$$
\begin{aligned}
& E=K+U \\
& \mathcal{L}=K-U \\
& \mathcal{A}=\int \mathcal{L} d t
\end{aligned}
$$

# Introduction to Mechanics <br> Including Gravity \& Kinetic Theory 

John F. Lindner<br>Physics Department<br>The College of Wooster

2015 January 4

## Contents

List of Tables ..... 7
List of Figures ..... 9
1 Energy ..... 11
1.1 Free Fall ..... 11
1.2 Free Fall Energy ..... 13
1.3 Momentum ..... 14
1.4 Quantitative Evolution ..... 16
1.5 Qualitative Evolution ..... 16
1.5.1 Flat Earth Gravity ..... 16
1.5.2 Mass and Spring ..... 17
1.5.3 Simple Pendulum ..... 19
1.6 Limitations ..... 19
Problems ..... 20
2 Action ..... 21
2.1 Wave-Particle Duality ..... 21
2.1.1 Classical Waves ..... 22
2.1.2 Young's Double Slit Experiment ..... 23
2.1.3 Matter Waves ..... 24
2.1.4 Sum Over Paths ..... 24
2.1.5 Recovering Classical Mechanics ..... 25
2.2 Free Fall Action ..... 26
2.2.1 Stationary vs. Least ..... 27
2.2.2 Simple Examples ..... 28
2.2.3 Global to Local ..... 28
Problems ..... 32
3 Calculus in a Nutshell ..... 35
3.1 Fundamental Theorem ..... 35
3.2 Derivatives \& Antiderivatives ..... 37
3.3 Compound Operations ..... 38
Problems ..... 41
4 Lagrange's Equations ..... 43
4.1 Differential Equations of Motion ..... 43
4.2 Numerical General Solution ..... 45
4.3 Example Initial Value Problems ..... 45
4.3.1 Flat Earth Gravity ..... 46
4.3.2 Mass and Spring ..... 46
4.3.3 Simple Pendulum ..... 47
4.4 Conservation Laws from Symmetries ..... 48
Problems ..... 50
5 Vectors in a Nutshell ..... 51
5.1 Vectors \& Coordinates ..... 51
5.2 Vector Addition ..... 53
5.3 Vector Multiplication ..... 54
5.3.1 Geometric Product ..... 54
5.3.2 Dot, Wedge, \& Cross Products ..... 55
5.3.3 Geometric Intepretation ..... 57
Problems ..... 60
6 Newton's Laws ..... 61
6.1 Translation ..... 61
6.1.1 Contact Acceleration ..... 63
6.1.2 Train ..... 64
6.1.3 Atwood Machine ..... 65
6.1.4 Incline ..... 66
6.1.5 Movable Incline ..... 67
6.2 Rotation ..... 69
6.2.1 Massive Pulley ..... 72
6.3 Circular Motion ..... 73
6.3 .1 Hill ..... 75
6.3.2 Slingshot ..... 75
6.3.3 Inertial Frames ..... 77
6.4 Work \& Impulse ..... 77
6.4.1 Hockey Puck ..... 78
6.5 Variable Mass Rockets ..... 78
Problems ..... 80
7 Gravity ..... 83
7.1 Universal Gravity ..... 83
7.2 Newton's Shell Theorems ..... 84
7.2.1 Interior Shell Theorem ..... 85
7.2.2 Exterior Shell Theorem ..... 85
7.3 Trans Earth Tunnel ..... 87
7.4 Near-Earth Gravity ..... 88
7.5 Kepler's Laws ..... 89
7.6 Binary Orbits ..... 91
Contents ..... 5
Problems ..... 93
8 Kinetic Theory ..... 95
8.1 Ideal Gas Law ..... 95
8.2 Mean Free Path. ..... 98
8.3 Compression \& Expansion ..... 100
8.4 Sound Speed ..... 102
8.4.1 General Sound Speed ..... 102
8.4.2 Isothermal Sound Speed ..... 103
8.4.3 Adiabatic Sound Speed ..... 103
Problems ..... 105
Appendices
A Notation ..... 107
B Measure \& Angles ..... 109
C Bibliography ..... 111

## List of Tables

3.1 Derivatives \& anti-derivatives ..... 37
3.2 Compound Differentiation ..... 39
6.1 Translation and rotation ..... 71
A. 1 Symbols ..... 108
B. 1 Planar \& solid angles ..... 110

## List of Figures

1.1 Galileo's experiment on Luna ..... 11
1.2 Free fall arithmetic ..... 12
1.3 Elastic collision spacetime diagrams ..... 15
1.4 Newton's cradle ..... 16
1.5 Free fall energy diagrams ..... 17
1.6 Oscillator energy diagrams ..... 18
1.7 Pendulum energy diagrams ..... 18
2.1 Electron interference ..... 21
2.2 Sinusoidal wave ..... 22
2.3 Young's double slit ..... 23
2.4 Many slits ..... 25
2.5 Many paths ..... 26
2.6 Action manipulator ..... 27
2.7 Simple action examples ..... 28
2.8 Perturbing free fall ..... 29
2.9 Force and potential ..... 31
3.1 Newton \& Leibniz ..... 35
3.2 Fundamental theorem of calculus ..... 36
4.1 History of action ..... 43
5.1 Baseball diamond ..... 51
5.2 Right-handed coordinates ..... 52
5.3 Vector addition ..... 53
5.4 Vector products ..... 54
5.5 Multiplication tables ..... 57
5.6 Magnitudes of the inner and outer products ..... 58
5.7 Projection and rejection ..... 58
6.1 Newton's Principia ..... 61
6.2 Contact acceleration ..... 64
6.3 Train acceleration ..... 64
6.4 Atwood machine ..... 65
List of Figures ..... 10
6.5 Incline ..... 67
6.6 Incline recoilng ..... 68
6.7 Rotational motion ..... 70
6.8 Massive pulley ..... 72
6.9 Circular motion ..... 74
6.10 Hill ..... 75
6.11 Slingshot ..... 76
6.12 Rocket ..... 78
7.1 Galileo to Kepler ..... 83
7.2 Shell Theorem Interior ..... 85
7.3 Shell Theorem Exterior ..... 86
7.4 Trans Earth Tunnel ..... 87
7.5 Gravitational Potential Energy ..... 89
7.6 Kepler's Laws ..... 90
7.7 Binary Orbits ..... 91
8.1 Solid, Liquid, Gas ..... 95
8.2 Temperature Scales ..... 96
8.3 Ideal Gas Law. ..... 97
8.4 Ideal Gas Model ..... 99
8.5 Isotherms \& Adiabats ..... 100
8.6 Sound Speed ..... 102
B. 1 Spherical Geometry ..... 109

## Chapter 1

## Energy

Constancy always accompanies change as motion conserves energy.


Figure 1.1: On 1971 August 2 on Luna's Hadley Plain, in the near vacuum at the lunar surface, Apollo 15 astronaut Dave Scott dropped a feather and a hammer side-by-side, and they hit the ground simultaneously [1]. In 2002, a survey of the most beautiful physics experiments by Physics World magazine ranked Galileo's experiment on the equality of falling bodies number two.

### 1.1 Free Fall

Drop a ball, and watch it fall. How does it move?

By 1632, Galileo Galilei concluded that objects fall identically regardless of their mass [2], provided air resistance is negligible. Some historians doubt that Galileo actually tested this idea by dropping different masses from the Leaning Tower of Pisa, partly because Aristotelians of his time purportedly performed the test to demonstrate that greater masses hit first! We now attribute this discrepancy to air resistance, and in 1971 Dave Scott definitively demonstrated Galileo's law of fall on the airless surface of the moon. Figure 1.1 is a video frame of his famous experiment.


Figure 1.2: Simple arithmetic patterns underlie free fall, which is straight in space (left) and parabolic in spacetime (right).

Galileo further concluded that a freely falling object obeys precise arithmetical laws, which are striking examples of patterns in natural phenomena. For example, the distances fallen in successive equal time intervals are proportional to the odd integers, and the cumulative distances fallen are proportional to the squares of the integers, as in Fig. 1.2. In modern and conventional notation, the upward space coordinate $s$ in meters $m$ depends depends quadratically on the time coordinate $t$ in seconds s according to

$$
\begin{equation*}
s[t]=-\frac{1}{2} g t^{2} \tag{1.1}
\end{equation*}
$$

where at Earth's surface the acceleration

$$
\begin{equation*}
g \approx 9.8 \frac{\mathrm{~m}}{\mathrm{~s}^{2}} \approx 22 \frac{\mathrm{mph}}{\mathrm{~s}} \tag{1.2}
\end{equation*}
$$

That's zero to 66 mph in just 3 s ; by comparison, a 2014 Corvette Stingray can accelerate from zero to 60 mph in 3.8 s . If you accidentally fall long enough to think "I'm falling", you're in grave danger.

The motion of a real projectile through air is actually quite complicated. However, there exist nomological machines - configurations of matter that behave simply - of which a particle falling in a vacuum is a paradigmatic example.

### 1.2 Free Fall Energy

Velocity is the rate of change of position with time (and speed is the velocity magnitude). Compute velocity by dividing a small change in space by the corresponding change in time to form the velocity derivative

$$
\begin{equation*}
v_{s}=\frac{d s}{d t}=\lim _{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t}=\lim _{\Delta t \rightarrow 0} \frac{s[t+\Delta t]-s[t]}{\Delta t} \tag{1.3}
\end{equation*}
$$

where square brackets $[\bullet]$ enclose function arguments and round parentheses (•) are reserved for grouping. For Galileo's Eq. 1.1 law of fall, the derivative

$$
\begin{align*}
v_{s} & =-\frac{1}{2} g \lim _{\Delta t \rightarrow 0} \frac{(t+\Delta t)^{2}-t^{2}}{\Delta t} \\
& =-\frac{1}{2} g \lim _{\Delta t \rightarrow 0} \frac{t^{2}+2 t \Delta t+\Delta t^{2}-t^{2}}{\Delta t} \\
& =-\frac{1}{2} g \lim _{\Delta t \rightarrow 0}(2 t+\Delta t) \\
& =-g t \tag{1.4}
\end{align*}
$$

so the velocity decreases linearly with time. Similarly acceleration is the rate of change of velocity with time. Compute it by

$$
\begin{equation*}
a_{s}=\frac{d v_{s}}{d t}=\lim _{\Delta t \rightarrow 0} \frac{v_{s}[t+\Delta t]-v_{s}[t]}{\Delta t}=-g \lim _{\Delta t \rightarrow 0} \frac{t+\Delta t-\hbar}{\Delta t}=-g \tag{1.5}
\end{equation*}
$$

so the acceleration is constant, independent of time.
Eliminate time $t$ from the Eq. 1.1 and Eq. 1.4 position and velocity to get

$$
\begin{equation*}
s=-\frac{1}{2} g\left(-\frac{v_{s}}{g}\right)^{2}=-\frac{v_{s}^{2}}{2 g} \tag{1.6}
\end{equation*}
$$

