},t) + i\pi(\vec{x},t))\,\big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big) - \big(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\big)\,(\omega\phi(\vec{x},t) + i\pi(\vec{x},t))\ \Big)$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'}\,\big(i\omega[\pi(\vec{x},t), \phi(\vec{x}\,',t)] - i\omega[\phi(\vec{x},t), \pi(\vec{x}\,',t)]\big)$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'}\, 2\omega\, \delta^3(\vec{x} - \vec{x}\,')$$

$$= 2\omega \int d^3x\, e^{i(k-k')\cdot x} = 2\omega\, e^{i(\omega-\omega')t} \int d^3x\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} = 2\omega\, e^{i(\omega-\omega')t}(2\pi)^3\delta^3(\vec{k}-\vec{k}')$$

Unit Fourier transform is a Dirac delta

$$[a(\vec{k}), a^\dagger(\vec{k}')] = a(\vec{k})a^\dagger(\vec{k}') - a^\dagger(\vec{k}')a(\vec{k})$$

$$= \int d^3x\, e^{ik\cdot x} \left(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\right) \int d^3x'\, e^{-ik'\cdot x'} \left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right)$$

$$- \int d^3x'\, e^{-ik'\cdot x'} \left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right) \int d^3x\, e^{ik\cdot x} \left(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\right)$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'} \Big(\left(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\right) \left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right) - \left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right) \left(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\right) \Big)$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'} \left(i\omega[\pi(\vec{x},t), \phi(\vec{x}\,',t)] - i\omega[\phi(\vec{x},t), \pi(\vec{x}\,',t)]\right)$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'} 2\omega\, \delta^3(\vec{x} - \vec{x}\,')$$

$$= 2\omega \int d^3x\, e^{i(k-k')\cdot x} = 2\omega\, e^{i(\omega-\omega')t} \int d^3x\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} = 2\omega\, e^{i(\omega-\omega')t} (2\pi)^3 \delta^3(\vec{k} - \vec{k}') = 2\omega(2\pi)^3\delta^3(\vec{k} - \vec{k}')$$

$$\omega = \sqrt{\vec{k}^2 + m^2}\ \&\ \vec{k} = \vec{k}' \ \rightarrow\ \omega = \omega'$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = a(\vec{k})a^\dagger(\vec{k}') - a^\dagger(\vec{k}')a(\vec{k})$$

$$= \int d^3x\, e^{ik\cdot x}\left(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\right)\int d^3x'\, e^{-ik'\cdot x'}\left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right)$$

$$- \int d^3x'\, e^{-ik'\cdot x'}\left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right)\int d^3x\, e^{ik\cdot x}\left(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\right)$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'}\Big(\left(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\right)\left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right) - \left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right)\left(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\right)\Big)$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'}\left(i\omega[\pi(\vec{x},t), \phi(\vec{x}\,',t)] - i\omega[\phi(\vec{x},t), \pi(\vec{x}\,',t)]\right)$$

$$= \int d^3x' d^3x\, e^{ik\cdot x - ik'\cdot x'} 2\omega\, \delta^3(\vec{x} - \vec{x}\,')$$

$$= 2\omega \int d^3x\, e^{i(k-k')\cdot x} = 2\omega\, e^{i(\omega-\omega')t}\int d^3x\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} = 2\omega\, e^{i(\omega-\omega')t}(2\pi)^3\delta^3(\vec{k}-\vec{k}') = 2\omega(2\pi)^3\delta^3(\vec{k}-\vec{k}')$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3 2\omega\, \delta^3(\vec{k}-\vec{k}')$$

Lorentz invariant combination $\omega\,\delta^3(\vec{k}) = E\,\delta^3(\vec{p})$ is the same for all observers

$$[a(\vec{k}), a^\dagger(\vec{k}')] = a(\vec{k})a^\dagger(\vec{k}') - a^\dagger(\vec{k}')a(\vec{k})$$

$$= \int d^3x\, e^{ik\cdot x}\, (\omega\phi(\vec{x},t) + i\pi(\vec{x},t)) \int d^3x'\, e^{-ik'\cdot x'}\, (\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t))$$

$$- \int d^3x'\, e^{-ik'\cdot x'}\, (\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)) \int d^3x\, e^{ik\cdot x}\, (\omega\phi(\vec{x},t) + i\pi(\vec{x},t))$$

$$= \int d^3x'\, d^3x\, e^{ik\cdot x - ik'\cdot x'} \Big((\omega\phi(\vec{x},t) + i\pi(\vec{x},t))\,(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)) - (\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t))\,(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)) \Big)$$

$$= \int d^3x'\, d^3x\, e^{ik\cdot x - ik'\cdot x'} \left(i\omega[\pi(\vec{x},t), \phi(\vec{x}\,',t)] - i\omega[\phi(\vec{x},t), \pi(\vec{x}\,',t)]\right)$$

$$= \int d^3x'\, d^3x\, e^{ik\cdot x - ik'\cdot x'}\, 2\omega\, \delta^3(\vec{x} - \vec{x}\,')$$

$$= 2\omega \int d^3x\, e^{i(k-k')\cdot x} = 2\omega\, e^{i(\omega-\omega')t} \int d^3x\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} = 2\omega\, e^{i(\omega-\omega')t}(2\pi)^3\delta^3(\vec{k}-\vec{k}') = 2\omega(2\pi)^3\delta^3(\vec{k}-\vec{k}')$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3 2\omega\, \delta^3(\vec{k}-\vec{k}')$$

Else $[a, a] = 0 = [a^\dagger, a^\dagger]$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = a(\vec{k})a^\dagger(\vec{k}') - a^\dagger(\vec{k}')a(\vec{k})$$

$$= \int d^3x\, e^{ik\cdot x}\,(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)) \int d^3x'\, e^{-ik'\cdot x'}\,\left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right)$$

$$- \int d^3x'\, e^{-ik'\cdot x'}\,\left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right) \int d^3x\, e^{ik\cdot x}\,(\omega\phi(\vec{x},t) + i\pi(\vec{x},t))$$

$$= \int d^3x'\,d^3x\, e^{ik\cdot x - ik'\cdot x'} \Big(\left(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\right)\left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right) - \left(\omega\phi(\vec{x}\,',t) - i\pi(\vec{x}\,',t)\right)\left(\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\right) \Big)$$

$$= \int d^3x'\,d^3x\, e^{ik\cdot x - ik'\cdot x'} \left(i\omega[\pi(\vec{x},t), \phi(\vec{x}\,',t)] - i\omega[\phi(\vec{x},t), \pi(\vec{x}\,',t)]\right)$$

$$= \int d^3x'\,d^3x\, e^{ik\cdot x - ik'\cdot x'}\, 2\omega\,\delta^3(\vec{x} - \vec{x}\,')$$

$$= 2\omega \int d^3x\, e^{i(k-k')\cdot x} = 2\omega\, e^{i(\omega-\omega')t} \int d^3x\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} = 2\omega\, e^{i(\omega-\omega')t}(2\pi)^3\delta^3(\vec{k} - \vec{k}') = 2\omega(2\pi)^3\delta^3(\vec{k} - \vec{k}')$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3 2\omega\,\delta^3(\vec{k} - \vec{k}') \quad \blacksquare$$

Hamiltonian

$$\mathcal{L} = \frac{1}{2}\partial^\alpha \phi \, \partial_\alpha \phi - \frac{1}{2} m^2 \phi^2$$

Recall Lagrangian density for real fields

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right)$$

Spacetime split

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} - \mathcal{L}$$

Hamilton density is a Legendre transformation of the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} - \mathcal{L} \;\; = \dot{\phi}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \mathcal{L}$$

Momentum density operator $\pi = \partial_0\phi = \dot{\phi}$

$$\mathcal{L} = \frac{1}{2}\partial^\alpha \phi \, \partial_\alpha \phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} - \mathcal{L} \; = \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} \; = \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

Explicitly

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} - \mathcal{L} \ = \dot{\phi}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \mathcal{L} \ = \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

Compare mechanics where $L = \frac{1}{2}m\dot{x}^2 - V(x) \ \rightarrow \ H = p\frac{\partial L}{\partial\dot{x}} - L = \frac{p^2}{2m} + V(x)$

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} - \mathcal{L} = \dot{\phi}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \mathcal{L} = \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$H = \int d^3x\,\mathcal{H}$$

Hamiltonian is the spatial integral of the Hamiltonian density

$$\mathcal{L} = \frac{1}{2}\partial^\alpha \phi\, \partial_\alpha \phi - \frac{1}{2} m^2 \phi^2 = \frac{1}{2}\left(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} - \mathcal{L} \;\; = \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} \; = \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$H = \int d^3x\, \mathcal{H} \;\; = \frac{1}{2}\int d^3x \left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

Explicitly

$$\mathcal{L} = \frac{1}{2}\partial^\alpha \phi \, \partial_\alpha \phi - \frac{1}{2} m^2 \phi^2 = \frac{1}{2}\left(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} - \mathcal{L} \;\; = \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} \; = \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$H = \int d^3x \, \mathcal{H} \;\; = \frac{1}{2}\int d^3x \left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

Recall field operator expansion

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} - \mathcal{L} \; = \dot{\phi}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \mathcal{L} = \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$H = \int d^3x\,\mathcal{H} \; = \frac{1}{2}\int d^3x\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$\pi(x) = \partial_t\phi(x) = -\int\frac{d^3k}{(2\pi)^3 2\omega}i\omega\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

Time derivative, where $\partial_t e^{\pm ik\cdot x} = \pm i\omega\, e^{\pm ik\cdot x}$

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\dot\phi^2 - (\vec\nabla\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} - \mathcal{L} \;= \dot\phi\frac{\partial\mathcal{L}}{\partial\dot\phi} - \mathcal{L} \;= \frac{1}{2}\left(\pi^2 + (\vec\nabla\phi)^2 + m^2\phi^2\right)$$

$$H = \int d^3x\,\mathcal{H} \;= \frac{1}{2}\int d^3x\left(\pi^2 + (\vec\nabla\phi)^2 + m^2\phi^2\right)$$

$$\pi(x)^2 = \left(\int\frac{d^3k}{(2\pi)^3 2\omega}i\omega\left(a(\vec k)e^{-ik\cdot x} - a^\dagger(\vec k)e^{ik\cdot x}\right)\right)\cdot\left(\int\frac{d^3k'}{(2\pi)^3 2\omega'}i\omega'\left(a(\vec k')e^{-ik'\cdot x} - a^\dagger(\vec k')e^{ik'\cdot x}\right)\right)$$

Square

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\dot\phi^2 - (\vec\nabla\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} - \mathcal{L} \;= \dot\phi\frac{\partial\mathcal{L}}{\partial\dot\phi} - \mathcal{L} = \frac{1}{2}\left(\pi^2 + (\vec\nabla\phi)^2 + m^2\phi^2\right)$$

$$H = \int d^3x\,\mathcal{H} \;= \frac{1}{2}\int d^3x\left(\pi^2 + (\vec\nabla\phi)^2 + m^2\phi^2\right)$$

$$\pi(x)^2 = \int\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(-\omega\omega'\left(a(\vec k)e^{-ik\cdot x} - a^\dagger(\vec k)e^{ik\cdot x}\right)\left(a(\vec k')e^{-ik'\cdot x} - a^\dagger(\vec k')e^{ik'\cdot x}\right)\right)$$

Consolidate integrals

$$\mathcal{L} = \frac{1}{2}\partial^\alpha \phi\, \partial_\alpha \phi - \frac{1}{2} m^2 \phi^2 = \frac{1}{2}\left(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} - \mathcal{L} \;\; = \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} \;= \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$H = \int d^3x\, \mathcal{H} \;\; = \frac{1}{2}\int d^3x \left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$\pi(x)^2 = \int \frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(-\omega\omega' \left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} - a^\dagger(\vec{k}')e^{ik'\cdot x}\right)\right)$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

Recall field operator expansion

$$\mathcal{L} = \frac{1}{2}\partial^\alpha \phi\, \partial_\alpha \phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} - \mathcal{L} \;\; = \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} \; = \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$H = \int d^3x\, \mathcal{H} \;\; = \frac{1}{2}\int d^3x \left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$\pi(x)^2 = \int \frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(-\omega\omega'\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} - a^\dagger(\vec{k}')e^{ik'\cdot x}\right)\right)$$

$$\vec{\nabla}\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} i\vec{k}\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

Gradient, where $\vec{\nabla}e^{\pm ik\cdot x} = \mp i\vec{k}\, e^{\pm ik\cdot x}$

$$\mathcal{L} = \frac{1}{2}\partial^\alpha \phi\, \partial_\alpha \phi - \frac{1}{2} m^2 \phi^2 = \frac{1}{2}\left(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} - \mathcal{L} \;= \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} \;= \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$H = \int d^3x\, \mathcal{H} \;= \frac{1}{2}\int d^3x \left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$\pi(x)^2 = \int \frac{d^3k}{(2\pi)^3 2\omega} \frac{d^3k'}{(2\pi)^3 2\omega'} \left(-\omega\omega' \left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} - a^\dagger(\vec{k}')e^{ik'\cdot x}\right)\right)$$

$$\vec{\nabla}\phi(x)\cdot\vec{\nabla}\phi(x) = \left(\int \frac{d^3k}{(2\pi)^3 2\omega} i\vec{k}\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)\right)\cdot\left(\int \frac{d^3k'}{(2\pi)^3 2\omega'} i\vec{k}'\left(a(\vec{k}')e^{-ik'\cdot x} - a^\dagger(\vec{k}')e^{ik'\cdot x}\right)\right)$$

Square

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} - \mathcal{L} \;= \dot{\phi}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \mathcal{L} \;= \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$H = \int d^3x\,\mathcal{H} \;= \frac{1}{2}\int d^3x\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$\pi(x)^2 = \int \frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(-\omega\omega'\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} - a^\dagger(\vec{k}')e^{ik'\cdot x}\right)\right)$$

$$(\vec{\nabla}\phi(x))^2 = \int \frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(-\vec{k}\cdot\vec{k}'\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} - a^\dagger(\vec{k}')e^{ik'\cdot x}\right)\right)$$