Multiply through by the mass $m$ and rearrange to find

$$
\begin{equation*}
0=\frac{1}{2} m v_{s}^{2}+m g s \tag{1.7}
\end{equation*}
$$

This is a profound result: as the object falls, its position $s$ and velocity $v_{s}$ are continually changing, yet the Eq. 1.7 combination is constant; an invariant core organizes the continuous change. More generally, if the object is thrown vertically with an initial velocity $v_{0}$ from an initial position $s_{0}$,

$$
\begin{equation*}
\frac{1}{2} m v_{0}^{2}+m g s_{0}=\frac{1}{2} m v_{s}^{2}+m g s \tag{1.8}
\end{equation*}
$$

Write this constant-of-the-motion as the energy

$$
\begin{equation*}
E=K+U \tag{1.9}
\end{equation*}
$$

and identify the kinetic energy

$$
\begin{equation*}
K=\frac{1}{2} m v_{s}^{2} \tag{1.10}
\end{equation*}
$$

and the flat-Earth gravitational potential energy

$$
\begin{equation*}
U=m g s \tag{1.11}
\end{equation*}
$$

The symbol $U$ resembles a potential well or valley. Constant total energy and its decomposition into time-varying kinetic and potential parts are at the core of classical mechanics.

Although total energy is always conserved, this particular decomposition is useful only under certain (very important) conditions: For the Eq. 1.10 kinetic energy, the speeds most be small compared to the constant speed of light, $v \ll$ $c=3.0 \times 10^{8} \mathrm{~m} / \mathrm{s} \approx 10^{9} \mathrm{kph}$; for the Eq. 1.11 potential energy, the distance above Earth's surface must be small compared to Earth's radius, $s \ll R_{\oplus} \approx 6400 \mathrm{~km}$.

### 1.3 Momentum

Energy conservation and the principle of relativity imply a second conserved quantity. Consider a direct "head-on" collision of two objects of mass $m_{a}$ and $m_{b}$. If the objects are free, the potential energy vanishes, $U=0$. If energy is not lost to other forms (such as sound or heat), the collision is elastic and the total kinetic energy $K$ is the same before and after. If accent marks denote quantities after the collision, then

$$
\begin{align*}
K & =K^{\prime}  \tag{1.12a}\\
K_{a}+K_{b} & =K_{a}^{\prime}+K_{b}^{\prime}  \tag{1.12b}\\
\frac{1}{2} m_{a} v_{a}^{2}+\frac{1}{2} m_{b} v_{b}^{2} & =\frac{1}{2} m_{a} v_{a}^{\prime 2}+\frac{1}{2} m_{b} v_{b}^{\prime 2} \tag{1.12c}
\end{align*}
$$

so the sum of the masses times the squares of the velocities are the same before and after. The notation $v_{a}$ is short for $v_{a s}$, the component of object $a$ 's motion in the $s$ direction, and so on.

Imagine two observers in relative motion recording the collision, as in the spacetime diagrams of Fig. 1.3. By the principle of relativity, both record the conservation of kinetic energy. Specifically, if the first observer records Eq. 1.12 c and is moving at a velocity $v_{r}$ relative to the second observer, then the second observer records

$$
\begin{equation*}
\frac{1}{2} m_{a}\left(v_{a}+v_{r}\right)^{2}+\frac{1}{2} m_{b}\left(v_{b}+v_{r}\right)^{2}=\frac{1}{2} m_{a}\left(v_{a}^{\prime}+v_{r}\right)^{2}+\frac{1}{2} m_{b}\left(v_{b}^{\prime}+v_{r}\right)^{2} . \tag{1.13}
\end{equation*}
$$



Figure 1.3: Spacetime diagrams of a direct, elastic collision between two objects according to two observers in relative motion. (Although the motion is onedimensional in space, it is two-dimensional in spacetime.)

Expand the binomials and use Eq. 1.12 c to simplify to

$$
\begin{equation*}
m_{a} v_{a} v_{r}+m_{b} v_{b} v_{r}=m_{a} v_{a}^{\prime} v_{r}+m_{b} v_{b}^{\prime} v_{r} . \tag{1.14}
\end{equation*}
$$

If $v_{r} \neq 0$, then

$$
\begin{align*}
m_{a} v_{a}+m_{b} v_{b} & =m_{a} v_{a}^{\prime}+m_{b} v_{b}^{\prime}  \tag{1.15a}\\
p_{a}+p_{b} & =p_{a}^{\prime}+p_{b}^{\prime}  \tag{1.15b}\\
p_{s} & =p_{s}^{\prime} \tag{1.15c}
\end{align*}
$$

so the sum of the masses times the velocities are the same before and after!
Generically, a mass $m$ with velocity $v_{s}$ has momentum

$$
\begin{equation*}
p_{s}=m v_{s} \tag{1.16}
\end{equation*}
$$

in the $s$ direction. The conventional momentum symbol $p$ can stand for "punch": the greater an object's momentum, the greater its punch. (Likewise, the conventional kinetic energy symbol $K$ can stand for "kick": the greater and object's kinetic energy, the greater its kick.) The momentum is the velocity rate of change of the kinetic energy,

$$
\begin{equation*}
p_{s}=m v_{s}=m \frac{1}{2} \frac{d}{d v_{s}} v_{s}^{2}=\frac{d}{d v_{s}}\left(\frac{1}{2} m v_{s}^{2}\right)=\frac{d K}{d v_{s}} . \tag{1.17}
\end{equation*}
$$

Kinetic energy and momentum conservation work together to predict the outcome of elastic collisions. Consider Newton's cradle, which consists of a series of swinging spheres that are close but not initially touching and collide only in pairs, as shown schematically in Fig. 1.4 For each collision, energy conservation alone allows multiple outcomes from which momentum conservation selects a unique result.


Figure 1.4: Schematic diagram of elastic collisions in Newton's cradle, before (left) and after (right). All of the collisions conserve kinetic energy, but only the one's boxed in yellow also conserve momentum, and they are the ones that happen.

### 1.4 Quantitative Evolution

Consider an object moving in 1 spatial dimension (or $1+1=2$ spacetime dimensions). Rewrite the Eq. 1.9 energy decomposition as

$$
\begin{equation*}
\frac{1}{2} m v_{s}^{2}=K=E-U \tag{1.18}
\end{equation*}
$$

and solve for the velocity

$$
\begin{equation*}
\frac{d s}{d t}=v_{s}= \pm \sqrt{\frac{2 K}{m}} \tag{1.19}
\end{equation*}
$$

For each time step $d t$, the corresponding space step

$$
\begin{equation*}
d s=v d t= \pm \sqrt{\frac{2 K}{m}} d t= \pm \sqrt{\frac{2}{m}(E-U[s])} d t \tag{1.20}
\end{equation*}
$$

where again the notation $U[s]$ is a reminder that the potential energy is a function of the space coordinate. Given an initial condition, such as $s_{0}=0$ and $v_{0}>0$, use a computer to apply Eq. 1.20 repeatedly to step the position $s$ forward in time and simulate the object's motion. If the time step is small, this is a good quantitative solution.

### 1.5 Qualitative Evolution

To qualitatively study the dynamics, energy diagrams plot total energy $E$ and potential energy $U$ versus position $s$.

### 1.5.1 Flat Earth Gravity

For free fall the linear potential energy is large for high objects and small for low objects, as in Fig. 1.5. The difference between the total and potential


Figure 1.5: Free fall energy diagram (top) and state space (bottom) for three different initial conditions. Horizontal colored lines (top) are total energies $E$.
energy curves is the kinetic energy $K$, which vanishes when the curves intersect at a turning point. Such diagrams are often paired with state space diagrams of velocity $v_{s}$ versus position $s$, as it requires both position and velocity to determine the future (and past) of the object. The free fall state space paths are always pieces of parabolas.

### 1.5.2 Mass and Spring

Another paradigmatic dynamical system in classical mechanics is the simple harmonic oscillator, which consists of a mass attached to an idealized spring. While a linear potential energy models free fall near Earth's surface, a quadratic potential energy models the simple harmonic oscillator. If $s$ is the displacement of the mass from its equilibrium position and $\kappa$ determines the stiffness of the spring, then the potential energy

$$
\begin{equation*}
U=\frac{1}{2} \kappa s^{2} \tag{1.21}
\end{equation*}
$$

is large at large (positive or negative) displacements, small at small displacements, and minimum at zero displacement. Figure 1.6 illustrates energy and state space diagrams for simple harmonic motion. The state space trajectories are ellipses, so the position and velocity vary sinusoidally but $90^{\circ}$ out of phase. The Eq. 1.21 parabolic potential energy is extremely useful because it well approximates generic potential energy minima.


Figure 1.6: Simple harmonic oscillator energy diagram (top) and state space (bottom) for three different initial conditions. The state-space trajectories are ellipses.


Figure 1.7: Simple pendulum energy diagram (top) and state space (bottom) for three different initial conditions. A separatrix (dashed blue curve) separates high speed rotation (left and right) and low speed libration (center).

### 1.5.3 Simple Pendulum

A final paradigmatic dynamical system is the simple pendulum, which consists of a mass swinging in a vertical circle under gravity. Instead of a linear or quadratic potential energy, a sinusoidal potential energy models the simple pendulum. Assume a mass $m$ at a distance $\ell$ from a fixed pivot swings through an angle $\theta$ from downward. Differentiate the arc length from downward

$$
\begin{equation*}
s=\ell \theta \tag{1.22}
\end{equation*}
$$

to get

$$
\begin{equation*}
v_{s}=\frac{d s}{d t}=\ell \frac{d \theta}{d t}=\ell \omega_{\theta} \tag{1.23}
\end{equation*}
$$

where the angular velocity $\omega_{\theta}=d \theta / d t$. The kinetic energy

$$
\begin{equation*}
K=\frac{1}{2} m v_{\theta}^{2}=\frac{1}{2} m\left(\ell \omega_{\theta}\right)^{2}=\frac{1}{2} m \ell^{2} \omega_{\theta}^{2}=\frac{1}{2} I \omega_{\theta}^{2} \tag{1.24}
\end{equation*}
$$

where the rotational inertia

$$
\begin{equation*}
I=m \ell^{2} \tag{1.25}
\end{equation*}
$$

The potential energy depends on the height $h$ of the mass, and so

$$
\begin{equation*}
U=m g h=m g(\ell-\ell \cos \theta)=m g \ell(1-\cos \theta) \tag{1.26}
\end{equation*}
$$

is large for angles near unstable $180^{\circ}$ (upward) and small for angles near stable $0^{\circ}$ (downward) and is periodic every $360^{\circ}$. Figure 1.7 illustrates energy and state space diagrams for pendulum motion. For large positive or negative velocities, the end-over-end motion is called rotation; for small velocities, the back-andforth motion is called libration.