Consolidate integrals

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\dot\phi^2 - (\vec\nabla\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} - \mathcal{L} \;= \dot\phi\frac{\partial\mathcal{L}}{\partial\dot\phi} - \mathcal{L} \;= \frac{1}{2}\left(\pi^2 + (\vec\nabla\phi)^2 + m^2\phi^2\right)$$

$$H = \int d^3x\,\mathcal{H} \;= \frac{1}{2}\int d^3x\left(\pi^2 + (\vec\nabla\phi)^2 + m^2\phi^2\right)$$

$$\pi(x)^2 = \int\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(-\omega\omega'\left(a(\vec k)e^{-ik\cdot x} - a^\dagger(\vec k)e^{ik\cdot x}\right)\left(a(\vec k')e^{-ik'\cdot x} - a^\dagger(\vec k')e^{ik'\cdot x}\right)\right)$$

$$(\vec\nabla\phi(x))^2 = \int\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(-\vec k\cdot\vec k'\left(a(\vec k)e^{-ik\cdot x} - a^\dagger(\vec k)e^{ik\cdot x}\right)\left(a(\vec k')e^{-ik'\cdot x} - a^\dagger(\vec k')e^{ik'\cdot x}\right)\right)$$

$$\phi(x) = \int\frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec k)e^{-ik\cdot x} + a^\dagger(\vec k)e^{ik\cdot x}\right)$$

Recall field operator expansion

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\dot\phi^2 - (\vec\nabla\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} - \mathcal{L} \;= \dot\phi\frac{\partial\mathcal{L}}{\partial\dot\phi} - \mathcal{L} \;= \frac{1}{2}\left(\pi^2 + (\vec\nabla\phi)^2 + m^2\phi^2\right)$$

$$H = \int d^3x\,\mathcal{H} \;= \frac{1}{2}\int d^3x\left(\pi^2 + (\vec\nabla\phi)^2 + m^2\phi^2\right)$$

$$\pi(x)^2 = \int\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(-\omega\omega'\left(a(\vec k)e^{-ik\cdot x} - a^\dagger(\vec k)e^{ik\cdot x}\right)\left(a(\vec k')e^{-ik'\cdot x} - a^\dagger(\vec k')e^{ik'\cdot x}\right)\right)$$

$$(\vec\nabla\phi(x))^2 = \int\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(-\vec k\cdot\vec k'\left(a(\vec k)e^{-ik\cdot x} - a^\dagger(\vec k)e^{ik\cdot x}\right)\left(a(\vec k')e^{-ik'\cdot x} - a^\dagger(\vec k')e^{ik'\cdot x}\right)\right)$$

$$\phi(x)^2 = \left(\int\frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec k)e^{-ik\cdot x} + a^\dagger(\vec k)e^{ik\cdot x}\right)\right)^2$$

Square

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\dot\phi^2 - (\vec\nabla\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} - \mathcal{L} \;= \dot\phi\frac{\partial\mathcal{L}}{\partial\dot\phi} - \mathcal{L} \;= \frac{1}{2}\left(\pi^2 + (\vec\nabla\phi)^2 + m^2\phi^2\right)$$

$$H = \int d^3x\,\mathcal{H} \;= \frac{1}{2}\int d^3x\left(\pi^2 + (\vec\nabla\phi)^2 + m^2\phi^2\right)$$

$$\pi(x)^2 = \int\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(-\omega\omega'\left(a(\vec k)e^{-ik\cdot x} - a^\dagger(\vec k)e^{ik\cdot x}\right)\left(a(\vec k')e^{-ik'\cdot x} - a^\dagger(\vec k')e^{ik'\cdot x}\right)\right)$$

$$(\vec\nabla\phi(x))^2 = \int\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(-\vec k\cdot\vec k'\left(a(\vec k)e^{-ik\cdot x} - a^\dagger(\vec k)e^{ik\cdot x}\right)\left(a(\vec k')e^{-ik'\cdot x} - a^\dagger(\vec k')e^{ik'\cdot x}\right)\right)$$

$$\phi(x)^2 = \int\frac{d^3k}{(2\pi)^3 2\omega}\left(a(\vec k)e^{-ik\cdot x} + a^\dagger(\vec k)e^{ik\cdot x}\right)\int\frac{d^3k'}{(2\pi)^3 2\omega'}\left(a(\vec k')e^{-ik'\cdot x} + a^\dagger(\vec k')e^{ik'\cdot x}\right)$$

Expand, remembering that $\omega^2 = m^2 + \vec k^2$

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} - \mathcal{L} \;= \dot{\phi}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \mathcal{L} \;= \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$H = \int d^3x\,\mathcal{H} \;= \frac{1}{2}\int d^3x\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$\pi(x)^2 = \int \frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(-\omega\omega'\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} - a^\dagger(\vec{k}')e^{ik'\cdot x}\right)\right)$$

$$(\vec{\nabla}\phi(x))^2 = \int \frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(-\vec{k}\cdot\vec{k}'\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} - a^\dagger(\vec{k}')e^{ik'\cdot x}\right)\right)$$

$$m^2\phi(x)^2 = \int \frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(m^2\left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} + a^\dagger(\vec{k}')e^{ik'\cdot x}\right)\right)$$

Consolidate the integrals & premultiply

$$\mathcal{L} = \frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right)$$

$$\mathcal{H} = \pi\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} - \mathcal{L} \;= \dot{\phi}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \mathcal{L} \;= \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$H = \int d^3x\,\mathcal{H} \;= \frac{1}{2}\int d^3x\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$\pi(x)^2 = \int\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(-\omega\omega'\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} - a^\dagger(\vec{k}')e^{ik'\cdot x}\right)\right)$$

$$(\vec{\nabla}\phi(x))^2 = \int\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(-\vec{k}\cdot\vec{k}'\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} - a^\dagger(\vec{k}')e^{ik'\cdot x}\right)\right)$$

$$m^2\phi(x)^2 = \int\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\left(m^2\left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} + a^\dagger(\vec{k}')e^{ik'\cdot x}\right)\right)$$

$$\begin{aligned}H = \frac{1}{2}\int d^3x\,\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\Big(&-\omega\omega'\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} - a^\dagger(\vec{k}')e^{ik'\cdot x}\right)\\ &-\vec{k}\cdot\vec{k}'\left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} - a^\dagger(\vec{k}')e^{ik'\cdot x}\right)\\ &+m^2\left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} + a^\dagger(\vec{k}')e^{ik'\cdot x}\right)\Big)\end{aligned}$$

Substitute

$$H = \frac{1}{2}\int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}\, \frac{d^3k'}{(2\pi)^3 2\omega'} \Big(\left(-\omega\omega' - \vec{k}\cdot\vec{k}' + m^2\right) a(\vec{k})a(\vec{k}')e^{-i(k+k')\cdot x}$$
$$+ \left(\omega\omega' + \vec{k}\cdot\vec{k}' + m^2\right) a(\vec{k})a^\dagger(\vec{k}')e^{-i(k-k')\cdot x}$$
$$+ \left(\omega\omega' + \vec{k}\cdot\vec{k}' + m^2\right) a^\dagger(\vec{k})a(\vec{k}')e^{i(k-k')\cdot x}$$
$$+ \left(-\omega\omega' - \vec{k}\cdot\vec{k}' + m^2\right) a^\dagger(\vec{k})a^\dagger(\vec{k}')e^{i(k+k')\cdot x} \Big)$$

Factor and FOIL

$$H = \frac{1}{2}\int d^3x\, \frac{d^3k}{(2\pi)^3 2\omega}\, \frac{d^3k'}{(2\pi)^3 2\omega'} \Big(-\omega\omega' \left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} - a^\dagger(\vec{k}')e^{ik'\cdot x}\right)$$
$$-\vec{k}\cdot\vec{k}' \left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} - a^\dagger(\vec{k}')e^{ik'\cdot x}\right)$$
$$+m^2 \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)\left(a(\vec{k}')e^{-ik'\cdot x} + a^\dagger(\vec{k}')e^{ik'\cdot x}\right) \Big)$$

$$
\begin{aligned}
H = \frac{1}{2}\int d^3x\,\frac{d^3k}{(2\pi)^3 2\omega}\,\frac{d^3k'}{(2\pi)^3 2\omega'}\Big(&\left(-\omega\omega' - \vec{k}\cdot\vec{k}' + m^2\right) a(\vec{k})a(\vec{k}')e^{-i(k+k')\cdot x}\\
&+\left(\ \ \omega\omega' + \vec{k}\cdot\vec{k}' + m^2\right) a(\vec{k})a^\dagger(\vec{k}')e^{-i(k-k')\cdot x}\\
&+\left(\ \ \omega\omega' + \vec{k}\cdot\vec{k}' + m^2\right) a^\dagger(\vec{k})a(\vec{k}')e^{i(k-k')\cdot x}\\
&+\left(-\omega\omega' - \vec{k}\cdot\vec{k}' + m^2\right) a^\dagger(\vec{k})a^\dagger(\vec{k}')e^{i(k+k')\cdot x}\Big)
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\,\frac{d^3k'}{(2\pi)^3 2\omega'}(2\pi)^3\Big(&\left(-\omega\omega' - \vec{k}\cdot\vec{k}' + m^2\right) a(\vec{k})a(\vec{k}')e^{-i(\omega+\omega')}\delta^3(\vec{k}+\vec{k}')\\
&+\left(\ \ \omega\omega' + \vec{k}\cdot\vec{k}' + m^2\right) a(\vec{k})a^\dagger(\vec{k}')e^{-i(\omega-\omega')}\delta^3(\vec{k}-\vec{k}')\\
&+\left(\ \ \omega\omega' + \vec{k}\cdot\vec{k}' + m^2\right) a^\dagger(\vec{k})a(\vec{k}')e^{i(\omega-\omega')}\delta^3(\vec{k}-\vec{k}')\\
&+\left(-\omega\omega' - \vec{k}\cdot\vec{k}' + m^2\right) a^\dagger(\vec{k})a^\dagger(\vec{k}')e^{i(\omega+\omega')}\delta^3(\vec{k}+\vec{k}')\Big)
\end{aligned}
$$

Unit Fourier transform is the Dirac delta, so $\int d^3x\, e^{\pm i\vec{c}\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{c})$

$$
\begin{aligned}
H = \frac{1}{2}\int d^3x \, \frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'} \Big(& \left(-\omega\omega' - \vec{k}\cdot\vec{k}' + m^2\right) a(\vec{k})a(\vec{k}')e^{-i(k+k')\cdot x} \\
& + \left(\ \ \omega\omega' + \vec{k}\cdot\vec{k}' + m^2\right) a(\vec{k})a^\dagger(\vec{k}')e^{-i(k-k')\cdot x} \\
& + \left(\ \ \omega\omega' + \vec{k}\cdot\vec{k}' + m^2\right) a^\dagger(\vec{k})a(\vec{k}')e^{i(k-k')\cdot x} \\
& + \left(-\omega\omega' - \vec{k}\cdot\vec{k}' + m^2\right) a^\dagger(\vec{k})a^\dagger(\vec{k}')e^{i(k+k')\cdot x} \Big)
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}(2\pi)^3 \Big(& \left(-\omega\omega' - \vec{k}\cdot\vec{k}' + m^2\right) a(\vec{k})a(\vec{k}')e^{-i(\omega+\omega')}\delta^3(\vec{k}+\vec{k}') \\
& + \left(\ \ \omega\omega' + \vec{k}\cdot\vec{k}' + m^2\right) a(\vec{k})a^\dagger(\vec{k}')e^{-i(\omega-\omega')}\delta^3(\vec{k}-\vec{k}') \\
& + \left(\ \ \omega\omega' + \vec{k}\cdot\vec{k}' + m^2\right) a^\dagger(\vec{k})a(\vec{k}')e^{i(\omega-\omega')}\delta^3(\vec{k}-\vec{k}') \\
& + \left(-\omega\omega' - \vec{k}\cdot\vec{k}' + m^2\right) a^\dagger(\vec{k})a^\dagger(\vec{k}')e^{i(\omega+\omega')}\delta^3(\vec{k}+\vec{k}') \Big)
\end{aligned}
$$

$$
= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}(2\pi)^3 \left(\omega\omega' + \vec{k}\cdot\vec{k}' + m^2\right)\left(a(\vec{k})a^\dagger(\vec{k}') + a^\dagger(\vec{k})a(\vec{k}')\right) e^{-i(\omega-\omega')}\delta^3(\vec{k}-\vec{k}')
$$

As $\vec{k} = \vec{k}' \to \omega = \omega'$ and $\omega^2 = m^2 + \vec{k}^2$

$$H = \frac{1}{2}\int d^3x \frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\Big(\Big(-\omega\omega' - \vec{k}\cdot\vec{k}' + m^2\Big) a(\vec{k})a(\vec{k}')e^{-i(k+k')\cdot x}$$
$$+ \Big(\ \ \omega\omega' + \vec{k}\cdot\vec{k}' + m^2\Big) a(\vec{k})a^\dagger(\vec{k}')e^{-i(k-k')\cdot x}$$
$$+ \Big(\ \ \omega\omega' + \vec{k}\cdot\vec{k}' + m^2\Big) a^\dagger(\vec{k})a(\vec{k}')e^{i(k-k')\cdot x}$$
$$+ \Big(-\omega\omega' - \vec{k}\cdot\vec{k}' + m^2\Big) a^\dagger(\vec{k})a^\dagger(\vec{k}')e^{i(k+k')\cdot x}\Big)$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}(2\pi)^3\Big(\Big(-\omega\omega' - \vec{k}\cdot\vec{k}' + m^2\Big) a(\vec{k})a(\vec{k}')e^{-i(\omega+\omega')}\delta^3(\vec{k}+\vec{k}')$$
$$+ \Big(\ \ \omega\omega' + \vec{k}\cdot\vec{k}' + m^2\Big) a(\vec{k})a^\dagger(\vec{k}')e^{-i(\omega-\omega')}\delta^3(\vec{k}-\vec{k}')$$
$$+ \Big(\ \ \omega\omega' + \vec{k}\cdot\vec{k}' + m^2\Big) a^\dagger(\vec{k})a(\vec{k}')e^{i(\omega-\omega')}\delta^3(\vec{k}-\vec{k}')$$
$$+ \Big(-\omega\omega' - \vec{k}\cdot\vec{k}' + m^2\Big) a^\dagger(\vec{k})a^\dagger(\vec{k}')e^{i(\omega+\omega')}\delta^3(\vec{k}+\vec{k}')\Big)$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}(2\pi)^3 \Big(\omega\omega' + \vec{k}\cdot\vec{k}' + m^2\Big)\Big(a(\vec{k})a^\dagger(\vec{k}') + a^\dagger(\vec{k})a(\vec{k}')\Big)\, e^{-i(\omega-\omega')}\delta^3(\vec{k}-\vec{k}')$$
$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\frac{1}{(2\pi)^3 2\omega}(2\pi)^3\, 2\omega^2 \Big(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})\Big)$$

Delta sift with $\vec{k} = -\vec{k}' \rightarrow \omega = \omega'$ and $\omega^2 = m^2 + \vec{k}^2$

$$H = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2\omega}\, \omega \left(a(\vec{k}) a^\dagger(\vec{k}) + a^\dagger(\vec{k}) a(\vec{k}) \right)$$

Cancel common factors

$$H = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2\omega} \frac{1}{(2\pi)^3 2\omega} (2\pi)^3\, 2\omega^2 \left(a(\vec{k}) a^\dagger(\vec{k}) + a^\dagger(\vec{k}) a(\vec{k}) \right)$$

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\left(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})\right)$$

$(ab)^\dagger = b^\dagger a^\dagger \rightarrow H^\dagger = H$ is hermitian

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\,\frac{1}{(2\pi)^3 2\omega}(2\pi)^3\,2\omega^2\left(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})\right)$$

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\left(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})\right)$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\left(2a^\dagger(\vec{k})a(\vec{k}) + (2\pi)^3 2\omega\,\delta^3(0)\right)$$

Apply commutator $a(\vec{k})a^\dagger(\vec{k}') - a^\dagger(\vec{k}')a(\vec{k})a^\dagger(\vec{k}') = [a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3 2\omega\,\delta^3(\vec{k} - \vec{k}')$

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\left(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})\right)$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\left(2a^\dagger(\vec{k})a(\vec{k}) + (2\pi)^3 2\omega\,\delta^3(0)\right)$$

$$\mathcal{N}(H) = :H: = H_n = \int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\, a^\dagger(\vec{k})a(\vec{k})$$

Shifting the energy by a (finite or infinite) constant doesn't change the dynamics

$$H = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2\omega}\, \omega \left(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k}) \right)$$

$$= \frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2\omega}\, \omega \left(2a^\dagger(\vec{k})a(\vec{k}) + (2\pi)^3 2\omega\, \delta^3(0) \right)$$

$$\mathcal{N}(H) = :H: = H_n = \int \frac{d^3k}{(2\pi)^3 2\omega}\, \omega\, a^\dagger(\vec{k})a(\vec{k})$$

The resulting Hamiltonian is normal-ordered

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\left(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})\right)$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\left(2a^\dagger(\vec{k})a(\vec{k}) + (2\pi)^3 2\omega\,\delta^3(0)\right)$$

$$\mathcal{N}(H) = :H: = H_n = \int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\,a^\dagger(\vec{k})a(\vec{k}) \quad \blacksquare$$

Energy Spectrum

$$H_n|E\rangle = E|E\rangle$$

Energy eigenvalue equation

$$H_n|E\rangle = E|E\rangle$$

$$a(\vec{k})|E\rangle$$

What is the energy of this state?

$H_n|E\rangle = E|E\rangle$

$$H_n a(\vec{k})|E\rangle = \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k}')a(\vec{k})|E\rangle$$

Apply the Hamiltonian operator expansion

$H_n|E\rangle = E|E\rangle$

$$H_n a(\vec{k})|E\rangle = \int \frac{d^3k'}{(2\pi)^3 2\omega'} \, \omega' a^\dagger(\vec{k}')a(\vec{k}')a(\vec{k})|E\rangle$$
$$= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \, \omega' a^\dagger(\vec{k}')a(\vec{k})a(\vec{k}')|E\rangle$$

Apply commutator $[a(\vec{k}), a(\vec{k}')] = 0$

$H_n|E\rangle = E|E\rangle$

$$H_n a(\vec{k})|E\rangle = \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k}')a(\vec{k})|E\rangle$$

$$= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k})a(\vec{k}')|E\rangle$$

$$= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' \left(a(\vec{k})a^\dagger(\vec{k}') - (2\pi)^3 2\omega\, \delta^3(\vec{k}' - \vec{k})\right) a(\vec{k}')|E\rangle$$

Apply commutator $[a^\dagger(\vec{k}'), a(\vec{k})] = -(2\pi)^3 2\omega\, \delta^3(\vec{k}' - \vec{k})$

$H_n|E\rangle = E|E\rangle$

$$
\begin{aligned}
H_n a(\vec{k})|E\rangle &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k}')a(\vec{k})|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k})a(\vec{k}')|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' \left(a(\vec{k})a^\dagger(\vec{k}') - (2\pi)^3 2\omega\, \delta^3(\vec{k}' - \vec{k})\right) a(\vec{k}')|E\rangle \\
&= a(\vec{k}) \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k}')|E\rangle - \omega\, a(\vec{k})|E\rangle
\end{aligned}
$$

Dirac delta sift on second term

$$H_n|E\rangle = E|E\rangle$$

$$\begin{aligned}
H_n a(\vec{k})|E\rangle &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega' a^\dagger(\vec{k}')a(\vec{k}')a(\vec{k})|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega' a^\dagger(\vec{k}')a(\vec{k})a(\vec{k}')|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega' \left(a(\vec{k})a^\dagger(\vec{k}') - (2\pi)^3 2\omega\,\delta^3(\vec{k}'-\vec{k})\right) a(\vec{k}')|E\rangle \\
&= a(\vec{k}) \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega' a^\dagger(\vec{k}')a(\vec{k}')|E\rangle - \omega\, a(\vec{k})|E\rangle \\
&= a(\vec{k})H_n|E\rangle - \omega\, a(\vec{k})|E\rangle
\end{aligned}$$

Hamiltonian operator expansion on first term

$$H_n|E\rangle = E|E\rangle$$

$$\begin{aligned}
H_n a(\vec{k})|E\rangle &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega' a^\dagger(\vec{k}')a(\vec{k}')a(\vec{k})|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega' a^\dagger(\vec{k}')a(\vec{k})a(\vec{k}')|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega' \left(a(\vec{k})a^\dagger(\vec{k}') - (2\pi)^3 2\omega\,\delta^3(\vec{k}' - \vec{k})\right) a(\vec{k}')|E\rangle \\
&= a(\vec{k}) \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega' a^\dagger(\vec{k}')a(\vec{k}')|E\rangle - \omega\, a(\vec{k})|E\rangle \\
&= a(\vec{k})H_n|E\rangle - \omega\, a(\vec{k})|E\rangle
\end{aligned}$$

$$H_n a(\vec{k})|E\rangle = (E-\omega)a(\vec{k})|E\rangle$$

As $H_n|E\rangle = E|E\rangle$

$$H_n|E\rangle = E|E\rangle$$

$$\begin{aligned}
H_n a(\vec{k})|E\rangle &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega'\, a^\dagger(\vec{k}')a(\vec{k}')a(\vec{k})|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega'\, a^\dagger(\vec{k}')a(\vec{k})a(\vec{k}')|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega' \left(a(\vec{k})a^\dagger(\vec{k}') - (2\pi)^3 2\omega\,\delta^3(\vec{k}' - \vec{k})\right) a(\vec{k}')|E\rangle \\
&= a(\vec{k}) \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega'\, a^\dagger(\vec{k}')a(\vec{k}')|E\rangle - \omega\, a(\vec{k})|E\rangle \\
&= a(\vec{k})H_n|E\rangle - \omega\, a(\vec{k})|E\rangle
\end{aligned}$$

$$H_n a(\vec{k})|E\rangle = (E-\omega)a(\vec{k})|E\rangle$$

a annihilates an energy quantum $\hbar\omega$

$$H_n|E\rangle = E|E\rangle$$

$$\begin{aligned}
H_n a(\vec{k})|E\rangle &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k}')a(\vec{k})|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k})a(\vec{k}')|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' \left(a(\vec{k})a^\dagger(\vec{k}') - (2\pi)^3 2\omega\, \delta^3(\vec{k}' - \vec{k})\right) a(\vec{k}')|E\rangle \\
&= a(\vec{k}) \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k}')|E\rangle - \omega\, a(\vec{k})|E\rangle \\
&= a(\vec{k})H_n|E\rangle - \omega\, a(\vec{k})|E\rangle
\end{aligned}$$

$$H_n a(\vec{k})|E\rangle = (E-\omega)a(\vec{k})|E\rangle$$

a annihilates an energy quantum ω in natural units

$$H_n|E\rangle = E|E\rangle$$

Energy eigenvalue equation

$H_n|E\rangle = E|E\rangle$

$$a^\dagger(\vec{k})|E\rangle$$

What is the energy of this state?

$$H_n|E\rangle = E|E\rangle$$

$$H_n a^\dagger(\vec{k})|E\rangle = \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega' a^\dagger(\vec{k}')a(\vec{k}')a^\dagger(\vec{k})|E\rangle$$

Apply the Hamiltonian operator expansion

$$H_n|E\rangle = E|E\rangle$$

$$\begin{aligned} H_n a^\dagger(\vec{k})|E\rangle &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k}')a^\dagger(\vec{k})|E\rangle \\ &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}') \left(a^\dagger(\vec{k})a(\vec{k}') + (2\pi)^3 2\omega\, \delta^3(\vec{k}' - \vec{k}) \right) |E\rangle \end{aligned}$$

Apply commutator $[a(\vec{k}'), a^\dagger(\vec{k})] = (2\pi)^3 2\omega\, \delta^3(\vec{k}' - \vec{k})$

$$H_n|E\rangle = E|E\rangle$$

$$\begin{aligned}
H_n a^\dagger(\vec{k})|E\rangle &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega'\, a^\dagger(\vec{k}')a(\vec{k}')a^\dagger(\vec{k})|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega'\, a^\dagger(\vec{k}')\left(a^\dagger(\vec{k})a(\vec{k}') + (2\pi)^3 2\omega\, \delta^3(\vec{k}' - \vec{k})\right)|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega'\, a^\dagger(\vec{k}')a^\dagger(\vec{k})a(\vec{k}')|E\rangle + \omega\, a^\dagger(\vec{k})|E\rangle
\end{aligned}$$

Dirac delta sift on second term

$H_n|E\rangle = E|E\rangle$

$$
\begin{aligned}
H_n a^\dagger(\vec{k})|E\rangle &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k}')a^\dagger(\vec{k})|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}') \left(a^\dagger(\vec{k})a(\vec{k}') + (2\pi)^3 2\omega\, \delta^3(\vec{k}' - \vec{k})\right) |E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a^\dagger(\vec{k})a(\vec{k}')|E\rangle + \omega\, a^\dagger(\vec{k})|E\rangle \\
&= a^\dagger(\vec{k}) \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k}')|E\rangle + \omega\, a^\dagger(\vec{k})|E\rangle
\end{aligned}
$$

Apply commutator $[a^\dagger(\vec{k}'), a^\dagger(\vec{k})] = 0$

$H_n|E\rangle = E|E\rangle$

$$\begin{aligned}
H_n a^\dagger(\vec{k})|E\rangle &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega'\,a^\dagger(\vec{k}')a(\vec{k}')a^\dagger(\vec{k})|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega'\,a^\dagger(\vec{k}')\left(a^\dagger(\vec{k})a(\vec{k}') + (2\pi)^3 2\omega\,\delta^3(\vec{k}'-\vec{k})\right)|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega'\,a^\dagger(\vec{k}')a^\dagger(\vec{k})a(\vec{k}')|E\rangle + \omega\,a^\dagger(\vec{k})|E\rangle \\
&= a^\dagger(\vec{k})\int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega'\,a^\dagger(\vec{k}')a(\vec{k}')|E\rangle + \omega\,a^\dagger(\vec{k})|E\rangle \\
&= a^\dagger(\vec{k})H_n|E\rangle + \omega\,a^\dagger(\vec{k})|E\rangle
\end{aligned}$$