### 1.6 Limitations

Conservation of energy can predict the motion of only a fraction of mechanical systems, typically those of low dimensionality. Seek a more fundamental principle: the principle of stationary action.

## Problems

1. Use explicit conversion factors to reexpress $g \approx 9.8 \mathrm{~m} / \mathrm{s}^{2}$ in $\mathrm{mph} / \mathrm{s}$.
2. Assuming the Eq. 1.1 law of fall, show algebraically that the distances fallen in successive equal time intervals are proportional to the odd integers. Hint: Compute $\Delta s_{n}=s_{n}-s_{n-1}$, where $s_{n}=s[n \Delta t]$.
3. Derive an equation describing the space rate of change of velocity for a falling object. Why do you think Galileo rejected this acceleration definition in favor of the time rate of change of velocity? Hint: If $y=x^{r}$, then $d y / d x=r x^{r-1}$, even for non-integer exponents $r$.
4. Derive the Eq. 1.8 law of energy conservation by assuming the most general quadratic law of fall, $s=s_{0}+v_{0} t-(1 / 2) g t^{2}$. What do the constants $s_{0}$ and $v_{0}$ represent physically?
5. Verify that kinetic energy is conserved for each of the Fig. 1.4 possible outcomes, but momentum is only conserved for the yellow boxed outcomes.
6. Prove that the simple harmonic oscillator state space ovals of Fig. 1.6 are ellipses. Assuming the states are initially at the large dots, is the motion clockwise or counterclockwise?
7. Derive a formula for the time $t$ needed by a mass $m$ to fall a distance $h$.
8. Throw three identical stones off a cliff with the same speed: one almost vertically upward, one horizontally, and one vertically downward. Neglecting air friction, which stone hits the ground with the greatest speed? Hint: By equating the sum of the kinetic and potential energies initially and finally, derive a formula for the final speed in terms of the initial speed and the height.
9. Throw a baseball up into the air. Including air friction, does the ball spend more time going up or coming down? Repeat on the lunar surface. Hint: Compare the energy of the ball at one height as it goes up and comes down, accounting for the loss of energy to the air.
10. Sketch energy and state space diagrams for a bistable oscillator with potential energy $U=a s^{2} / 2-b s^{4} / 4$, where $a, b>0$. Sketch a separatrix curve in the state space separating two qualitatively different kinds of motion.

## Chapter 2

## Action

Motion "stationizes" action - motion is such that action is stationary!


Figure 2.1: Electrons passing through a double slit buildup a wave-like interference pattern, which is a probability distribution for their arrivals [3]. In 2002, a survey of the most beautiful physics experiments by PhysicsWorld ranked Feynman's double slit experiment with electrons number one.

### 2.1 Wave-Particle Duality

Subatomic entities or quanta like electrons and photons act like particles in some contexts and waves in others, as in Fig. 2.1. Such wave-particle duality is central to quantum mechanics and contains the key to unlock classical mechanics. Understand billiard balls by first understanding electrons, not the other way around. Begin by understanding classical waves.

### 2.1.1 Classical Waves

Consider a sinusoidal traveling wave, with zeros separating maxima and minima, like the crests and troughs in ripples on a pond. The wave's period $T$ is the time between maxima (or minima) at a fixed position $s$; the wave's wavelength $\lambda$ is the distance between maxima (or minima) at a fixed time $t$. The wave travels a wavelength $\lambda$ in a period $T$, as in Fig. 2.2. so its velocity magnitude or speed

$$
\begin{equation*}
v=\frac{\lambda}{T}=\lambda f>0 \tag{2.1}
\end{equation*}
$$

where $f$ is its frequency. Other widely used secondary parameters include the angular frequency or temporal frequency

$$
\begin{equation*}
\omega=\frac{2 \pi}{T}=2 \pi f \tag{2.2}
\end{equation*}
$$

and the wave number or spatial frequency magnitude

$$
\begin{equation*}
k=\frac{2 \pi}{\lambda} \tag{2.3}
\end{equation*}
$$

The wave height

$$
\begin{equation*}
h[s, t]=A \sin [\varphi[s, t]], \tag{2.4}
\end{equation*}
$$

where the amplitude $A$ is half the height between a maxim and a minimum, and the phase

$$
\begin{equation*}
\varphi[s, t]=2 \pi\left(\frac{s}{\lambda}-\frac{t}{T}\right)=k_{s} s-\omega t=k_{s}(s-v t) \tag{2.5}
\end{equation*}
$$

is the number of wave cycles elapsed since the origin, including fractional cycles, times $2 \pi$ radians or $360^{\circ}$. The subscript $s$ on the spatial frequency indicates a wave traveling in the $s$ direction.


Figure 2.2: Spacetime plot (left) and three spatial snapshots (right) of a sinusoidal traveling wave of amplitude $A$, wavelength $\lambda$, period $T$, and speed $v$.

At a fixed time, say $t=0$, the phase $\varphi[s, 0]=2 \pi s / \lambda$ increases by $2 \pi$ when the distance $s$ increases by one wavelength $\lambda$; at a fixed position, say $s=0$, the
phase $\varphi[0, t]=-2 \pi t / T$ decreases by $2 \pi$ when the time $t$ increases by one period $T$. Since the height $h[v t, t]=0=h[0,0]$, the initial zero is always at $s=v t$, and the wave travels in the positive $s$ direction at speed $v$.

### 2.1.2 Young's Double Slit Experiment

In 1803, Thomas Young demonstrated that light behaves like a wave when passing through sufficiently small and close holes, as in Fig. 2.3. If the paths from the holes to a distant screen differ by an odd number of half wavelengths, then the maxima of waves from one path arrive simultaneous with the minima from the other and destructively interfere causing darkness. If the paths differ by an even number of half wavelengths (or an integer number of wavelengths), then the maxima (and minima) of waves from both paths arrive simultaneously and constructively interfere causing brightness. The frequency of visible light, $f \sim 500 \mathrm{THz}$, is too high to observe directly; instead the eye is sensitive to the mean-square amplitude or irradiance $I \propto\left\langle A^{2}\right\rangle$.


Figure 2.3: Young's double slit experiment (left). If the paths from the slits to screen differ by three half wavelengths, for example, then the light arrives "crest to trough" and destructive interference causes darkness (right).

In the 1900s, Young's experiment was repeated with very faint light - so faint that the "graininess" of light became apparent, and the probability to detect individual grains of light or photons was found to be proportional to the wave's amplitude squared, $\mathcal{P} \propto\langle A\rangle^{2}$. Even when the light was detected photon-byphoton, so there was only one particle at the slits at any given time, the same interference pattern accumulated. It became clear that the classical concepts of "wave" and "particle", either separately or collectively, did not exhaustively describe light.

### 2.1.3 Matter Waves

In the early 1900s, at the birth of quantum mechanics, Albert Einstein first proposed that particles are associated with waves, and Louis de Broglie then suggested that waves are associated with particles. Specifically, Einstein argued that light is emitted or absorbed in packets or quanta, now called photons, whose energies are proportional to the light's frequency,

$$
\begin{equation*}
E=\hbar \omega=\left(\frac{h}{2 \pi}\right)(2 \pi f)=h f \tag{2.6}
\end{equation*}
$$

where Planck's constant

$$
\begin{equation*}
h=2 \pi \hbar \approx 6.6 \times 10^{-34} \mathrm{~J} \mathrm{~s}=0.66 \frac{\mathrm{zJ}}{\mathrm{THz}} \tag{2.7}
\end{equation*}
$$

is the rate of change of photon energy with frequency. By symmetry, de Broglie later argued that particles, like electrons, should be associated with waves whose spatial frequencies are proportional to the particles' momenta,

$$
\begin{equation*}
p_{s}=\hbar k_{s} \tag{2.8}
\end{equation*}
$$

or in terms of momentum magnitude,

$$
\begin{equation*}
p=\hbar k=\left(\frac{h}{2 \pi}\right)\left(\frac{2 \pi}{\lambda}\right)=\frac{h}{\lambda} \tag{2.9}
\end{equation*}
$$

where Planck's reduced constant is also the rate of change of photon momentum with inverse wavelength.

As Richard Feynman famously emphasized in an early 1960s thought experiment [4, wave-particle duality means that Young's double slit experiment should also work with electrons, and indeed it does. By 2000, Feynman's thought experiment had been realized using beams of electrons, atoms, small molecules, and even buckyballs ( $\mathrm{C}_{60}$ "soccer ball" molecules).

### 2.1.4 Sum Over Paths

How can photons or electrons buildup an interference pattern, especially if they pass through the slits one by one? In Feynman's sum-over-paths approach to quantum mechanics, each quantum takes both paths through the double slit experiment and interferes with itself according to the phase difference between the paths.

Consider an electron or other quantum moving through space [5] with the Eq. 2.5 phase

$$
\begin{equation*}
\varphi=k_{s} s-\omega t \tag{2.10}
\end{equation*}
$$

Its rate of change with time is

$$
\begin{equation*}
\frac{d \varphi}{d t}=k_{s} \frac{d s}{d t}-\omega=k_{s} v_{s}-\omega \tag{2.11}
\end{equation*}
$$

where $v_{s}=d s / d t$ is the quantum's velocity. Multiply by Planck's reduced constant $\hbar$ and substitute the Eq. 2.8 Einstein-de Broglie relations to write

$$
\begin{equation*}
\hbar \frac{d \varphi}{d t}=\hbar k_{s} v_{s}-\hbar \omega=p_{s} v_{s}-E \tag{2.12}
\end{equation*}
$$

But the combination

$$
\begin{equation*}
p_{s} v_{s}-E=m v_{s}^{2}-\frac{1}{2} m v_{s}^{2}-U=\frac{1}{2} m v_{s}^{2}-U=K-U=\mathcal{L} \tag{2.13}
\end{equation*}
$$

where $\mathcal{L}$ is the Lagrangian. Hence, the difference in the kinetic and potential energies determines the rate of change with time of the quantum's phase,

$$
\begin{equation*}
\hbar \frac{d \varphi}{d t}=\mathcal{L}=K-U \tag{2.14}
\end{equation*}
$$

The total phase accumulated over the quantum's path is the sum or integral

$$
\begin{equation*}
\hbar \varphi=\int \hbar d \varphi=\int \hbar \frac{d \varphi}{d t} d t=\int \mathcal{L} d t=\mathcal{A} \tag{2.15}
\end{equation*}
$$

where $\mathcal{A}$ is the classical action for the path. The total phase $\varphi=\mathcal{A} / \hbar$ is precisely the path's action in units of the quantum of action, Planck's reduced constant $\hbar$.