Hamiltonian operator expansion on first term

$$H_n|E\rangle = E|E\rangle$$

$$\begin{aligned} H_n a^\dagger(\vec{k})|E\rangle &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k}')a^\dagger(\vec{k})|E\rangle \\ &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')\left(a^\dagger(\vec{k})a(\vec{k}') + (2\pi)^3 2\omega\, \delta^3(\vec{k}' - \vec{k})\right)|E\rangle \\ &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a^\dagger(\vec{k})a(\vec{k}')|E\rangle + \omega\, a^\dagger(\vec{k})|E\rangle \\ &= a^\dagger(\vec{k}) \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k}')|E\rangle + \omega\, a^\dagger(\vec{k})|E\rangle \\ &= a^\dagger(\vec{k})H_n|E\rangle + \omega\, a^\dagger(\vec{k})|E\rangle \end{aligned}$$

$$H_n a^\dagger(\vec{k})|E\rangle = (E+\omega)a^\dagger(\vec{k})|E\rangle$$

As $H_n|E\rangle = E|E\rangle$

$$H_n|E\rangle = E|E\rangle$$

$$\begin{aligned}
H_n a^\dagger(\vec{k})|E\rangle &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k}')a^\dagger(\vec{k})|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')\left(a^\dagger(\vec{k})a(\vec{k}') + (2\pi)^3 2\omega\, \delta^3(\vec{k}' - \vec{k})\right)|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a^\dagger(\vec{k})a(\vec{k}')|E\rangle + \omega\, a^\dagger(\vec{k})|E\rangle \\
&= a^\dagger(\vec{k}) \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k}')|E\rangle + \omega\, a^\dagger(\vec{k})|E\rangle \\
&= a^\dagger(\vec{k})H_n|E\rangle + \omega\, a^\dagger(\vec{k})|E\rangle
\end{aligned}$$

$$H_n a^\dagger(\vec{k})|E\rangle = (E+\omega)a^\dagger(\vec{k})|E\rangle$$

$a^\dagger$ creates an energy quantum $\hbar\omega$

$$H_n|E\rangle = E|E\rangle$$

$$\begin{aligned}
H_n a^\dagger(\vec{k})|E\rangle &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega'\, a^\dagger(\vec{k}')a(\vec{k}')a^\dagger(\vec{k})|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega'\, a^\dagger(\vec{k}')\left(a^\dagger(\vec{k})a(\vec{k}') + (2\pi)^3 2\omega\,\delta^3(\vec{k}'-\vec{k})\right)|E\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega'\, a^\dagger(\vec{k}')a^\dagger(\vec{k})a(\vec{k}')|E\rangle + \omega\, a^\dagger(\vec{k})|E\rangle \\
&= a^\dagger(\vec{k})\int \frac{d^3k'}{(2\pi)^3 2\omega'}\,\omega'\, a^\dagger(\vec{k}')a(\vec{k}')|E\rangle + \omega\, a^\dagger(\vec{k})|E\rangle \\
&= a^\dagger(\vec{k})H_n|E\rangle + \omega\, a^\dagger(\vec{k})|E\rangle
\end{aligned}$$

$$H_n a^\dagger(\vec{k})|E\rangle = (E+\omega)a^\dagger(\vec{k})|E\rangle$$

$a^\dagger$ creates an energy quantum ω in natural units

$$H_n|E\rangle = E|E\rangle$$

$$\begin{aligned} H_n a^\dagger(\vec{k})|E\rangle &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k}')a^\dagger(\vec{k})|E\rangle \\ &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}') \left(a^\dagger(\vec{k})a(\vec{k}') + (2\pi)^3 2\omega\, \delta^3(\vec{k}' - \vec{k})\right) |E\rangle \\ &= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a^\dagger(\vec{k})a(\vec{k}')|E\rangle + \omega\, a^\dagger(\vec{k})|E\rangle \\ &= a^\dagger(\vec{k}) \int \frac{d^3k'}{(2\pi)^3 2\omega'}\, \omega' a^\dagger(\vec{k}')a(\vec{k}')|E\rangle + \omega\, a^\dagger(\vec{k})|E\rangle \\ &= a^\dagger(\vec{k})H_n|E\rangle + \omega\, a^\dagger(\vec{k})|E\rangle \end{aligned}$$

$$H_n a^\dagger(\vec{k})|E\rangle = (E + \omega)a^\dagger(\vec{k})|E\rangle$$

$$H_n a(\vec{k})|E\rangle = (E - \omega)a(\vec{k})|E\rangle$$

a and $a^\dagger$ annihilate and create energy quanta

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k})$$

Normal-ordered Hamiltonian with normalized measure $\underline{d^3k}$ and number operator $N(\vec{k})$

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\, a^\dagger(\vec{k})a(\vec{k})$$

Expanding

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\ a^\dagger(\vec{k})a(\vec{k})$$

Canceling

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\, a^\dagger(\vec{k})a(\vec{k})$$

$$\langle\Psi|H_n|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\,\langle\Psi|a^\dagger(\vec{k})a(\vec{k})|\Psi\rangle$$

Energy expectation of a state

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\, \omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\ a^\dagger(\vec{k})a(\vec{k})$$

$$\langle\Psi|H_n|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \langle\Psi|a^\dagger(\vec{k})a(\vec{k})|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \|a(\vec{k})|\Psi\rangle\|^2 > 0$$

Energy expectation of a state is positive, as it is the sum of squared norms

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\, \omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\ a^\dagger(\vec{k})a(\vec{k})$$

$$\langle\Psi|H_n|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \langle\Psi|a^\dagger(\vec{k})a(\vec{k})|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \|a(\vec{k})|\Psi\rangle\|^2 > 0$$

Hence a lowest or ground or vacuum state $|0\rangle$ exists

$$H_n = \int \underline{d^3k} \; \hbar\omega_{\vec{k}} \; N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega} \, \omega \, a^\dagger(\vec{k}) a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2} \, a^\dagger(\vec{k}) a(\vec{k})$$

$$\langle \Psi | H_n | \Psi \rangle = \int \frac{d^3k}{(2\pi)^3 2} \langle \Psi | a^\dagger(\vec{k}) a(\vec{k}) | \Psi \rangle = \int \frac{d^3k}{(2\pi)^3 2} \| a(\vec{k}) | \Psi \rangle \|^2 > 0$$

$$a(\vec{k}) | 0 \rangle = 0$$

Annihilate the vacuum state

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\, \omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\ a^\dagger(\vec{k})a(\vec{k})$$

$$\langle\Psi|H_n|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \langle\Psi|a^\dagger(\vec{k})a(\vec{k})|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \|a(\vec{k})|\Psi\rangle\|^2 > 0$$

$$a(\vec{k})|0\rangle = 0$$

$$a^\dagger(\vec{k})|0\rangle = |\vec{k}\rangle$$

Create a particle of momentum $\vec{k}$ from the vacuum state

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\ \omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\ a^\dagger(\vec{k})a(\vec{k})$$

$$\langle\Psi|H_n|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \langle\Psi|a^\dagger(\vec{k})a(\vec{k})|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \|a(\vec{k})|\Psi\rangle\|^2 > 0$$

$$a(\vec{k})|0\rangle = 0$$

$$a^\dagger(\vec{k})|0\rangle = |\vec{k}\rangle$$

$$\frac{1}{\sqrt{n(\vec{k})!}}a^\dagger(\vec{k})^{n(\vec{k})}|0\rangle = |n(\vec{k})\rangle$$

Create n particles of momentum $\vec{k}$ from the vacuum state

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\ a^\dagger(\vec{k})a(\vec{k})$$

$$\langle\Psi|H_n|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \langle\Psi|a^\dagger(\vec{k})a(\vec{k})|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \|a(\vec{k})|\Psi\rangle\|^2 > 0$$

$$a(\vec{k})|0\rangle = 0$$

$$a^\dagger(\vec{k})|0\rangle = |\vec{k}\rangle$$

$$\frac{1}{\sqrt{n(\vec{k})!}}a^\dagger(\vec{k})^{n(\vec{k})}|0\rangle = |n(\vec{k})\rangle$$

Pre-factor normalizes the state as for the quantum simple harmonic oscillator

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\ \omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\ a^\dagger(\vec{k})a(\vec{k})$$

$$\langle\Psi|H_n|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \langle\Psi|a^\dagger(\vec{k})a(\vec{k})|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \|a(\vec{k})|\Psi\rangle\|^2 > 0$$

$$a(\vec{k})|0\rangle = 0$$

$$a^\dagger(\vec{k})|0\rangle = |\vec{k}\rangle$$

$$\frac{1}{\sqrt{n(\vec{k})!}}a^\dagger(\vec{k})^{n(\vec{k})}|0\rangle = |n(\vec{k})\rangle$$

Pre-factor normalizes the state as for the quantum simple harmonic oscillator

$$a^\dagger|n\rangle = c|n+1\rangle$$

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\, \omega\, a^\dagger(\vec{k}) a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\, a^\dagger(\vec{k}) a(\vec{k})$$

$$\langle\Psi|H_n|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\, \langle\Psi|a^\dagger(\vec{k}) a(\vec{k})|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\, \|a(\vec{k})|\Psi\rangle\|^2 > 0$$

$$a(\vec{k})|0\rangle = 0$$

$$a^\dagger(\vec{k})|0\rangle = |\vec{k}\rangle$$

$$\frac{1}{\sqrt{n(\vec{k})!}}\, a^\dagger(\vec{k})^{n(\vec{k})}|0\rangle = |n(\vec{k})\rangle$$

Pre-factor normalizes the state as for the quantum simple harmonic oscillator

$$a^\dagger|n\rangle = c|n+1\rangle \rightarrow \langle n|a = \langle n+1|c^*$$

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\, \omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\ a^\dagger(\vec{k})a(\vec{k})$$

$$\langle\Psi|H_n|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \langle\Psi|a^\dagger(\vec{k})a(\vec{k})|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \|a(\vec{k})|\Psi\rangle\|^2 > 0$$

$$a(\vec{k})|0\rangle = 0$$

$$a^\dagger(\vec{k})|0\rangle = |\vec{k}\rangle$$

$$\frac{1}{\sqrt{n(\vec{k})!}}a^\dagger(\vec{k})^{n(\vec{k})}|0\rangle = |n(\vec{k})\rangle$$

Pre-factor normalizes the state as for the quantum simple harmonic oscillator

$$a^\dagger|n\rangle = c|n+1\rangle \to \langle n|a = \langle n+1|c^*$$

$$\langle n+1|c^* c|n+1\rangle = \langle n|aa^\dagger|n\rangle$$

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\ \omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\ a^\dagger(\vec{k})a(\vec{k})$$

$$\langle\Psi|H_n|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \langle\Psi|a^\dagger(\vec{k})a(\vec{k})|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \|a(\vec{k})|\Psi\rangle\|^2 > 0$$

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$$a^\dagger|n\rangle = c|n+1\rangle \rightarrow \langle n|a = \langle n+1|c^*$$

$$\langle n+1|c^* c|n+1\rangle = \langle n|aa^\dagger|n\rangle = \langle n|a^\dagger a + 1|n\rangle$$

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\, N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\, a^\dagger(\vec{k})a(\vec{k})$$

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$$|c|^2\langle n+1|n+1\rangle = \langle n|N|n\rangle + \langle n|n\rangle$$

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\ \omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\ a^\dagger(\vec{k})a(\vec{k})$$

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$$|c|^2\langle n+1|n+1\rangle = \langle n|N|n\rangle + \langle n|n\rangle$$

$$|c|^2 = n+1$$

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\ \omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\ a^\dagger(\vec{k})a(\vec{k})$$

$$\langle\Psi|H_n|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \langle\Psi|a^\dagger(\vec{k})a(\vec{k})|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \|a(\vec{k})|\Psi\rangle\|^2 > 0$$

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$$|c|^2\langle n+1|n+1\rangle = \langle n|N|n\rangle + \langle n|n\rangle$$
$$|c|^2 = n+1 \leftarrow c = \sqrt{n+1}$$

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\ \omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\ a^\dagger(\vec{k})a(\vec{k})$$

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$$|c|^2 = n+1 \leftarrow c = \sqrt{n+1}$$

$$a^\dagger|0\rangle = \sqrt{1}\,|1\rangle$$

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\, a^\dagger(\vec{k})a(\vec{k})$$

$$\langle\Psi|H_n|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\,\langle\Psi|a^\dagger(\vec{k})a(\vec{k})|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\,\|a(\vec{k})|\Psi\rangle\|^2 > 0$$

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$$|c|^2 = n+1 \leftarrow c = \sqrt{n+1}$$

$$a^\dagger|0\rangle = \sqrt{1}\,|1\rangle$$

$$a^{\dagger 2}|0\rangle = \sqrt{1}\sqrt{2}\,|2\rangle$$

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\, a^\dagger(\vec{k})a(\vec{k})$$

$$\langle\Psi|H_n|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\,\langle\Psi|a^\dagger(\vec{k})a(\vec{k})|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\,\|a(\vec{k})|\Psi\rangle\|^2 > 0$$

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$$|c|^2\langle n+1|n+1\rangle = \langle n|N|n\rangle + \langle n|n\rangle$$
$$|c|^2 = n+1 \leftarrow c = \sqrt{n+1}$$
$$a^\dagger|0\rangle = \sqrt{1}\,|1\rangle$$
$$a^{\dagger 2}|0\rangle = \sqrt{1}\sqrt{2}\,|2\rangle$$
$$a^{\dagger n}|0\rangle = \sqrt{1\cdot 2\cdot 3\cdots n}\,|n\rangle = \sqrt{n!}\,|n\rangle$$

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\ a^\dagger(\vec{k})a(\vec{k})$$

$$\langle\Psi|H_n|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \langle\Psi|a^\dagger(\vec{k})a(\vec{k})|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\ \|a(\vec{k})|\Psi\rangle\|^2 > 0$$

$$a(\vec{k})|0\rangle = 0$$

$$a^\dagger(\vec{k})|0\rangle = |\vec{k}\rangle$$

$$\frac{1}{\sqrt{n(\vec{k})!}}a^\dagger(\vec{k})^{n(\vec{k})}|0\rangle = |n(\vec{k})\rangle$$

$$\prod_{i=1}^{K} \frac{a^\dagger(\vec{k}_i)^{n(\vec{k}_i)}}{\sqrt{n(\vec{k}_i!)}}|0\rangle = |n(\vec{k}_1)\cdots n(\vec{k}_K)\rangle$$

Create a general Fock state of $n(\vec{k}_i)$ particles of momentum $\vec{k}_i$ from the vacuum state

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\, \omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\, a^\dagger(\vec{k})a(\vec{k})$$

$$\langle\Psi|H_n|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\, \langle\Psi|a^\dagger(\vec{k})a(\vec{k})|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\, \|a(\vec{k})|\Psi\rangle\|^2 > 0$$

$$a(\vec{k})|0\rangle = 0$$

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$$\frac{1}{\sqrt{n(\vec{k})!}} a^\dagger(\vec{k})^{n(\vec{k})}|0\rangle = |n(\vec{k})\rangle$$

$$\prod_{i=1}^{K} \frac{a^\dagger(\vec{k}_i)^{n(\vec{k}_i)}}{\sqrt{n(\vec{k}_i!)}}|0\rangle = |n(\vec{k}_1)\cdots n(\vec{k}_K)\rangle$$