If, for example, the total phases for two paths differ by $\pi$ radians, corresponding to a half a cycle, destructive interference results; if the total phases for two paths differ by $2 \pi$ radians, corresponding to a full cycle, constructive interference results.


Figure 2.4: Imagine space filled with infinitely many barriers containing infinitely many slits - so they aren't there any more!

### 2.1.5 Recovering Classical Mechanics

Replace Young's two slits with many slits. Now quanta like photons and electrons take many interfering paths through the slits, from source to screen. Next
imagine filling space with many barriers each with many slits - or even infinitely many barriers with infinitely many slits, as in Fig. 2.4 .

Conclude that quanta explores all possible paths, including complicated zigzag paths or even backward-in-time paths. Most of the paths will interfere destructively and won't contribute to the probability amplitude, as in Fig. 2.5. For example, for any zig-zag path, another path with an extra zig or zag will accumulate an extra $\pi$ phase shift to cancel it.

However, there is typically a special path whose phase and action $\varphi=\mathcal{A} / \hbar$ are stationary with respect to small path variations, like the black path in Fig. 2.5. A narrow bundle of paths about this path interferes constructively to increase the probability of such a path. The special path is the classical path. In many important cases, such as the motion of large masses, the bundle is very thin making a path like the classical path nearly certain. Thus, the observed motion is such that action is stationary: motion "stationizes" action.


Figure 2.5: An electron explores all paths between fixed initial and final points. The wiggly paths interfere destructively, but the narrow pencil of paths about the classical path interfere constructively, because differences among their phases - and hence actions - are negligible.

### 2.2 Free Fall Action

Returning to the Chapter 1 problem of free fall, take the difference of the kinetic and potential energies to form the Lagrangian

$$
\begin{equation*}
\mathcal{L}=K-U \tag{2.16}
\end{equation*}
$$

Consider the area bounded by a plot of the Lagrangian versus time, as on the right of Fig. 2.6. Compute this area by decomposing it into narrow rectangles of height $\mathcal{L}$ and width $d t$ and summing from $t=0$ to $t=T$. The resulting action integral is simply the mean height $\langle\mathcal{L}\rangle$ times the total time $T$ or

$$
\begin{equation*}
\mathcal{A}=\sum_{\Delta t \rightarrow 0} \mathcal{L} \Delta t=\int \mathcal{L} d t=\left(\frac{1}{T} \int_{0}^{T} \mathcal{L} d t\right) T=\langle\mathcal{L}\rangle T=\langle K-U\rangle T \tag{2.17}
\end{equation*}
$$

Thus, the action is proportional to the average difference between the kinetic and potential energies over the path. Like the potential energy, the actual value of the action is determined only up to an additive constant. But for free fall, as in Fig. 2.6, infinitesimal path changes never alter it, while finite path changes always increase it.


Figure 2.6: Actual spacetime free fall path $s[t]$ (top) conserves energy $E$ (center) and minimizes action $\mathcal{A}$ (right). Changing the path (bottom) varies the energy (center) and increases the action (right).

### 2.2.1 Stationary vs. Least

While the action is always stationary for variations about the actual path, in important examples it is sometimes also least. For an analogy, consider an undulating two-dimensional surface. At stationary points the surface has vanishing slope; maxima are like hill tops, minima are like depression bottoms, and saddles are maxima in some directions and minima in other directions. Likewise, for sufficiently short sections of any path, the action is always a minimum, but for longer sections, it can be a saddle, so that it's a minimum for some variations and a maximum for others. The action is never a maximum for all variations 6. Historically the idea of "least action" has suggested the economy and elegance of nature. Represent stationarity symbolically by

$$
\begin{equation*}
\delta \mathcal{A}=0 \tag{2.18}
\end{equation*}
$$

as a reminder that the differences in action $\delta \mathcal{A}$ between the real path and nearby virtual trajectories are negligible.

### 2.2.2 Simple Examples

To get a sense of how this works, first consider a free object with a vanishing potential energy $U=0$ moving between two places in a fixed time, as on the left of Fig. 2.7, so that the Lagrangian $\mathcal{L}=K$ and the action $\mathcal{A}=\int K d t=\langle K\rangle T=$ $(1 / 2) m\left\langle v_{s}^{2}\right\rangle T$. Consider two velocities $v_{1}$ and $v_{2}$. Quarter the inequality

$$
\begin{equation*}
0 \leq\left(v_{1}-v_{2}\right)^{2}=2\left(v_{1}^{2}+v_{2}^{2}\right)-\left(v_{1}+v_{2}\right)^{2} \tag{2.19}
\end{equation*}
$$

and rearrange to prove that the square of the mean is a lower bound for the mean of the square,

$$
\begin{equation*}
\left\langle v_{s}\right\rangle^{2}=\left(\frac{v_{1}+v_{2}}{2}\right)^{2} \leq \frac{v_{1}^{2}+v_{2}^{2}}{2}=\left\langle v_{s}^{2}\right\rangle \tag{2.20}
\end{equation*}
$$

The inequality is "saturated" and equality happens when the velocities are equal, $v_{1}=v_{2}$. Because the path's endpoints fix the left-side mean, this also minimizes the right-side mean square. More generally, constant velocity $v_{s}=v_{0}$ minimizes the mean square velocity $\left\langle v_{s}^{2}\right\rangle$, the mean kinetic energy $\langle K\rangle$, and the action $\mathcal{A}$, exactly as expected for a free object.


Figure 2.7: Straight spacetime path minimizes the action $\mathcal{A}=\langle K-U\rangle T$ for a free object (left), while a parabolic path minimizes the action for a gravitationally bound object (right).

Next consider an object under gravity with the Eq. 1.11 potential energy moving between two places in a fixed time, as on the right of Fig. 2.7. Because the action is the mean difference between the kinetic and potential energies, moving higher decreases the action by increasing the potential energy. However, moving higher also increases the action by increasing kinetic energy near the start and stop. The actual path is a compromise that adds potential energy without adding too much kinetic energy.

### 2.2.3 Global to Local

Energy conservancy and action stationarity are key physical insights. The action principle is a global, integral condition, but it implies a local, differential
constraint: Isaac Newton's second law of motion [7.
Imagine perturbing the spacetime path of an object under gravity [8, 9]. Leaving the endpoints fixed, raise the midpoint of a short path segment of duration $2 d t$ by a height $d s$, as in Fig. 2.8. The segment's action

$$
\begin{equation*}
A=\int \mathcal{L} d t=\mathcal{L}_{b} d t+\mathcal{L}_{a} d t=\left(\mathcal{L}_{b}+\mathcal{L}_{a}\right) d t \tag{2.21}
\end{equation*}
$$

where the subscripts $b$ and $a$ indicate before and after the midpoint. Since the action is stationary with respect to small variations,

$$
\begin{equation*}
0=\delta A=\delta\left(\mathcal{L}_{b}+\mathcal{L}_{a}\right) d t \tag{2.22}
\end{equation*}
$$

and since the duration $d t \neq 0$,

$$
\begin{equation*}
0=\delta\left(\mathcal{L}_{b}+\mathcal{L}_{a}\right)=\delta\left(K_{b}+K_{a}\right)-\delta\left(U_{b}+U_{a}\right) \tag{2.23}
\end{equation*}
$$



Figure 2.8: Perturbing a short segment of a free fall path (left) with closeup approximating the path and its perturbation as straight line segments (right).

The kinetic energy changes because the slopes, and hence the velocities, before and after the midpoint change. Specifically, before the midpoint the slope increases by

$$
\begin{equation*}
\delta v_{b}=+\frac{d s}{d t} \tag{2.24}
\end{equation*}
$$

and after the midpoint the slope decreases by

$$
\begin{equation*}
\delta v_{a}=-\frac{d s}{d t} \tag{2.25}
\end{equation*}
$$

Generically, by Eq. 1.17, kinetic energy changes with velocity like

$$
\begin{equation*}
\delta K=p_{s} \delta v_{s} \tag{2.26}
\end{equation*}
$$

so the kinetic energy change in the variation is

$$
\begin{equation*}
\delta\left(K_{b}+K_{a}\right)=p_{b} \delta v_{a}+p_{a} \delta v_{b}=\left(p_{b}-p_{a}\right) \frac{d s}{d t}=-\delta p_{s} \frac{d s}{d t}=-\frac{d p_{s}}{d t} \delta s \tag{2.27}
\end{equation*}
$$

where $\delta s / \delta t=d s / d t$ in the limit where $\delta t \rightarrow 0$, and so on.
The potential energy changes because the average heights before and after the midpoint change. Specifically, before and after the midpoint the mean height change

$$
\begin{equation*}
\langle\delta s\rangle=\frac{\delta s}{2} \tag{2.28}
\end{equation*}
$$

so the potential energy change in the variation is

$$
\begin{equation*}
\delta\left(U_{b}+U_{a}\right)=m g \frac{\delta s}{2}+m g \frac{\delta s}{2}=m g \delta s \tag{2.29}
\end{equation*}
$$

Substitute the Eq. 2.27 and Eq. 2.29 changes into the Eq. 2.23 non-change to find

$$
\begin{equation*}
0=-\frac{d p_{s}}{d t} \delta s-m g \delta s \tag{2.30}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{d p_{s}}{d t} \delta s=-m g \delta s \tag{2.31}
\end{equation*}
$$

Thus, the gravitational force

$$
\begin{equation*}
f_{s}=\frac{d p_{s}}{d t}=-m g \tag{2.32}
\end{equation*}
$$

is the rate of change of momentum with time. This is an example of Newton's second law of motion, a differential constraint on the momentum that is the local version of the global variational constraint on the action.

Similarly, for a generic potential energy function $U[s]$, the Eq. 2.23 nonchange becomes

$$
\begin{equation*}
0=-\frac{d p_{s}}{d t} d s-d U=-\frac{d p_{s}}{d t} d s-\frac{d U}{d s} d s \tag{2.33}
\end{equation*}
$$

so the corresponding generic force

$$
\begin{equation*}
f_{s}=\frac{d p_{s}}{d t}=-\frac{d U}{d s} \tag{2.34}
\end{equation*}
$$

The force is proportional to the rate of change of potential energy with distance. Geometrically, the force is the negative slope of the potential energy function, as in Fig. 2.9. And so a ball rolls down hill.

For the quadratic Eq. 1.21 simple harmonic oscillator potential function, the linear force

$$
\begin{equation*}
f_{s}=-\frac{d}{d s}\left(\frac{1}{2} \kappa s^{2}\right)=-\kappa s \tag{2.35}
\end{equation*}
$$

which is Hooke's law. The larger the stretch $s>0$ or squeeze $s<0$ of the spring, the larger the force magnitude, but always in the opposite direction.


Figure 2.9: Linear (left) and quadratic (right) potentials and their force functions. Force is minus the gradient of the potential energy.

## Problems

1. Reexpress Planck's constant $h \approx 0.66 \mathrm{zJ} / \mathrm{THz}$ in abol/Tw to emphasize its alternate role as the rate of change of photon momentum with spatial frequency. Assume $1 \mathrm{bol}=10^{-5} \mathrm{~kg} \mathrm{~m} / \mathrm{s}$ and $1 \mathrm{w}=1 \mathrm{~m}^{-1}$. (The "bole" has been proposed as a unit of linear momentum; the symbol "w", which is an upside-down letter " $m$ ", is pronounced "me" and represents an inverse meter.)
2. In the Fig. 2.3 schematic of Young's experiment, show that the $n$th bright band or fringe is a distance $n \lambda D / d$ from the central fringe, where $d$ is the small distance between the slits, $D \gg d$ is the large distance from the slits to the screen, and $\lambda$ is the light wavelength. Hint: In Fig. 2.3 $\tan \theta \sim \sin \theta \sim \theta \ll 1$, and the paths from the slits to the $n$th fringe are nearly parallel.
3. Write the equation for a sinusoidal wave of amplitude $a$ and wavelength $\Lambda$ traveling in the negative $x$ direction at a speed $c$.
4. If $n$ water wave crests and troughs pass a point in time $\tau$, and the horizontal distance between crest and the nearest trough is a distance $\ell$, what is the wave's speed?
5. Download the "EnergyActionHW" Mathematica manipulator. Click and drag the spacetime events to minimize the action. What happens to the energy? Include a screenshot of your best result with your solution.
6. Using dimensionless variables, consider an object freely falling between $s=0$ at $t=0$ and $s=-1 / 2$ at $t=1$. Assume the object follows the path $s=-(1 / 2) t^{a}$ with kinetic energy $K=(1 / 2)(d s / d t)^{2}$ and potential energy $U=s$. Compute the action $\mathcal{A}$ for $a=1,2,3$. Which is smallest and why? Hint: Use the Chapter 1 Problem 3 hint to find the velocities and then integrate term-by-term using $\int_{0}^{1} t^{b} d t=1 /(b+1)$.

7. Use Eq. 2.9 to compute the de Broglie wavelength $\lambda$ of an apple falling from a tree. Make intelligent estimates of the mass $m$ and speed $v$ of the apple. Why would it be difficult to observe a double slit interference with the apple?
8. According to Eq. 2.34 to what force $f_{s}$ does the Eq. 1.21 simple harmonic oscillator potential energy function $U[s]$ correspond? Sketch a graph of the force and potential energy versus position.

## Chapter 3

## Calculus in a Nutshell

Infinitesimal calculus is a cornerstone of physics and mathematics.


Figure 3.1: Isaac Newton and Gottfried Leibniz independently and intuitively developed the calculus in the mid 1600s. Augustin-Louis Cauchy and others rigorously refined the calculus in the 1800s and later.

### 3.1 Fundamental Theorem

The fundamentals of infinitesimal calculus were discovered in the 1600s by Isaac Newton and Gottried Liebniz of Fig. 3.1. For Newton, the fundamental problem was twofold: given positions $s[t]$, find the corresponding velocities $v_{s}[t]$, and given velocities $v[t]$, find the corresponding positions $s[t]$. For Leibniz, in his famous notation, these are the problems of differentiation $v_{s}=d s / d t$ and
integration $s=\int v_{s} d t$, or the problem of slopes and areas. Differentiation and integration undo one another, which is obvious from Newton's dynamical perspective but perhaps not from Leibniz's geometric perspective.


Figure 3.2: Differentiation and integration relate slopes (top) and areas (bottom).

Consider the geometry of the position $s[t]$ and velocity $v_{s}[t]$ plots in Fig. 3.2 . Intuitively, during short "infinitesimal" times $d t$, the bottom velocity plot sweeps out small differential areas $d s=v_{s} d t$, and during longer times sweeps out large cumulative areas

$$
\begin{equation*}
s=\int d s=\int v_{s} d t \tag{3.1}
\end{equation*}
$$

that are the sum or integral of these differential areas. Correspondingly, in a short time $d t$, the slope of the position plot is the velocity

$$
\begin{equation*}
\frac{d s}{d t}=\frac{s[t+d t]-s[t]}{d t}=v_{s}[t] . \tag{3.2}
\end{equation*}
$$

Combine these results to show that the integral of the derivative is the function itself,

$$
\begin{equation*}
s=\int \frac{d s}{d t} d t \tag{3.3}
\end{equation*}
$$

which means that the integral of a function is its anti-derivative. More precisely, the area between two times $t_{1}$ and $t_{2}$ swept out by the velocity $v_{s}[t]$ is

$$
\begin{equation*}
s\left[t_{2}\right]-s\left[t_{1}\right]=\int_{t_{1}}^{t_{2}} d s=\int_{t_{1}}^{t_{2}} \frac{d s}{d t} d t=\int_{t_{1}}^{t_{2}} v_{s} d t \tag{3.4}
\end{equation*}
$$

Differentiate both sides with respect to $t_{2}$ to show conversely that the derivative of the integral is the function itself,

$$
\begin{equation*}
v_{s}\left[t_{2}\right]=\frac{d}{d t_{2}} \int_{t_{1}}^{t_{2}} v_{s}[t] d t \tag{3.5}
\end{equation*}
$$

### 3.2 Derivatives \& Antiderivatives

Since the derivative of a constant is zero, two function that differ by a constant share the same derivative. Thus, their antiderivatives or indefinite integrals are not unique. For example, from Chapter 1. we know

$$
\begin{equation*}
\frac{d}{d t}\left(t^{2}\right)=2 t \tag{3.6}
\end{equation*}
$$

but also

$$
\begin{equation*}
\frac{d}{d t}\left(t^{2}+1\right)=2 t+0=2 t \tag{3.7}
\end{equation*}
$$

Hence the antiderivative

$$
\begin{equation*}
\int 2 t d t=t^{2}+C \tag{3.8}
\end{equation*}
$$

where $C$ is an undefined constant. Fortunately, we will often deal with definite integrals whose limits fix the constant. For example,

$$
\begin{equation*}
\int_{3}^{4} 2 t d t=\left.t^{2}\right|_{3} ^{4}=4^{2}-3^{2}=7 \tag{3.9}
\end{equation*}
$$

Table 3.1: Basic derivatives and antiderivatives without integration constants.

| Antiderivative | Function | Derivative | Graphs |
| :--- | :--- | :--- | :--- |
| $\int f[x] d x$ | $f[x]$ | $\frac{d}{d x} f[x]$ |  |
| $\frac{x^{n+1}}{n+1}$ | $x^{n}$ | $n x^{n-1}$ |  |
| $-\cos x$ | $\sin x$ | $\cos x$ | $\exp x$ |
| $\exp x$ | $\exp x$ | $\log x$ | $\frac{1}{x}$ |
| $x \log x-x$ |  |  |  |

Table 3.1 contains a list of important derivatives and antiderivatives. As is common in theoretical physics and pure mathematics, $\log x=\log _{e} x=\ln x$ is the inverse of $\exp x=e^{x}$.

### 3.3 Compound Operations

Because differentiation is a linear operation, the derivative of the sum of two functions $f[x]$ and $g[x]$ is simply the sum of the derivatives,

$$
\begin{equation*}
\frac{d}{d x}(a f+b g)=a \frac{d f}{d x}+b \frac{d g}{d x} \tag{3.10}
\end{equation*}
$$

where $a$ and $b$ are constants. If $x$ changes by $\delta x$, then the product of the functions $f g$ changes by

$$
\begin{align*}
\delta(f g) & =(f+\delta f)(g+\delta g)-f g \\
& =\delta f g+f \delta g+\delta f \delta g \tag{3.11}
\end{align*}
$$

and so

$$
\begin{equation*}
\frac{\delta(f g)}{\delta x}=\frac{\delta f}{\delta x} g+f \frac{\delta g}{\delta x}+\frac{\delta f}{\delta x} \delta g \tag{3.12}
\end{equation*}
$$

In the limit $\delta x \rightarrow 0$, both $\delta f \rightarrow 0$ and $\delta g \rightarrow 0$ such that the derivative of a product is

$$
\begin{equation*}
\frac{d}{d x}(f g)=\frac{d f}{d x} g+f \frac{d g}{d x} \tag{3.13}
\end{equation*}
$$

For example,

$$
\begin{equation*}
\frac{d}{d t}\left(s_{0}+v_{0} t+\frac{1}{2} g t^{2}\right)=v_{0}+g t \tag{3.14}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d}{d t}\left(t^{2} \sin t\right)=2 t \sin t+t^{2} \cos t \tag{3.15}
\end{equation*}
$$

and, from Table 3.1,

$$
\begin{equation*}
\frac{d}{d x}(x \log x-x)=1 \log x+x \frac{1}{x}-1=\log x \tag{3.16}
\end{equation*}
$$

The derivative of a composition of functions, such as the function of a function $f[g[x]]$, is simply

$$
\begin{equation*}
\frac{d f}{d x}=\frac{d f}{d g} \frac{d g}{d x}=\frac{d f}{d g} \frac{d g}{d x} \tag{3.17}
\end{equation*}
$$

which is known as the chain rule. Thus, the derivative of a composition is the derivative of the outer function with respect to its argument (the inner function) times the derivative of the argument. For example,

$$
\begin{equation*}
\frac{d}{d t}(A \sin [\omega t])=A \cos [\omega t] \frac{d}{d t}(\omega t)=\omega A \cos [\omega t] \tag{3.18}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d}{d t}\left(I_{0} e^{\kappa t}\right)=I_{0} e^{\kappa t} \frac{d}{d t}(\kappa t)=\kappa I_{0} e^{\kappa t} \tag{3.19}
\end{equation*}
$$