These particles are bosons because the creation operators commute

$$H_n = \int \underline{d^3k}\ \hbar\omega_{\vec{k}}\ N(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\, a^\dagger(\vec{k})a(\vec{k}) = \int \frac{d^3k}{(2\pi)^3 2}\, a^\dagger(\vec{k})a(\vec{k})$$

$$\langle\Psi|H_n|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\,\langle\Psi|a^\dagger(\vec{k})a(\vec{k})|\Psi\rangle = \int \frac{d^3k}{(2\pi)^3 2}\,\|a(\vec{k})|\Psi\rangle\|^2 > 0$$

$$a(\vec{k})|0\rangle = 0$$

$$a^\dagger(\vec{k})|0\rangle = |\vec{k}\rangle$$

$$\frac{1}{\sqrt{n(\vec{k})!}}a^\dagger(\vec{k})^{n(\vec{k})}|0\rangle = |n(\vec{k})\rangle$$

$$\prod_{i=1}^{K}\frac{a^\dagger(\vec{k}_i)^{n(\vec{k}_i)}}{\sqrt{n(\vec{k}_i!)}}|0\rangle = |n(\vec{k}_1)\cdots n(\vec{k}_K)\rangle \ \blacksquare$$

Feynman Propagator

$$(\Box^2 + m^2)\phi = 0$$

Recall the free Klein-Gordon equation

$$(\Box^2 + m^2)\phi(x) = J(x)$$

Add a source "current" $J(x) = J(\vec{x}, t)$

$(\Box^2 + m^2)\phi(x) = J(x)$

Compare Schrödinger equation $i\hbar\partial_t\psi(\vec{x}) = -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{x}) + V(\vec{x})\psi(\vec{x})$

$(\Box^2 + m^2)\phi(x) = J(x)$

Compare Schrödinger equation $\left(i\hbar\partial_t + \frac{\hbar^2}{2m}\nabla^2\right)\psi(\vec{x}) = V(\vec{x})\psi(\vec{x})$

$$(\Box^2 + m^2)\phi(x) = J(x)$$

Compare Schrödinger equation $\left(i\hbar\partial_t + \frac{\hbar^2}{2m}\nabla^2\right)\psi(\vec{x}) = V(\vec{x})\psi(\vec{x}) \equiv J(\vec{x})$

$$(\Box^2 + m^2)\phi(x) = J(x)$$

$$(\Box^2 + m^2)G(x - x') = -\delta^4(x - x')$$

"Green's function" propagator is the solution for a point source

$$(\Box^2 + m^2)\phi(x) = J(x)$$

$$(\Box^2 + m^2)G(x - x') = -\delta^4(x - x')$$

$$\phi(x) = \phi_0(x) - \int d^4x'\, G(x - x')J(x')$$

General solution is the superposition with the source, where $\phi_0(x)$ solves the free equation

$$(\Box^2 + m^2)\phi(x) = J(x)$$

$$(\Box^2 + m^2)G(x - x') = -\delta^4(x - x')$$

$$\phi(x) = \phi_0(x) - \int d^4x'\, G(x - x')J(x')$$

$$(\Box^2 + m^2)\phi(x) = (\Box^2 + m^2)\phi_0(x) - (\Box^2 + m^2)\int d^4x'\, G(x - x')J(x')$$

Check by applying $\Box^2 + m^2$ to both sides, where $\Box^2 = \partial_t^2 - \nabla_x^2$

$$(\Box^2 + m^2)\phi(x) = J(x)$$

$$(\Box^2 + m^2)G(x - x') = -\delta^4(x - x')$$

$$\phi(x) = \phi_0(x) - \int d^4x'\, G(x - x')J(x')$$

$$(\Box^2 + m^2)\phi(x) = (\Box^2 + m^2)\phi_0(x) - (\Box^2 + m^2)\int d^4x'\, G(x - x')J(x') = 0 + \int d^4x'\, \delta^4(x - x')J(x')$$

Free and point solutions

$$(\Box^2 + m^2)\phi(x) = J(x)$$

$$(\Box^2 + m^2)G(x - x') = -\delta^4(x - x')$$

$$\phi(x) = \phi_0(x) - \int d^4x'\, G(x - x')J(x')$$

$$(\Box^2 + m^2)\phi(x) = (\Box^2 + m^2)\phi_0(x) - (\Box^2 + m^2)\int d^4x'\, G(x - x')J(x') = 0 + \int d^4x'\, \delta^4(x - x')J(x') = J(x)$$

Dirac delta sifting

$$(\Box^2 + m^2)\phi(x) = J(x)$$

$$(\Box^2 + m^2)G(x - x') = -\delta^4(x - x')$$

$$\phi(x) = \phi_0(x) - \int d^4x' \, G(x - x')J(x')$$

$$(\Box^2 + m^2)\phi(x) = (\Box^2 + m^2)\phi_0(x) - (\Box^2 + m^2) \int d^4x' \, G(x - x')J(x') = 0 + \int d^4x' \, \delta^4(x - x')J(x') = J(x)$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4} \, e^{-ik\cdot(x-x')} \tilde{G}(k)$$

Fourier expand into momentum states whose components k^μ are independent, so generically $k^0 \neq \omega$

$$(\Box^2 + m^2)\phi(x) = J(x)$$

$$(\Box^2 + m^2)G(x - x') = -\delta^4(x - x')$$

$$\phi(x) = \phi_0(x) - \int d^4x'\, G(x - x')J(x')$$

$$(\Box^2 + m^2)\phi(x) = (\Box^2 + m^2)\phi_0(x) - (\Box^2 + m^2)\int d^4x'\, G(x - x')J(x') = 0 + \int d^4x'\, \delta^4(x - x')J(x') = J(x)$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')}G(k)$$

Simpler notation is common but slightly ambiguous

$$(\Box^2 + m^2)\phi(x) = J(x)$$

$$(\Box^2 + m^2)G(x - x') = -\delta^4(x - x')$$

$$\phi(x) = \phi_0(x) - \int d^4x'\, G(x - x')J(x')$$

$$(\Box^2 + m^2)\phi(x) = (\Box^2 + m^2)\phi_0(x) - (\Box^2 + m^2)\int d^4x'\, G(x - x')J(x') = 0 + \int d^4x'\, \delta^4(x - x')J(x') = J(x)$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')}G(k)$$

$$\delta^4(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')}$$

Fourier expand the Dirac delta point source into momentum states

$$(\Box^2 + m^2)\phi(x) = J(x)$$

$$(\Box^2 + m^2)G(x - x') = -\delta^4(x - x')$$

$$\phi(x) = \phi_0(x) - \int d^4x'\, G(x - x')J(x')$$

$$(\Box^2 + m^2)\phi(x) = (\Box^2 + m^2)\phi_0(x) - (\Box^2 + m^2)\int d^4x'\, G(x - x')J(x') = 0 + \int d^4x'\, \delta^4(x - x')J(x') = J(x)$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')}G(k)$$

$$\delta^4(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')}$$

$$(\Box^2 + m^2)\int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')}G(k) = -\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')}$$

Substitute into propagator Green's function differential equation

$$(\Box^2 + m^2)\phi(x) = J(x)$$

$$(\Box^2 + m^2)G(x - x') = -\delta^4(x - x')$$

$$\phi(x) = \phi_0(x) - \int d^4x'\, G(x - x')J(x')$$

$$(\Box^2 + m^2)\phi(x) = (\Box^2 + m^2)\phi_0(x) - (\Box^2 + m^2)\int d^4x'\, G(x - x')J(x') = 0 + \int d^4x'\, \delta^4(x - x')J(x') = J(x)$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')}G(k)$$

$$\delta^4(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')}$$

$$(\Box^2 + m^2)\int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')}G(k) = -\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')}$$

$$\int \frac{d^4k}{(2\pi)^4}\,(-k^2 + m^2)e^{-ik\cdot(x-x')}G(k) = -\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')}$$

Differentiate with $\Box^2 = \partial_t^2 - \nabla_x^2$

$$(\Box^2 + m^2)\phi(x) = J(x)$$

$$(\Box^2 + m^2)G(x - x') = -\delta^4(x - x')$$

$$\phi(x) = \phi_0(x) - \int d^4x'\, G(x - x')J(x')$$

$$(\Box^2 + m^2)\phi(x) = (\Box^2 + m^2)\phi_0(x) - (\Box^2 + m^2)\int d^4x'\, G(x - x')J(x') = 0 + \int d^4x'\, \delta^4(x - x')J(x') = J(x)$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')}G(k)$$

$$\delta^4(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')}$$

$$(\Box^2 + m^2)\int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')}G(k) = -\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')}$$

$$\int \frac{d^4k}{(2\pi)^4}\,(-k^2 + m^2)e^{-ik\cdot(x-x')}G(k) = -\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')}$$

$$(-k^2 + m^2)G(k) = -1$$

Compare integrands

$$G(k) = \frac{1}{k^2 - m^2}$$

Solve for the momentum space propagator

$$(-k^2 + m^2)G(k) = -1$$

$$G(k) = \frac{1}{k^2 - m^2} = \frac{1}{(k^0)^2 - \vec{k}^{\,2} - m^2}$$

Spacetime split

$$G(k) = \frac{1}{k^2 - m^2} = \frac{1}{(k^0)^2 - \vec{k}^{\,2} - m^2} = \frac{1}{(k^0)^2 - \omega_+^2}, \quad \omega_+ = +\sqrt{\vec{k}^{\,2} + m^2} > 0$$

Substitute the positive frequency

$$G(k) = \frac{1}{k^2 - m^2} = \frac{1}{(k^0)^2 - \vec{k}^{\,2} - m^2} = \frac{1}{(k^0)^2 - \omega_+^2}, \quad \omega_+ = +\sqrt{\vec{k}^{\,2} + m^2} > 0$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} G(k)$$

Coordinate space Green's function propagator is the Fourier transform of the momentum space propagator

$$G(k) = \frac{1}{k^2 - m^2} = \frac{1}{(k^0)^2 - \vec{k}^{\,2} - m^2} = \frac{1}{(k^0)^2 - \omega_+^2}, \quad \omega_+ = +\sqrt{\vec{k}^{\,2} + m^2} > 0$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} G(k)$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} \frac{1}{(k^0)^2 - \omega_+^2}$$

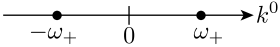
Substitute

$$G(k) = \frac{1}{k^2 - m^2} = \frac{1}{(k^0)^2 - \vec{k}^{\,2} - m^2} = \frac{1}{(k^0)^2 - \omega_+^2}, \quad \omega_+ = +\sqrt{\vec{k}^{\,2} + m^2} > 0$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} G(k)$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} \frac{1}{(k^0)^2 - \omega_+^2}$$

Integrand diverges at two poles $k^0 = \pm\omega_+$ on the real axis

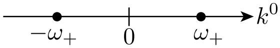


$$G(k) = \frac{1}{k^2 - m^2} = \frac{1}{(k^0)^2 - \vec{k}^{\,2} - m^2} = \frac{1}{(k^0)^2 - \omega_+^2}, \quad \omega_+ = +\sqrt{\vec{k}^{\,2} + m^2} > 0$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} G(k)$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} \frac{1}{(k^0)^2 - \omega_+^2}$$

$$\Delta_F(k) = \frac{1}{k^2 - m^2 + i\epsilon'}$$



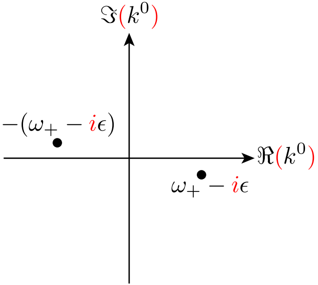
Feynman propagator complexifies the Green's function ...

$$G(k) = \frac{1}{k^2 - m^2} = \frac{1}{(k^0)^2 - \vec{k}^{\,2} - m^2} = \frac{1}{(k^0)^2 - \omega_+^2}, \quad \omega_+ = +\sqrt{\vec{k}^{\,2} + m^2} > 0$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} G(k)$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} \frac{1}{(k^0)^2 - \omega_+^2}$$

$$\Delta_F(k) = \frac{1}{k^2 - m^2 + i\epsilon'}$$

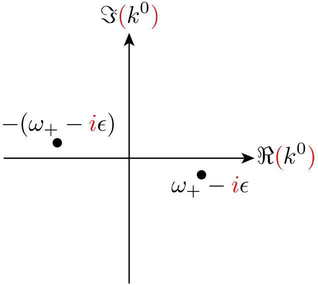


by moving the poles infinitesimally off the real axis

$$G(k) = \frac{1}{k^2 - m^2} = \frac{1}{(k^0)^2 - \vec{k}^{\,2} - m^2} = \frac{1}{(k^0)^2 - \omega_+^2}, \quad \omega_+ = +\sqrt{\vec{k}^{\,2} + m^2} > 0$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} G(k)$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} \frac{1}{(k^0)^2 - \omega_+^2}$$



$$\Delta_F(k) = \frac{1}{k^2 - m^2 + i\epsilon'} = \frac{1}{(k^0)^2 - (\omega_+ - i\epsilon)^2} + \mathcal{O}(\epsilon^2)$$

where $\epsilon' = \epsilon\, 2\omega_+ \downarrow 0$

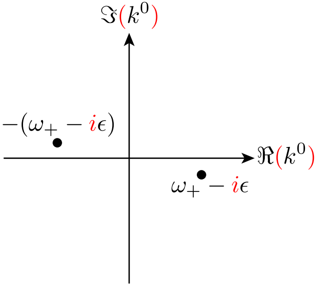
$$G(k) = \frac{1}{k^2 - m^2} = \frac{1}{(k^0)^2 - \vec{k}^{\,2} - m^2} = \frac{1}{(k^0)^2 - \omega_+^2}, \quad \omega_+ = +\sqrt{\vec{k}^{\,2} + m^2} > 0$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} G(k)$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} \frac{1}{(k^0)^2 - \omega_+^2}$$

$$\Delta_F(k) = \frac{1}{k^2 - m^2 + i\epsilon'} = \frac{1}{(k^0)^2 - (\omega_+ - i\epsilon)^2} + \mathcal{O}(\epsilon^2)$$

$$= \frac{1}{2\omega_+}\left(\frac{1}{k^0 - (\omega_+ - i\epsilon)} - \frac{1}{k^0 + (\omega_+ - i\epsilon)}\right) + \mathcal{O}(\epsilon)$$