Table 3.2: Basic differentiation rules.

| Rule | Form |
| :--- | :--- |
| sum | $\frac{d}{d x}(a f+b g)=a \frac{d f}{d x}+b \frac{d g}{d x}$ |
| product | $\frac{d}{d x}(f g)=\frac{d f}{d x} g+f \frac{d g}{d x}$ |
| composition or "chain" | $\frac{d f}{d x}=\frac{d f}{d g} \frac{d g}{d x}$ |
| partial | $\frac{\partial f}{\partial x}=\left.\frac{d f}{d x}\right\|_{y}$ |

The partial derivative of a function of two variables $f[x, y]$ with respect to one of them $x$ is simply the ordinary derivative with the other variable $y$ held constant,

$$
\begin{equation*}
\frac{\partial f}{\partial x}=\left.\frac{d f}{d x}\right|_{y}=\frac{d}{d x} f[x, y=\text { const }] \tag{3.20}
\end{equation*}
$$

For example, if

$$
\begin{equation*}
h[s, t]=A \sin \left[k_{s} s-\omega t\right] \tag{3.21}
\end{equation*}
$$

then the partial derivatives

$$
\begin{equation*}
\frac{\partial h}{\partial s}=A \cos \left[k_{s} s-\omega t\right] \frac{\partial}{\partial s}\left(k_{s} s-\omega t\right)=+k_{s} A \cos \left[k_{s} s-\omega t\right] \tag{3.22}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial h}{\partial t}=A \cos \left[k_{s} s-\omega t\right] \frac{\partial}{\partial t}\left(k_{s} s-\omega t\right)=-\omega_{s} A \cos \left[k_{s} s-\omega t\right] \tag{3.23}
\end{equation*}
$$

If $x$ and $y$ are independent variables,

$$
\begin{equation*}
\frac{\partial x}{\partial x}=1 \tag{3.24}
\end{equation*}
$$

but

$$
\begin{equation*}
\frac{\partial y}{\partial x}=0 \tag{3.25}
\end{equation*}
$$

as the latter is the rate of change of $y$ with $x$ assuming all other variables including $y$ are held constant. The pseudo-letters partial $\partial$ and nabla $\nabla$ may
be pronounced "del" in analogy with the Greek letters $\delta$ and $\Delta$, which are pronounced "delta".

Second derivatives are simply derivatives of derivatives. In Leibniz's classic notation

$$
\begin{equation*}
\frac{d}{d x}\left(\frac{d}{d x}(f)\right)=\frac{d}{d x} \frac{d}{d x} f=\left(\frac{d}{d x}\right)^{2} f=\frac{d^{2}}{d x^{2}} f=\frac{d^{2} f}{d x^{2}} \tag{3.26}
\end{equation*}
$$

and similarly for higher orders. Table 3.2 summarizes the basic differentiation rules.

## Problems

1. Compute the following derivatives.
(a) $\frac{d}{d x}\left(3 x^{2}+4 x^{3}\right)$
(b) $\frac{d}{d \phi}(3 \sin \phi+4 \cos \phi)$
(c) $\frac{d}{d t}(3 \exp t-2 \log t)$
(d) $\frac{d}{d z}(3 \exp z \log z)$
(e) $\frac{d}{d s}(2 \sin k s \cos k s)$
2. Compute the following partial derivatives.
(a) $\frac{\partial}{\partial x}\left(3 x^{2} y^{3}\right)$
(b) $\frac{\partial}{\partial y}\left(3 x^{2} y^{3}\right)$
(c) $\frac{\partial}{\partial s}\left(3 s^{2}+e^{s^{3} t^{2}}\right)$
(d) $\frac{\partial}{\partial t}\left(3 s^{2}+e^{s^{3} t^{2}}\right)$
3. Compute the following anti-derivatives or indefinite integrals. Check each one by differentiating the result.
(a) $\int x^{2} d x$
(b) $\int(\cos \theta+3 \sin \theta) d \theta$
(c) $\int 3 e^{3 t} d t$
4. Compute the following areas or definite integrals. Hint: In Eq. 3.4, s is the antiderivative of $v_{s}$.
(a) $\int_{0}^{3} x^{3} d x$
(b) $\int_{-\pi}^{\pi}(\sin \theta-\cos \theta) d \theta$
(c) $\int_{0}^{\infty} 4 e^{-2 t} d t$

## Chapter 4

## Lagrange's Equations

Lagrange's equations are the formal differential expression of stationary action.


Figure 4.1: Postage stamp history of mechanics: from Newton's forces (1600s) to Lagrange's differential equations (1700s) to Hamilton's integral action principle (1800s) to Feynman's sum-over-paths quantum mechanics (1900s).

### 4.1 Differential Equations of Motion

Newton's classical mechanics was successively reformulated by Lagrange and Hamilton in the action principle elucidated by Feynman's quantum mechanics,
as summarized in Fig. 4.1. Elementary calculus can reduce the requirement of stationary action to equations involving derivatives that both describe motion and are especially well-suited for computer solution [10, 11]. These Lagrange or Euler-Lagrange equations generalize the Eq. 2.34 force equation.

First replace continuous time $t$ with discrete or stroboscopic time $t_{n}=n d t$, where $n$ is an integer and $d t$ is a small time step. Assume the Lagrangian $\mathcal{L}\left[s, v_{s}\right]$ depends only on space and velocity, with $s_{n}=s\left[t_{n}\right]$ and

$$
\begin{equation*}
v_{n}=\frac{d s_{n}}{d t}=\frac{s\left[t_{n}+d t\right]-s\left[t_{n}\right]}{d t}=\frac{s_{n+1}-s_{n}}{d t} \tag{4.1}
\end{equation*}
$$

so the action

$$
\begin{equation*}
\mathcal{A}=\int \mathcal{L} d t=\sum_{n} \mathcal{L}\left[s_{n}, v_{n}\right] d t=\sum_{n} \mathcal{L}\left[s_{n}, \frac{s_{n+1}-s_{n}}{d t}\right] d t \tag{4.2}
\end{equation*}
$$

Any one point, say $n=8$, appears just twice in the action sum,

$$
\begin{align*}
\mathcal{A} & =\cdots+\mathcal{L}\left[s_{7}, v_{7}\right] d t+\mathcal{L}\left[s_{8}, v_{8}\right] d t+\cdots \\
& =\cdots+\mathcal{L}\left[s_{7}, \frac{s_{8}-s_{7}}{d t}\right] d t+\mathcal{L}\left[s_{8}, \frac{s_{9}-s_{8}}{d t}\right] d t+\cdots \tag{4.3}
\end{align*}
$$

Enforce the principle of stationary action by demanding that the rate of change $\mathcal{A}$ with $s_{8}$ be zero. Use the Eq. 3.17 composition rule to expand and get

$$
\begin{equation*}
0=\frac{\partial \mathcal{A}}{\partial s_{8}}=\left(0+\frac{\partial \mathcal{L}}{\partial v_{7}} \frac{\partial v_{7}}{\partial s_{8}}+\frac{\partial \mathcal{L}}{\partial s_{8}}+\frac{\partial \mathcal{L}}{\partial v_{8}} \frac{\partial v_{8}}{\partial s_{8}}+0\right) d t \tag{4.4}
\end{equation*}
$$

Cancel the time step and evaluate the derivatives of the velocities to show

$$
\begin{equation*}
0=\frac{\partial \mathcal{L}}{\partial v_{7}}\left(\frac{1-0}{d t}\right)+\frac{\partial \mathcal{L}}{\partial s_{8}}+\frac{\partial \mathcal{L}}{\partial v_{8}}\left(\frac{0-1}{d t}\right) \tag{4.5}
\end{equation*}
$$

Isolate and implement the derivative definition to find

$$
\begin{equation*}
\frac{\partial \mathcal{L}}{\partial s_{8}}=\frac{\partial \mathcal{L} / \partial v_{8}-\partial \mathcal{L} / \partial v_{7}}{d t}=\frac{d}{d t}\left(\frac{\partial \mathcal{L}}{\partial v_{7}}\right)=\frac{d}{d t} \frac{\partial \mathcal{L}}{\partial v_{7}} \tag{4.6}
\end{equation*}
$$

Because the times $t_{7}$ and $t_{8}$ are infinitesimally close, the general Lagrange differential equation is

$$
\begin{equation*}
\frac{\partial \mathcal{L}}{\partial s}=\frac{d}{d t} \frac{\partial \mathcal{L}}{\partial v_{s}} \tag{4.7}
\end{equation*}
$$

where $v_{s}=d s / d t$ is the velocity in the $s$ direction.
In the variational calculus, the action $\mathcal{A}[s[t]]$ depends on the path $s[t]$ and its corresponding velocity $v_{s}[t]=d s[t] / d t$, but the Lagrangian $\mathcal{L}\left[s_{n}, v_{n}\right]=$ $\mathcal{L}\left[s\left[t_{n}\right], v\left[t_{n}\right]\right]$ depends on the values of the positions $s_{n}$ and velocities $v_{n}$, which are independent at any time $t_{n}$ because they may arise from completely different virtual paths. The stationary action requirement removes their independence during the derivation to select one actual path.

Equation 4.7 is the simplest example of a Lagrange equation. It easily generalizes to multiple particles moving in multiple dimensions. Cast it in the form of Newton's second law

$$
\begin{equation*}
f_{s}=\frac{d p_{s}}{d t} \tag{4.8}
\end{equation*}
$$

where the generalized force

$$
\begin{equation*}
f_{s}=\frac{\partial \mathcal{L}}{\partial s} \tag{4.9}
\end{equation*}
$$

and the generalized momentum

$$
\begin{equation*}
p_{s}=\frac{\partial \mathcal{L}}{\partial v_{s}} \tag{4.10}
\end{equation*}
$$

which reduces to $p_{s}=d K / d v_{s}=m v_{s}$ in simple cases, in agreement with Eq. 1.17 .