Partial fraction decomposition as $\epsilon \downarrow 0$

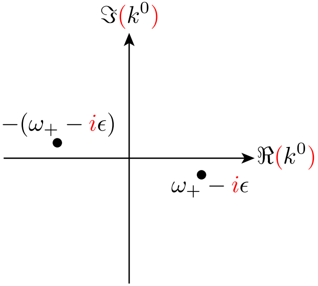
$$G(k) = \frac{1}{k^2 - m^2} = \frac{1}{(k^0)^2 - \vec{k}^2 - m^2} = \frac{1}{(k^0)^2 - \omega_+^2}, \quad \omega_+ = +\sqrt{\vec{k}^2 + m^2} > 0$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} G(k)$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} \frac{1}{(k^0)^2 - \omega_+^2}$$

$$\Delta_F(k) = \frac{1}{k^2 - m^2 + i\epsilon'} = \frac{1}{(k^0)^2 - (\omega_+ - i\epsilon)^2} + \mathcal{O}(\epsilon^2)$$

$$= \frac{1}{2\omega_+}\left(\frac{1}{k^0 - (\omega_+ - i\epsilon)} - \frac{1}{k^0 + (\omega_+ - i\epsilon)}\right) + \mathcal{O}(\epsilon)$$



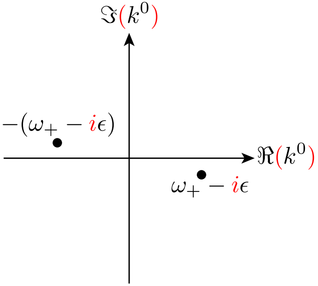
$$\Delta_F(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} \Delta_F(k)$$

Coordinate space Feynman propagator is the Fourier transform of the momentum space propagator

$$G(k) = \frac{1}{k^2 - m^2} = \frac{1}{(k^0)^2 - \vec{k}^{\,2} - m^2} = \frac{1}{(k^0)^2 - \omega_+^2}, \quad \omega_+ = +\sqrt{\vec{k}^{\,2} + m^2} > 0$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} G(k)$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} \frac{1}{(k^0)^2 - \omega_+^2}$$



$$\Delta_F(k) = \frac{1}{k^2 - m^2 + i\epsilon'} = \frac{1}{(k^0)^2 - (\omega_+ - i\epsilon)^2} + \mathcal{O}(\epsilon^2)$$

$$= \frac{1}{2\omega_+}\left(\frac{1}{k^0 - (\omega_+ - i\epsilon)} - \frac{1}{k^0 + (\omega_+ - i\epsilon)}\right) + \mathcal{O}(\epsilon)$$

$$\Delta_F(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} \Delta_F(k)$$

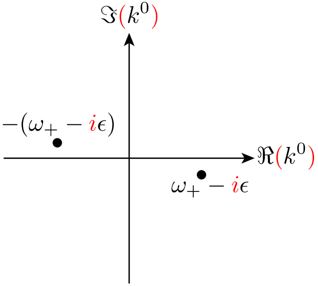
$$= \int \frac{d^3k}{(2\pi)^3 2\omega_+}\, e^{-ik\cdot(x-x')} \left(\frac{1}{k^0 - (\omega_+ - i\epsilon)} - \frac{1}{k^0 + (\omega_+ - i\epsilon)}\right)$$

Substitute

$$G(k) = \frac{1}{k^2 - m^2} = \frac{1}{(k^0)^2 - \vec{k}^{\,2} - m^2} = \frac{1}{(k^0)^2 - \omega_+^2}, \quad \omega_+ = +\sqrt{\vec{k}^{\,2} + m^2} > 0$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} G(k)$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} \frac{1}{(k^0)^2 - \omega_+^2}$$



$$\Delta_F(k) = \frac{1}{k^2 - m^2 + i\epsilon'} = \frac{1}{(k^0)^2 - (\omega_+ - i\epsilon)^2} + \mathcal{O}(\epsilon^2)$$

$$= \frac{1}{2\omega_+}\left(\frac{1}{k^0 - (\omega_+ - i\epsilon)} - \frac{1}{k^0 + (\omega_+ - i\epsilon)}\right) + \mathcal{O}(\epsilon)$$

$$\Delta_F(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} \Delta_F(k)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_+}\, e^{-ik\cdot(x-x')} \left(\frac{1}{k^0 - (\omega_+ - i\epsilon)} - \frac{1}{k^0 + (\omega_+ - i\epsilon)}\right)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_+}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{dk^0}{2\pi}\, e^{-ik^0(t-t')} \left(\frac{1}{k^0 - (\omega_+ - i\epsilon)} - \frac{1}{k^0 + (\omega_+ - i\epsilon)}\right)$$

Spacetime split with $k \cdot x = k^0 t - \vec{k} \cdot \vec{x}$

$$\int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega - (\omega_+ - i\epsilon)}$$

Extend the first $\omega = k^0$ real integral ...

$$\Delta_F(x - x') = \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{dk^0}{2\pi} e^{-ik^0(t-t')} \left(\frac{1}{k^0 - (\omega_+ - i\epsilon)} - \frac{1}{k^0 + (\omega_+ - i\epsilon)} \right)$$

$$\int_A \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega-(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)}$$

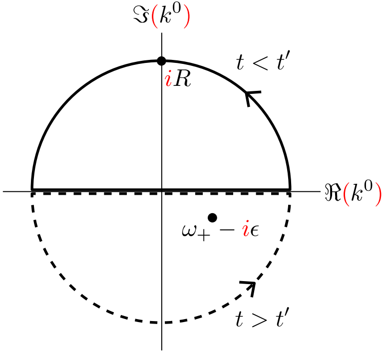
to a closed contour C in the complex plane

$$\int_A \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega-(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)}$$

where $z \in \mathbb{C}$ and $\omega \in \mathbb{R}$

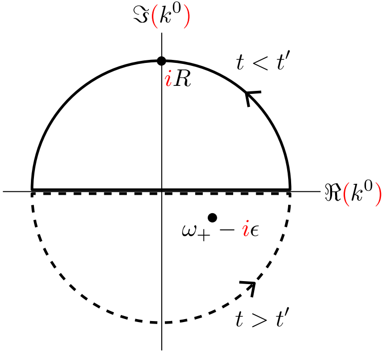
$$\int_A \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega-(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)}$$

Contour segment A is a circular arc of radius $R \uparrow \infty$



$$\int_A \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega-(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)} = 2\pi i \sum \text{Res}$$

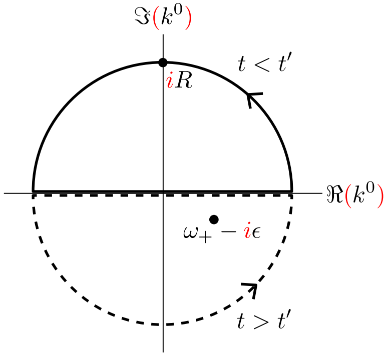
The contour integral is proportional to the sum of the residues of the integrand inside the contour



$$\int_A \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega-(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)} = 2\pi i \sum \mathrm{Res}$$

$$0 + \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega-(\omega_+ - i\epsilon)} = 2\pi i\, 0, \qquad t < t',\ \epsilon \downarrow 0,\ R \uparrow \infty$$

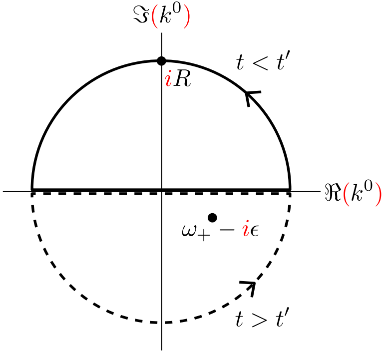
For $t - t' < 0$, close contour from above so the A integral vanishes as $R \uparrow \infty$



$$\int_A \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z - (\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega - (\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z - (\omega_+ - i\epsilon)} = 2\pi i \sum \mathrm{Res}$$

$$0 + \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega - (\omega_+ - i\epsilon)} = 2\pi i\, 0, \qquad t < t',\ \epsilon \downarrow 0,\ R \uparrow \infty$$

No poles inside the contour leaves no residue

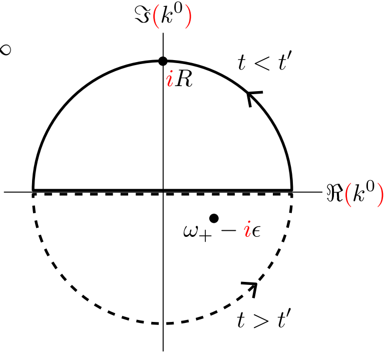


$$\int_A \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega-(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)} = 2\pi i \sum \mathrm{Res}$$

$$0 + \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega-(\omega_+ - i\epsilon)} = 2\pi i\, 0, \qquad t < t',\ \epsilon \downarrow 0,\ R \uparrow \infty$$

$$0 + \int_{+\infty}^{-\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega-(\omega_+ - i\epsilon)} = 2\pi i\, \frac{1}{2\pi} e^{-i\omega_+(t-t')}, \qquad t > t',\ \epsilon \downarrow 0,\ R \uparrow \infty$$

For $t - t' > 0$, close contour from below so the A integral vanishes as $R \uparrow \infty$



$$\int_A \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega-(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)} = 2\pi i \sum \mathrm{Res}$$

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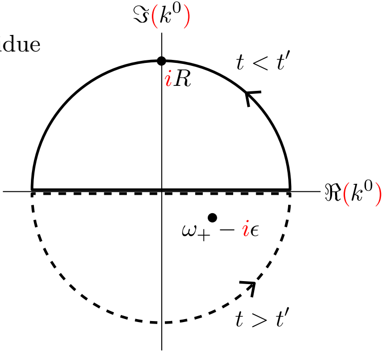
$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$



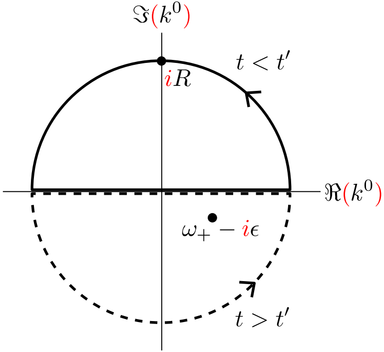
$$\int_A \frac{dz}{2\pi}\frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega-(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi}\frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)} = 2\pi i \sum \mathrm{Res}$$

$$0 + \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega-(\omega_+ - i\epsilon)} = 2\pi i\, 0, \qquad t < t',\ \epsilon \downarrow 0,\ R \uparrow \infty$$

$$0 + \int_{+\infty}^{-\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega-(\omega_+ - i\epsilon)} = 2\pi i\, \frac{1}{2\pi} e^{-i\omega_+(t-t')}, \qquad t > t',\ \epsilon \downarrow 0,\ R \uparrow \infty$$

$$\int_{+\infty}^{-\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega-(\omega_+ - i\epsilon)} = \begin{cases} 0, & t < t' \\ -ie^{-i\omega_+(t-t')}, & t > t' \end{cases}$$

Consolidate results



$$\int_A \frac{dz}{2\pi}\frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega-(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi}\frac{e^{-iz(t-t')}}{z-(\omega_+ - i\epsilon)} = 2\pi i \sum \mathrm{Res}$$

$$0 + \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega-(\omega_+ - i\epsilon)} = 2\pi i\, 0, \qquad t < t',\ \epsilon \downarrow 0,\ R \uparrow \infty$$

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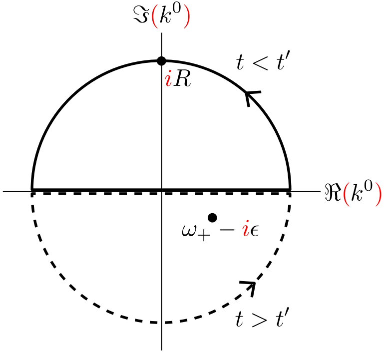
$$\int_{+\infty}^{-\infty} \frac{dk^0}{2\pi}\frac{e^{-ik^0(t-t')}}{k^0-(\omega_+ - i\epsilon)} = -i\Theta(t-t')e^{-i\omega_+(t-t')}$$

Restore $\omega = k^0$ and introduce a step function $\Theta(t)$



$$\int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega + (\omega_+ - i\epsilon)}$$

Extend the second $\omega = k^0$ real integral ...

$$\Delta_F(x - x') = \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{dk^0}{2\pi} e^{-ik^0(t-t')} \left(\frac{1}{k^0 - (\omega_+ - i\epsilon)} - \frac{1}{k^0 + (\omega_+ - i\epsilon)} \right)$$

$$\int_A \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z+(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z+(\omega_+ - i\epsilon)}$$