### 4.2 Numerical General Solution

Coupled with initial condition $s[0]=s_{0}$ and $v[0]=v_{0}$, the Lagrange equation becomes an initial value problem for the spacetime path $s[t]$, which can sometime be solved exactly but more often must be solved numerically, especially by computer. By the Eq. 4.8 Newtonian form, in a short time $d t$, force changes momentum by $d p_{s}=f_{s} d t$ or, if mass $m$ is constant, acceleration changes velocity by $d v_{s}=a_{s} d t$, where $a_{s}=f_{s} / m$. Always velocity changes position by $d s=v_{s} d t$. This suggests the semi-implicit Euler or Euler-Cromer [12] algorithm for numerically approximating the path $s[t]$. First initialize velocity and position by

$$
\begin{align*}
v_{s} & \leftarrow v_{0},  \tag{4.11a}\\
s & \leftarrow s_{0} \tag{4.11b}
\end{align*}
$$

and then repeatedly update them with

$$
\begin{equation*}
a_{s} \leftarrow f_{s} / m \tag{4.12}
\end{equation*}
$$

and

$$
\begin{align*}
v_{s} & \leftarrow v_{s}+a_{s} d t  \tag{4.13a}\\
s & \leftarrow s+v_{s} d t . \tag{4.13b}
\end{align*}
$$

This approximate solution can be very good if the integration step $d t$ is very small, but a computer might take very long to execute the algorithm. In practice, a large fraction of the word's computer resources implement such algorithms by balancing accuracy against computation time.

### 4.3 Example Initial Value Problems

Use the Lagrange equations to create initial value problems for the Section 1.5 fundamental dynamical systems. Solve them exactly if possible.

### 4.3.1 Flat Earth Gravity

Toss a stone of mass $m$ from a height $s_{0}$ with a velocity $v_{0}$. The flat Earth Lagrangian

$$
\begin{equation*}
\mathcal{L}\left[s, v_{s}\right]=K\left[v_{s}\right]-U[s]=\frac{1}{2} m v_{s}^{2}-m g s . \tag{4.14}
\end{equation*}
$$

Substitute into the Eq. 4.7 Lagrange equation and differentiate to find

$$
\begin{equation*}
\frac{\partial}{\partial s}\left(\frac{1}{2} m v_{s}^{2}-m g s\right)=\frac{d}{d t} \frac{\partial}{\partial v_{s}}\left(\frac{1}{2} m v_{s}^{2}-m g s\right) \tag{4.15}
\end{equation*}
$$

or

$$
\begin{equation*}
(0-m g)=\frac{d}{d t}\left(m v_{s}-0\right) \tag{4.16}
\end{equation*}
$$

or

$$
\begin{equation*}
-\not \approx g=\not n \frac{d v_{s}}{d t} \tag{4.17}
\end{equation*}
$$

The resulting equation

$$
\begin{equation*}
-g=\frac{d v_{s}}{d t}=\frac{d}{d t} \frac{d s}{d t}=\left(\frac{d}{d t}\right)^{2} s=\frac{d^{2}}{d t^{2}} s=\frac{d^{2} s}{d t^{2}} \tag{4.18}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{d^{2} s}{d t^{2}}=-g \tag{4.19}
\end{equation*}
$$

Integrate once to get the linear velocity

$$
\begin{equation*}
\frac{d s}{d t}=v_{s}[t]=v_{0}-g t \tag{4.20}
\end{equation*}
$$

and twice to get quadratic position

$$
\begin{equation*}
s[t]=s_{0}+v_{0} t-\frac{1}{2} g t^{2} \tag{4.21}
\end{equation*}
$$

as expected. Differentiate to check the integration. Substitute $t=0$ to check the initial conditions, $s[0]=s_{0}$ and $v[0]=v_{0}$.

### 4.3.2 Mass and Spring

An ideal spring of stiffness $\kappa$ anchors a mass $m$ to a fixed point. Stretch the spring an initial distance $s_{0}$ and release it from rest. The corresponding simple harmonic oscillator Lagrangian

$$
\begin{equation*}
\mathcal{L}\left[s, v_{s}\right]=K\left[v_{s}\right]-U[s]=\frac{1}{2} m v_{s}^{2}-\frac{1}{2} \kappa s^{2} . \tag{4.22}
\end{equation*}
$$

Substitute into the Eq. 4.7 Lagrange equation and differentiate to find

$$
\begin{equation*}
\frac{\partial}{\partial s}\left(\frac{1}{2} m v_{s}^{2}-\frac{1}{2} \kappa s^{2}\right)=\frac{d}{d t} \frac{\partial}{\partial v_{s}}\left(\frac{1}{2} m v_{s}^{2}-\frac{1}{2} \kappa s^{2}\right) \tag{4.23}
\end{equation*}
$$

or

$$
\begin{equation*}
(0-\kappa s)=\frac{d}{d t}\left(m v_{s}-0\right) \tag{4.24}
\end{equation*}
$$

or

$$
\begin{equation*}
-\kappa s=m \frac{d v_{s}}{d t} \tag{4.25}
\end{equation*}
$$

The resulting simple harmonic oscillator differential equation

$$
\begin{equation*}
\frac{d^{2} s}{d t^{2}}+\frac{\kappa}{m} s=0 \tag{4.26}
\end{equation*}
$$

has the solution

$$
\begin{equation*}
s[t]=s_{0} \cos [\omega t] \tag{4.27}
\end{equation*}
$$

where the temporal frequency

$$
\begin{equation*}
\omega=\sqrt{\frac{\kappa}{m}} \tag{4.28}
\end{equation*}
$$

increases with spring stiffness and decreases with mass. Differentiate to check the integration. Substitute $t=0$ to check the initial conditions, $s[0]=s_{0}$ and $v[0]=0$.

### 4.3.3 Simple Pendulum

A mass $m$ moves in a circle of radius $\ell$. At time $t$ it is at an angle $\theta$ from downward moving with angular velocity $\omega_{\theta}=d \theta / d t$. Rotate the mass to an initial angle $\theta_{0}$ and release it from test. The corresponding pendulum Lagrangian

$$
\begin{equation*}
\mathcal{L}\left[\theta, \omega_{s}\right]=K\left[\omega_{\theta}\right]-U[\theta]=\frac{1}{2} m \ell^{2} \omega_{\theta}^{2}-m g \ell(1-\cos \theta) \tag{4.29}
\end{equation*}
$$

Substitute into the Eq. 4.7 Lagrange equation and differentiate to find

$$
\begin{equation*}
\frac{\partial}{\partial \theta}\left(\frac{1}{2} m \ell^{2} \omega_{\theta}^{2}-m g \ell(1-\cos \theta)\right)=\frac{d}{d t} \frac{\partial}{\partial \omega_{\theta}}\left(\frac{1}{2} m \ell^{2} \omega_{\theta}^{2}-m g \ell(1-\cos \theta)\right) \tag{4.30}
\end{equation*}
$$

or

$$
\begin{equation*}
(0-0-m g \ell \sin \theta)=\frac{d}{d t}\left(m \ell^{2} \omega_{\theta}-0\right) \tag{4.31}
\end{equation*}
$$

or

$$
\begin{equation*}
-m \nVdash g \ell \sin \theta=m \ell^{2} \frac{d \omega_{\theta}}{d \theta} . \tag{4.32}
\end{equation*}
$$

The resulting pendulum differential equation

$$
\begin{equation*}
\frac{d^{2} \theta}{d t^{2}}+\frac{g}{\ell} \sin \theta=0 \tag{4.33}
\end{equation*}
$$

infamously does not have an elementary solution. However, for small angles $\sin \theta \sim \theta \ll 1$ and

$$
\begin{equation*}
\frac{d^{2} \theta}{d t^{2}}+\frac{g}{\ell} \theta=0 \tag{4.34}
\end{equation*}
$$

has the solution

$$
\begin{equation*}
\theta[t]=\theta_{0} \cos [\omega t] \tag{4.35}
\end{equation*}
$$

where the temporal frequency

$$
\begin{equation*}
\omega=\sqrt{\frac{g}{\ell}} \tag{4.36}
\end{equation*}
$$

increases with gravitational acceleration and decreases with pendulum length. The frequency and the corresponding period

$$
\begin{equation*}
T=\frac{2 \pi}{\omega}=\sqrt{\frac{\ell}{g}} \tag{4.37}
\end{equation*}
$$

famously does not depend on time, as first observed by Galileo, thereby facilitating the pendulum clock. Differentiate to check the integration. Substitute $t=0$ to check the initial conditions, $\theta[0]=\theta_{0}$ and $\omega_{\theta}[0]=0$.

### 4.4 Conservation Laws from Symmetries

Something is symmetric if it is invariant under a transformation. For example, a ball is spherically symmetric because it is invariant under rotations about any axis through its center. A Lagrangian is time symmetric if it is invariant under time translations. Show that a system whose Lagrangian $\mathcal{L}$ does not explicitly depend on time conserves energy.

Consider a more general Lagrangian $\mathcal{L}\left[s[t], v_{s}[t], t\right]$ with both implicit and explicit time dependence (via, for example, a time varying potential energy $U[s, t])$. Use the Eq. 3.17 composition rule to expand the time derivative

$$
\begin{equation*}
\frac{d \mathcal{L}}{d t}=\frac{\partial \mathcal{L}}{\partial s} \frac{d s}{d t}+\frac{\partial \mathcal{L}}{\partial v_{s}} \frac{d v_{s}}{d t}+\frac{\partial \mathcal{L}}{\partial t} \tag{4.38}
\end{equation*}
$$

Substitute the Eq. 4.7 Lagrange equation and use the Eq. 3.13 product rule to condense the right side to

$$
\begin{equation*}
\frac{d \mathcal{L}}{d t}=\left(\frac{d}{d t} \frac{\partial \mathcal{L}}{\partial v_{s}}\right) v_{s}+\frac{\partial \mathcal{L}}{\partial v_{s}} \frac{d v_{s}}{d t}+\frac{\partial \mathcal{L}}{\partial t}=\frac{d}{d t}\left(\frac{\partial \mathcal{L}}{\partial v_{s}} v_{s}\right)+\frac{\partial \mathcal{L}}{\partial t} \tag{4.39}
\end{equation*}
$$

Collect terms and use the Eq. 3.10 sum rule to show

$$
\begin{equation*}
-\frac{\partial \mathcal{L}}{\partial t}=\frac{d}{d t}\left(\frac{\partial \mathcal{L}}{\partial v_{s}} v_{s}-\mathcal{L}\right)=\frac{d \mathcal{H}}{d t} \tag{4.40}
\end{equation*}
$$

where the Hamiltonian

$$
\begin{align*}
\mathcal{H} & =\frac{\partial \mathcal{L}}{\partial v_{s}} v_{s}-\mathcal{L} \\
& =\frac{\partial}{\partial v_{s}}\left(\frac{1}{2} m v_{s}^{2}-U[s, t]\right) v_{s}-\left(\frac{1}{2} m v_{s}^{2}-U[s, t]\right) \\
& =\left(m v_{s}-0\right) v_{s}-\frac{1}{2} m v_{s}^{2}+U \\
& =\frac{1}{2} m v_{s}^{2}+U \\
& =E \tag{4.41}
\end{align*}
$$

is simply the energy of the system. By Eq. 4.40, if the Lagrangian does not explicitly depend on time, $\partial \mathcal{L} / \partial t=0$, and both Hamiltonian and energy are conserved.