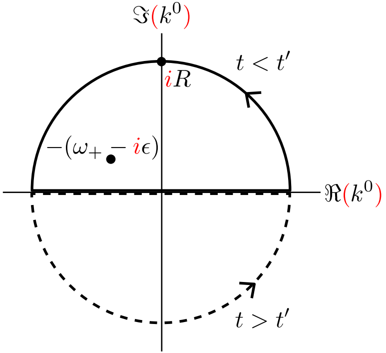
to a closed contour C in the complex plane

$$\int_A \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z+(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z+(\omega_+ - i\epsilon)}$$

where $z \in \mathbb{C}$ and $\omega \in \mathbb{R}$

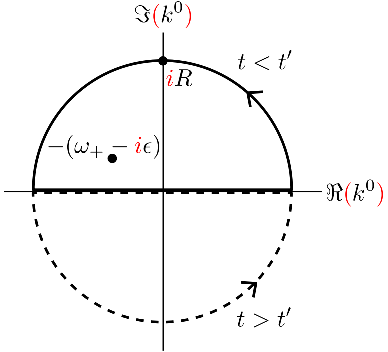
$$\int_A \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z + (\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega + (\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z + (\omega_+ - i\epsilon)}$$

Contour segment A is a circular arc of radius $R \uparrow \infty$



$$\int_A \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z+(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z+(\omega_+ - i\epsilon)} = 2\pi i \sum \mathrm{Res}$$

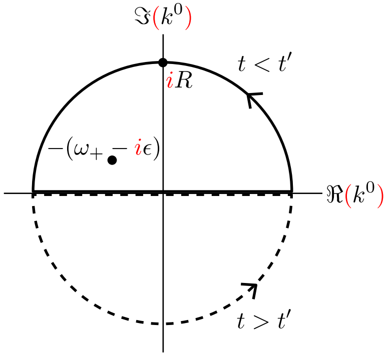
The contour integral is proportional to the sum of the residues of the integrand inside the contour



$$\int_A \frac{dz}{2\pi}\frac{e^{-iz(t-t')}}{z+(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi}\frac{e^{-iz(t-t')}}{z+(\omega_+ - i\epsilon)} = 2\pi i \sum \mathrm{Res}$$

$$0 + \int\limits_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = 2\pi i\,\frac{1}{2\pi}e^{-i\omega_+(t-t')}, \quad t < t',\ \epsilon \downarrow 0,\ R \uparrow \infty$$

For $t - t' < 0$, close contour from above so the A integral vanishes as $R \uparrow \infty$



$$\int_A \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z+(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z+(\omega_+ - i\epsilon)} = 2\pi i \sum \mathrm{Res}$$

$$0 + \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = 2\pi i \, \frac{1}{2\pi} e^{-i\omega_+(t-t')}, \quad t < t', \; \epsilon \downarrow 0, \; R \uparrow \infty$$

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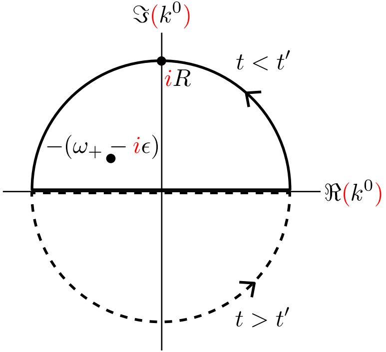
$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$

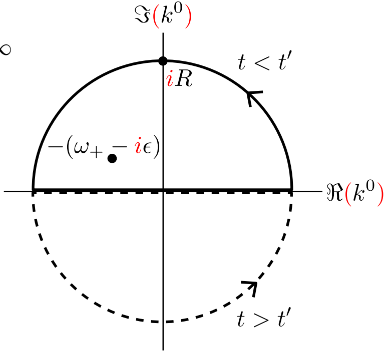


$$\int_A \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z+(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z+(\omega_+ - i\epsilon)} = 2\pi i \sum \mathrm{Res}$$

$$0 + \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = 2\pi i \, \frac{1}{2\pi} e^{-i\omega_+(t-t')}, \quad t < t', \ \epsilon \downarrow 0, \ R \uparrow \infty$$

$$0 + \int_{+\infty}^{-\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = 2\pi i \, 0, \quad t > t', \ \epsilon \downarrow 0, \ R \uparrow \infty$$

For $t - t' > 0$, close contour from below so the A integral vanishes as $R \uparrow \infty$

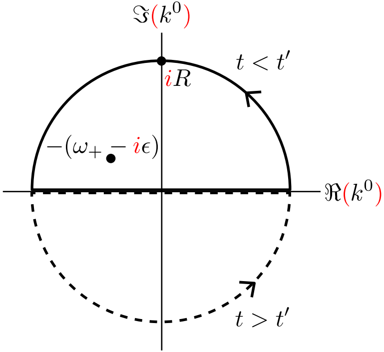


$$\int_A \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z+(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi} \frac{e^{-iz(t-t')}}{z+(\omega_+ - i\epsilon)} = 2\pi i \sum \mathrm{Res}$$

$$0 + \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = 2\pi i \, \frac{1}{2\pi} e^{-i\omega_+(t-t')}, \quad t < t', \ \epsilon \downarrow 0, \ R \uparrow \infty$$

$$0 + \int_{+\infty}^{-\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = 2\pi i \, 0, \qquad t > t', \ \epsilon \downarrow 0, \ R \uparrow \infty$$

No poles inside the contour leaves no residue

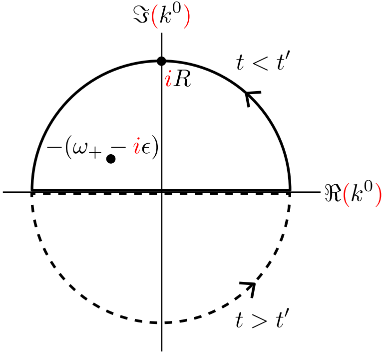


$$\int_A \frac{dz}{2\pi}\frac{e^{-iz(t-t')}}{z+(\omega_+ - i\epsilon)} + \int \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = \oint_C \frac{dz}{2\pi}\frac{e^{-iz(t-t')}}{z+(\omega_+ - i\epsilon)} = 2\pi i \sum \mathrm{Res}$$

$$0 + \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = 2\pi i \,\frac{1}{2\pi} e^{-i\omega_+(t-t')}, \quad t < t',\ \epsilon \downarrow 0,\ R \uparrow \infty$$

$$0 + \int_{+\infty}^{-\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = 2\pi i \, 0, \quad t > t',\ \epsilon \downarrow 0,\ R \uparrow \infty$$

$$\int_{+\infty}^{-\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega+(\omega_+ - i\epsilon)} = \begin{cases} ie^{-i\omega_+(t-t')}, & t < t' \\ 0, & t > t' \end{cases}$$

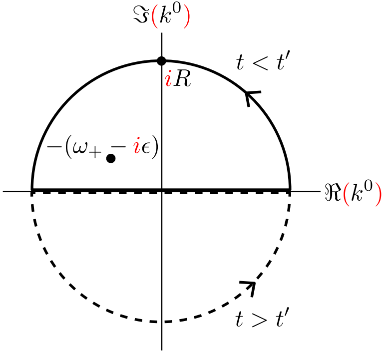


Consolidate results

$$\int_A \frac{dz}{2\pi}\,\frac{e^{-iz(t-t')}}{z+(\omega_+-i\epsilon)} + \int \frac{d\omega}{2\pi}\,\frac{e^{-i\omega(t-t')}}{\omega+(\omega_+-i\epsilon)} = \oint_C \frac{dz}{2\pi}\,\frac{e^{-iz(t-t')}}{z+(\omega_+-i\epsilon)} = 2\pi i \sum \mathrm{Res}$$

$$0+\int\limits_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\,\frac{e^{-i\omega(t-t')}}{\omega+(\omega_+-i\epsilon)} = 2\pi i\,\frac{1}{2\pi}e^{-i\omega_+(t-t')}, \qquad t<t',\ \epsilon\downarrow 0,\ R\uparrow\infty$$

$$0+\int\limits_{+\infty}^{-\infty} \frac{d\omega}{2\pi}\,\frac{e^{-i\omega(t-t')}}{\omega+(\omega_+-i\epsilon)} = 2\pi i\,0, \qquad t>t',\ \epsilon\downarrow 0,\ R\uparrow\infty$$



$$\int\limits_{+\infty}^{-\infty} \frac{d\omega}{2\pi}\,\frac{e^{-i\omega(t-t')}}{\omega+(\omega_+-i\epsilon)} = \begin{cases} ie^{-i\omega_+(t-t')}, & t<t' \\ 0, & t>t' \end{cases}$$

$$\int\limits_{+\infty}^{-\infty} \frac{dk^0}{2\pi}\,\frac{e^{-ik^0(t-t')}}{k^0+(\omega_+-i\epsilon)} = i\Theta(t'-t)e^{i\omega_+(t-t')}$$

Restore $\omega = k^0$ and introduce a step function $\Theta(t)$

$$\Delta_F(x-x') = \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{dk^0}{2\pi} e^{-ik^0(t-t')} \left(\frac{1}{k^0-(\omega_+ - i\epsilon)} - \frac{1}{k^0+(\omega_+ - i\epsilon)} \right)$$

Recall Feynman propagator

$$\Delta_F(x-x') = \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{dk^0}{2\pi} e^{-ik^0(t-t')} \left(\frac{1}{k^0 - (\omega_+ - i\epsilon)} - \frac{1}{k^0 + (\omega_+ - i\epsilon)} \right)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \left(-i\Theta(t-t') e^{i\omega_+(t'-t)} - i\Theta(t'-t) e^{i\omega_+(t-t')} \right)$$

Substitute

$$\Delta_F(x-x') = \int \frac{d^3k}{(2\pi)^3 2\omega_+}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{dk^0}{2\pi}\, e^{-ik^0(t-t')} \left(\frac{1}{k^0-(\omega_+-i\epsilon)} - \frac{1}{k^0+(\omega_+-i\epsilon)} \right)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_+}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \left(-i\Theta(t-t') e^{i\omega_+(t'-t)} - i\Theta(t'-t) e^{i\omega_+(t-t')} \right)$$

$$= -i\Theta(t-t') \int \frac{d^3k}{(2\pi)^3 2\omega_+}\, e^{i\omega_+(t'-t)i+\vec{k}\cdot(\vec{x}-\vec{x}')} - i\Theta(t'-t) \int \frac{d^3k}{(2\pi)^3 2\omega_+}\, e^{i\omega_+(t-t')+i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

Expand

$$\Delta_F(x-x') = \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{dk^0}{2\pi} e^{-ik^0(t-t')} \left(\frac{1}{k^0 - (\omega_+ - i\epsilon)} - \frac{1}{k^0 + (\omega_+ - i\epsilon)} \right)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \left(-i\Theta(t-t')e^{i\omega_+(t'-t)} - i\Theta(t'-t)e^{i\omega_+(t-t')} \right)$$

$$= -i\Theta(t-t') \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{i\omega_+(t'-t)i+\vec{k}\cdot(\vec{x}-\vec{x}')} - i\Theta(t'-t) \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{i\omega_+(t-t')+i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$= -i\Theta(t-t') \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{-i\omega_+(t-t')+i\vec{k}\cdot(\vec{x}-\vec{x}')} - i\Theta(t'-t) \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{i\omega_+(t-t')-i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

Replace $t \to t'$ Replace $\vec{k} \to -\vec{k}$, where $d^3k = k^2 d\Omega$

$$\Delta_F(x - x') = \int \frac{d^3k}{(2\pi)^3 2\omega_+} \, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{dk^0}{2\pi} \, e^{-ik^0(t-t')} \left(\frac{1}{k^0 - (\omega_+ - i\epsilon)} - \frac{1}{k^0 + (\omega_+ - i\epsilon)} \right)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_+} \, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \left(-i\Theta(t-t')e^{i\omega_+(t'-t)} - i\Theta(t'-t)e^{i\omega_+(t-t')} \right)$$

$$= -i\Theta(t-t') \int \frac{d^3k}{(2\pi)^3 2\omega_+} \, e^{i\omega_+(t'-t)i+\vec{k}\cdot(\vec{x}-\vec{x}')} - i\Theta(t'-t) \int \frac{d^3k}{(2\pi)^3 2\omega_+} \, e^{i\omega_+(t-t')+i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$= -i\Theta(t-t') \int \frac{d^3k}{(2\pi)^3 2\omega_+} \, e^{-i\omega_+(t-t')+i\vec{k}\cdot(\vec{x}-\vec{x}')} - i\Theta(t'-t) \int \frac{d^3k}{(2\pi)^3 2\omega_+} \, e^{i\omega_+(t-t')-i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$= -i \int \frac{d^3k}{(2\pi)^3 2\omega_+} \left(\Theta(t-t')e^{-ik\cdot(x-x')} + \Theta(t'-t)e^{ik\cdot(x-x')} \right)$$

Consolidate with $k \cdot x = \omega_+ t - \vec{k} \cdot \vec{x}$

$$\Delta_F(x-x') = \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{dk^0}{2\pi} e^{-ik^0(t-t')} \left(\frac{1}{k^0-(\omega_+-i\epsilon)} - \frac{1}{k^0+(\omega_+-i\epsilon)} \right)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \left(-i\Theta(t-t')e^{i\omega_+(t'-t)} - i\Theta(t'-t)e^{i\omega_+(t-t')} \right)$$

$$= -i\Theta(t-t') \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{i\omega_+(t'-t)i+\vec{k}\cdot(\vec{x}-\vec{x}')} - i\Theta(t'-t) \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{i\omega_+(t-t')+i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$= -i\Theta(t-t') \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{-i\omega_+(t-t')+i\vec{k}\cdot(\vec{x}-\vec{x}')} - i\Theta(t'-t) \int \frac{d^3k}{(2\pi)^3 2\omega_+} e^{i\omega_+(t-t')-i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$= -i \int \frac{d^3k}{(2\pi)^3 2\omega_+} \left(\Theta(t-t')e^{-ik\cdot(x-x')} + \Theta(t'-t)e^{ik\cdot(x-x')} \right)$$

$$i\Delta_F(x-x') = \int \frac{d^3k}{(2\pi)^3 2\omega_+} \left(\Theta(t-t')e^{-ik\cdot(x-x')} + \Theta(t'-t)e^{ik\cdot(x-x')} \right)$$

Multiply by i

$$\Delta_F(x-x') = \int \frac{d^3k}{(2\pi)^3 2\omega_+} \, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{dk^0}{2\pi} \, e^{-ik^0(t-t')} \left(\frac{1}{k^0 - (\omega_+ - i\epsilon)} - \frac{1}{k^0 + (\omega_+ - i\epsilon)} \right)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_+} \, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \left(-i\Theta(t-t') e^{i\omega_+(t'-t)} - i\Theta(t'-t) e^{i\omega_+(t-t')} \right)$$

$$= -i\Theta(t-t') \int \frac{d^3k}{(2\pi)^3 2\omega_+} \, e^{i\omega_+(t'-t)i + \vec{k}\cdot(\vec{x}-\vec{x}')} - i\Theta(t'-t) \int \frac{d^3k}{(2\pi)^3 2\omega_+} \, e^{i\omega_+(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$= -i\Theta(t-t') \int \frac{d^3k}{(2\pi)^3 2\omega_+} \, e^{-i\omega_+(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x}')} - i\Theta(t'-t) \int \frac{d^3k}{(2\pi)^3 2\omega_+} \, e^{i\omega_+(t-t') - i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$= -i \int \frac{d^3k}{(2\pi)^3 2\omega_+} \left(\Theta(t-t') e^{-ik\cdot(x-x')} + \Theta(t'-t) e^{ik\cdot(x-x')} \right)$$

$$i\Delta_F(x-x') = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(\Theta(t-t') e^{-ik\cdot(x-x')} + \Theta(t'-t) e^{ik\cdot(x-x')} \right)$$

Streamline notation

$$\phi(x) = \int \underline{d^3k} \left(a(\vec{k}) e^{-ik\cdot x} + a^\dagger(\vec{k}) e^{ik\cdot x} \right)$$

Recall real scalar field Fourier decomposition, where notationally $a^\dagger(\vec{k}) = a(\vec{k})^\dagger$

$$\phi(x) = \int \underline{d^3k} \left(a(\vec{k}) e^{-ik\cdot x} + a^\dagger(\vec{k}) e^{ik\cdot x} \right) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k}) e^{-ik\cdot x} + a^\dagger(\vec{k}) e^{ik\cdot x} \right)$$

Expand Lorentz invariant measure

$$\phi(x) = \int \underline{d}^3k \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\phi(x)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})|0\rangle e^{-ik\cdot x} + a^\dagger(\vec{k})|0\rangle e^{ik\cdot x}\right)$$

Apply real scalar field to the vacuum state

$$\phi(x) = \int \underline{d}^3k \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\phi(x)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})|0\rangle e^{-ik\cdot x} + a^\dagger(\vec{k})|0\rangle e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^3 2\omega}|\vec{k}\rangle e^{ik\cdot x}$$

and create a quantum at x with momentum $p = \hbar k$

$$\phi(x) = \int \underline{d^3k} \left(a(\vec{k}) e^{-ik \cdot x} + a^\dagger(\vec{k}) e^{ik \cdot x} \right) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k}) e^{-ik \cdot x} + a^\dagger(\vec{k}) e^{ik \cdot x} \right)$$

$$\phi(x)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})|0\rangle e^{-ik \cdot x} + a^\dagger(\vec{k})|0\rangle e^{ik \cdot x} \right) = \int \frac{d^3k}{(2\pi)^3 2\omega} |\vec{k}\rangle e^{ik \cdot x}$$

and create a quantum at x with momentum $p = k$ in natural units

$$\phi(x) = \int \underline{d^3k} \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\phi(x)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})|0\rangle e^{-ik\cdot x} + a^\dagger(\vec{k})|0\rangle e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^3 2\omega} |\vec{k}\rangle e^{ik\cdot x}$$

$$\langle 0|\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \langle\vec{k}| e^{-ik\cdot x}$$

Adjoint

$$\phi(x) = \int \underline{d^3k} \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\phi(x)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})|0\rangle e^{-ik\cdot x} + a^\dagger(\vec{k})|0\rangle e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^3 2\omega} |\vec{k}\rangle e^{ik\cdot x}$$

$$\langle 0|\phi(x') = \int \frac{d^3k'}{(2\pi)^3 2\omega'} \langle \vec{k}'| e^{-ik'\cdot x'}$$

Prime the variables

$$\phi(x) = \int \underline{d^3k} \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\phi(x)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})|0\rangle e^{-ik\cdot x} + a^\dagger(\vec{k})|0\rangle e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^3 2\omega} |\vec{k}\rangle e^{ik\cdot x}$$

$$\langle 0|\phi(x') = \int \frac{d^3k'}{(2\pi)^3 2\omega'} \langle \vec{k}'|e^{-ik'\cdot x'}$$

$$\langle 0|\phi(x')\phi(x)|0\rangle = \int \frac{d^3k'}{(2\pi)^3 2\omega'} \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik'\cdot x'} e^{ik\cdot x} \langle \vec{k}'|\vec{k}\rangle$$

Scalar product is the vacuum expectation

$$\phi(x) = \int \underline{d}^3k \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\phi(x)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})|0\rangle e^{-ik\cdot x} + a^\dagger(\vec{k})|0\rangle e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^3 2\omega} |\vec{k}\rangle e^{ik\cdot x}$$

$$\langle 0|\phi(x') = \int \frac{d^3k'}{(2\pi)^3 2\omega'} \langle \vec{k}'| e^{-ik'\cdot x'}$$

$$\begin{aligned}\langle 0|\phi(x')\phi(x)|0\rangle &= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik'\cdot x'} e^{ik\cdot x} \langle \vec{k}'|\vec{k}\rangle \\ &= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik'\cdot x' + ik\cdot x} (2\pi)^3 2\omega' \delta^3(\vec{k}' - \vec{k})\end{aligned}$$

Invariant orthonormalization $\langle \vec{k}|\vec{k}'\rangle = \langle 0|a(\vec{k})a^\dagger(\vec{k}')|0\rangle$

$$\begin{aligned}&= \langle 0|a^\dagger(\vec{k}')a(\vec{k}) + (2\pi)^3 2\omega' \delta^3(\vec{k}' - \vec{k})|0\rangle \\ &= 0 + (2\pi)^3 2\omega' \delta^3(\vec{k}' - \vec{k})\end{aligned}$$

$$\phi(x) = \int \underline{d^3k} \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\phi(x)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})|0\rangle e^{-ik\cdot x} + a^\dagger(\vec{k})|0\rangle e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^3 2\omega} |\vec{k}\rangle e^{ik\cdot x}$$

$$\langle 0|\phi(x') = \int \frac{d^3k'}{(2\pi)^3 2\omega'} \langle \vec{k}'|e^{-ik'\cdot x'}$$

$$\begin{aligned}
\langle 0|\phi(x')\phi(x)|0\rangle &= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik'\cdot x'} e^{ik\cdot x} \langle \vec{k}'|\vec{k}\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik'\cdot x' + ik\cdot x} (2\pi)^3 2\omega' \delta^3(\vec{k}' - \vec{k}) \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik\cdot(x'-x)}
\end{aligned}$$

Dirac delta sifting, where $\vec{k}' = \vec{k}$ forces $\omega' = \omega$

$$\phi(x) = \int \underline{d^3k} \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)$$

$$\phi(x)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\vec{k})|0\rangle e^{-ik\cdot x} + a^\dagger(\vec{k})|0\rangle e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^3 2\omega} |\vec{k}\rangle e^{ik\cdot x}$$

$$\langle 0|\phi(x') = \int \frac{d^3k'}{(2\pi)^3 2\omega'} \langle \vec{k}'| e^{-ik'\cdot x'}$$

$$\begin{aligned}
\langle 0|\phi(x')\phi(x)|0\rangle &= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik'\cdot x'} e^{ik\cdot x} \langle \vec{k}'|\vec{k}\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik'\cdot x' + ik\cdot x} (2\pi)^3 2\omega' \delta^3(\vec{k}' - \vec{k}) \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik\cdot(x'-x)}
\end{aligned}$$

$$\langle 0|\phi(x)\phi(x')|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega} e^{ik\cdot(x'-x)}$$

Swapping x and x'

$$i\Delta_F(x-x') = \int \frac{d^3k}{(2\pi)^3 2\omega}\left(\Theta(t-t')e^{-ik\cdot(x-x')} + \Theta(t'-t)e^{ik\cdot(x-x')}\right)$$

Recall Feynman propagator

$$i\Delta_F(x-x') = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(\Theta(t-t')e^{-ik\cdot(x-x')} + \Theta(t'-t)e^{ik\cdot(x-x')}\right)$$

$$= \Theta(t-t')\langle 0|\phi(x)\phi(x')|0\rangle + \Theta(t'-t)\langle 0|\phi(x')\phi(x)|0\rangle$$

Substitute

$$i\Delta_F(x-x') = \int \frac{d^3k}{(2\pi)^3 2\omega}\left(\Theta(t-t')e^{-ik\cdot(x-x')} + \Theta(t'-t)e^{ik\cdot(x-x')}\right)$$

$$= \Theta(t-t')\langle 0|\phi(x)\phi(x')|0\rangle + \Theta(t'-t)\langle 0|\phi(x')\phi(x)|0\rangle$$

$$i\Delta_F(x-x') = \langle 0|\mathcal{T}(\phi(x)\phi(x'))|0\rangle$$

Feynman propagator is the vacuum expectation of the time-ordered-product of the field operators

$$i\Delta_F(x-x') = \int \frac{d^3k}{(2\pi)^3 2\omega}\left(\Theta(t-t')e^{-ik\cdot(x-x')} + \Theta(t'-t)e^{ik\cdot(x-x')}\right)$$

$$= \Theta(t-t')\langle 0|\phi(x)\phi(x')|0\rangle + \Theta(t'-t)\langle 0|\phi(x')\phi(x)|0\rangle$$

$$i\Delta_F(x-x') = \langle 0|\mathcal{T}(\phi(x)\phi(x'))|0\rangle$$

$$i\Delta_F(x-x') = \langle 0|\phi(x)\phi(x')|0\rangle, \quad t > t'$$

Amplitude that a particle is created at time t', propagates from $\vec{x}\,'$ to $\vec{x}$, and is annihilated at a later time $t > t'$

$$i\Delta_F(x-x') = \int \frac{d^3k}{(2\pi)^3 2\omega}\left(\Theta(t-t')e^{-ik\cdot(x-x')} + \Theta(t'-t)e^{ik\cdot(x-x')}\right)$$

$$= \Theta(t-t')\langle 0|\phi(x)\phi(x')|0\rangle + \Theta(t'-t)\langle 0|\phi(x')\phi(x)|0\rangle$$

$$i\Delta_F(x-x') = \langle 0|\mathcal{T}(\phi(x)\phi(x'))|0\rangle$$

$$i\Delta_F(x-x') = \langle 0|\phi(x)\phi(x')|0\rangle, \quad t > t'$$

$$i\Delta_F(x-x') = \langle 0|\phi(x')\phi(x)|0\rangle, \quad t' > t$$

Amplitude that a particle is created at time t, propagates from $\vec{x}$ to $\vec{x}\,'$, and is annihilated at a later time $t' > t$

$$i\Delta_F(x-x') = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(\Theta(t-t')e^{-ik\cdot(x-x')} + \Theta(t'-t)e^{ik\cdot(x-x')}\right)$$

$$= \Theta(t-t')\langle 0|\phi(x)\phi(x')|0\rangle + \Theta(t'-t)\langle 0|\phi(x')\phi(x)|0\rangle$$

$$i\Delta_F(x-x') = \langle 0|\mathcal{T}(\phi(x)\phi(x'))|0\rangle$$

$$i\Delta_F(x-x') = \langle 0|\phi(x)\phi(x')|0\rangle, \quad t > t'$$

$$i\Delta_F(x-x') = \langle 0|\phi(x')\phi(x)|0\rangle, \quad t' > t$$

$$\mathcal{T}(A(x)B(x')) = \begin{cases} A(x)B(x'), & t > t' \\ B(x')A(x), & t' > t \end{cases}$$

Time ordering means earlier operators to the right of later operators (so applied earlier)

$$i\Delta_F(x-x') = \int \frac{d^3k}{(2\pi)^3 2\omega}\left(\Theta(t-t')e^{-ik\cdot(x-x')} + \Theta(t'-t)e^{ik\cdot(x-x')}\right)$$

$$= \Theta(t-t')\langle 0|\phi(x)\phi(x')|0\rangle + \Theta(t'-t)\langle 0|\phi(x')\phi(x)|0\rangle$$

$$i\Delta_F(x-x') = \langle 0|\mathcal{T}(\phi(x)\phi(x'))|0\rangle$$

$$i\Delta_F(x-x') = \langle 0|\phi(x)\phi(x')|0\rangle, \quad t > t'$$

$$i\Delta_F(x-x') = \langle 0|\phi(x')\phi(x)|0\rangle, \quad t' > t$$

$$\mathcal{T}(A(x)B(x')) = \begin{cases} A(x)B(x'), & t > t' \\ B(x')A(x), & t' > t \end{cases}$$

Normal ordering means annihilation operators to the right of creation operators, for comparison

$$i\Delta_F(x-x') = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(\Theta(t-t')e^{-ik\cdot(x-x')} + \Theta(t'-t)e^{ik\cdot(x-x')}\right)$$

$$= \Theta(t-t')\langle 0|\phi(x)\phi(x')|0\rangle + \Theta(t'-t)\langle 0|\phi(x')\phi(x)|0\rangle$$

$$i\Delta_F(x-x') = \langle 0|\mathcal{T}(\phi(x)\phi(x'))|0\rangle \ \blacksquare$$

$$i\Delta_F(x-x') = \langle 0|\phi(x)\phi(x')|0\rangle, \quad t > t'$$

$$i\Delta_F(x-x') = \langle 0|\phi(x')\phi(x)|0\rangle, \quad t' > t$$

$$\mathcal{T}(A(x)B(x')) = \begin{cases} A(x)B(x'), & t > t' \\ B(x')A(x), & t' > t \end{cases}$$

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