Similarly, and more simply, a system whose Lagrangian $\mathcal{L}$ does not explicitly depend on space conserves momentum. For example, the Lagrangian for a single free particle

$$
\begin{equation*}
\mathcal{L}=K=\frac{1}{2} m v_{s}^{2} \tag{4.42}
\end{equation*}
$$

implies a vanishing force

$$
\begin{equation*}
f_{s}=\frac{\partial \mathcal{L}}{\partial v_{s}}=0 \tag{4.43}
\end{equation*}
$$

and a constant momentum $p_{s}=m v_{s}$. More broadly, the intimate relation between symmetry principles and conservation laws was first proved by Emmy Noether 13 in the early 1900s.

## Problems

1. Verify the solutions to the Section 4.3 initial value problems by substituting into the differential equations and checking the initial conditions.
(a) The Eq. 4.21 flat Earth gravity.
(b) The Eq. 4.27 mass and spring.
(c) The Eq. 4.35 simple pendulum.
2. How does the Eq. 4.27 simple harmonic oscillator solution change if the mass is kicked from equilibrium with $s[0]=0$ and $v[0]=v_{0}$ rather than released from a stationary stretch?
3. Consider a massless spring of stiffness $\kappa$ suspending a mass $m$ vertically under flat Earth gravity $g$. Assume the spring's stretch is $\ell-\ell_{u}$, where $\ell$ is the spring length downward from its fixed end to its mass end and $\ell_{u}$ is the spring's unstretched length. Assume the mass's velocity is $v_{\ell}=d \ell / d t$.
(a) Write formulas for the system's kinetic and potential energies and its Lagrangian. Hint: The formulas for spring and gravitational potential energies are slight variations of earlier versions.
(b) Substitute into the Eq. 4.7 Lagrange equation, differentiate, and derive a second-order differential equation for the spring length $\ell[t]$.
(c) Show that when the acceleration vanishes, $d^{2} \ell / d t^{2}=0$, the spring length $\ell$ is the equilibrium length $\ell_{e}=\ell_{u}+m g / \kappa$.
(d) Guess a sinusoidal solution of the form $\ell=\ell_{e}+A \cos [\omega t]$, and derive a formula for the constant temporal frequency $\omega$ by substituting into the Lagrange equation.
(e) Compare the vertical spring under gravity to the horizontal spring (or the vertical spring without gravity). What is the same and what is different?

## Chapter 5

## Vectors in a Nutshell

Vector algebra helps generalize mechanics from one spatial dimension to three.


Figure 5.1: Arrows representing vectors for magnitudes and directions (left) and successive displacements (right) on a baseball diamond.

### 5.1 Vectors \& Coordinates

Physical displacements are models for vectors. They have both direction and magnitude, and they can be multiplied or scaled by numbers. They are often represented as arrows in Euclidean space.

As an example, from a baseball diamond's home plate, run 90 ft east to 1st base, then run 90 ft north to 2 nd base. Alternately, run about 130 ft northeast directly from home plate to 2 nd base, as in Fig. 5.1. Since the sum of the first two displacements is the third displacement, write

$$
\begin{equation*}
\vec{u}+\stackrel{\rightharpoonup}{v}=\vec{w} \tag{5.1}
\end{equation*}
$$

or

$$
\begin{equation*}
\vec{u}+\vec{v}=\vec{w} \tag{5.2}
\end{equation*}
$$

where the harpoons and arrows over the letters denote vectors. Also write

$$
\begin{equation*}
(90 \mathrm{ft}) \hat{e}+(90 \mathrm{ft}) \hat{n}=\vec{w} \tag{5.3}
\end{equation*}
$$

where $\hat{e}$ and $\hat{n}$ are unit vectors in the east and north directions and their coefficients are the corresponding components. The harpoon $\stackrel{\rightharpoonup}{\bullet}$ or arrow $\bullet$ notation suggests displacement with both magnitude and direction, while the hat or arrowhead notation $\hat{\bullet}$ suggests direction only. Read the equation $\vec{v}=3 \hat{x}$ as "v vector equals three x hat". By the Pythagorean theorem, the displacement magnitude

$$
\begin{equation*}
w=|\vec{w}|=\operatorname{mag}[\vec{w}]=\sqrt{(90 \mathrm{ft})^{2}+(90 \mathrm{ft})^{2}} \approx 130 \mathrm{ft} \tag{5.4}
\end{equation*}
$$

is the sum of the square of the components in the orthogonal east and north directions.


Figure 5.2: For right-handed coordinate systems, rotating $x$ into $y$ points a right-handed thumb (left perspective) in the $z$ direction. Rotating $x$ into $y$ also advances a right-handed screw (right projection) in the $z$ direction. Dot $\odot$ and cross $\otimes$ represent an arrow head and tail indicating "out" and "in".

In three dimensions, use an orthogonal coordinate system $\{x, y, z\}$ where the basis unit vectors $\{\hat{x}, \hat{y}, \hat{z}\}$ point in the direction of increasing coordinates. (Other common basis vector notations include $\left\{\hat{x}_{1}, \hat{x}_{2}, \hat{x}_{3}\right\}$, which generalizes easily to higher dimensions; $\left\{e_{1}, e_{2}, e_{3}\right\}$ from the German "einheit" for unit; and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ after Hamilton's quaternions.) By common convention, make the coordinate system right-handed so that rotating the fingers of a right hand $90^{\circ}$ from the $x$-axis to the $y$-axis points the thumb in the direction of the positive $z$-axis, as in Fig. 5.2. Similarly, rotating a right-handed screw from $x$ to $y$ advances it in $z$. Expand a generic vector in terms of the basis vectors like

$$
\begin{align*}
\vec{v} & =\hat{x} v_{x}+\hat{y} v_{y}+\hat{z} v_{z} \\
& =v_{x} \hat{x}+v_{y} \hat{y}+v_{z} \hat{z} \tag{5.5}
\end{align*}
$$

If the basis vectors are implicit, then just list the components like $\vec{v}=\left\{v_{x}, v_{y}, v_{z}\right\}$ or $\vec{v}=\left\langle v_{x}, v_{y}, v_{z}\right\rangle$.

### 5.2 Vector Addition

Add vectors like displacements. Geometrically, represent each vector as an arrow of the same direction and with a proportional magnitude. Draw the first arrow $\vec{u}$ from head to tail. Next draw the second arrow $\vec{v}$ from the head of the first arrow. Finally draw the sum arrow $\vec{w}=\vec{u}+\vec{v}$ from the tail of the first arrow to the head of the second arrow, as in Fig. 5.3 .

The reverse of a vector $\vec{v}$ is the negative of the original vector $-\vec{v}$. The difference of two vectors $\vec{u}-\vec{v}$ is the sum of the first vector and the reverse of the second $\vec{u}+(-\vec{v})$. Twice a vector $\vec{v}$ is simply the sum of the vector with itself $\vec{v}+\vec{v}=2 \vec{v}$, and similarly for other scalar multiples. Thus compute any linear combination.


Figure 5.3: Geometric addition, reverse, subtraction, and scalar multiplication of vectors. Vector addition readily checks vector subtraction.

Algebraically, compute

$$
\begin{equation*}
\vec{w}=a \vec{u}+b \vec{v} \tag{5.6}
\end{equation*}
$$

by combining the components, either separately like

$$
\begin{align*}
& w_{x}=a u_{x}+b v_{x}  \tag{5.7a}\\
& w_{y}=a u_{y}+b v_{y}  \tag{5.7b}\\
& w_{z}=a u_{z}+b v_{z} \tag{5.7c}
\end{align*}
$$

or arranged in columns like

$$
\left(\begin{array}{l}
w_{x}  \tag{5.8}\\
w_{y} \\
w_{z}
\end{array}\right)=a\left(\begin{array}{l}
u_{x} \\
u_{y} \\
u_{z}
\end{array}\right)+b\left(\begin{array}{l}
v_{x} \\
v_{y} \\
v_{z}
\end{array}\right)=\left(\begin{array}{l}
a u_{x}+b v_{x} \\
a u_{y}+b v_{y} \\
a u_{z}+b v_{z}
\end{array}\right)
$$

Vector scaling, or multiplication of vectors by scalars, enables the creation of unit vectors by normalization. For example,

$$
\begin{equation*}
\hat{v}=\frac{\vec{v}}{v}=\frac{1}{v} \vec{v}=\frac{1}{|\vec{v}|} \vec{v}=\frac{1}{\sqrt{v_{x}^{2}+v_{y}^{2}+v_{z}^{2}}} \vec{v} \tag{5.9}
\end{equation*}
$$

### 5.3 Vector Multiplication

A single geometric product decomposes into the classic dot and cross products.

### 5.3.1 Geometric Product

Compared to vector addition, vector multiplication is abstract with multiple kinds, as in Fig. 5.4 However, like vector addition, vector multiplication is very useful.


Figure 5.4: Vector dot, wedge, and cross products descend from the geometric product.

Denote the abstract geometric product [14] by the juxtaposition of two vectors $\vec{u} \vec{v}$. The geometric product is associative, so multiplication can be done in any order,

$$
\begin{equation*}
\vec{u}(\vec{v} \vec{w})=\vec{u} \vec{v} \vec{w}=(\vec{u} \vec{v}) \vec{w} \tag{5.10}
\end{equation*}
$$

but it is not necessarily commutative, so sometimes $\vec{u} \vec{v} \neq \vec{v} \vec{u}$. In particular, the geometric product of the basis vectors are antiysmmetric

$$
\begin{equation*}
\hat{x} \hat{y}=-\hat{y} \hat{x}, \quad \hat{x} \hat{z}=-\hat{z} \hat{x}, \quad \hat{y} \hat{z}=-\hat{z} \hat{y} \tag{5.11}
\end{equation*}
$$

but normalized

$$
\begin{equation*}
\hat{x} \hat{x}=1, \quad \hat{y} \hat{y}=1, \quad \hat{z} \hat{z}=1 . \tag{5.12}
\end{equation*}
$$

This abstract algebra enables the multiplication of any two vectors. For example, the geometric product of the vectors

$$
\begin{align*}
\vec{u} \vec{v} & =(\hat{x}+2 \hat{y})(\hat{x}-\hat{y}) \\
& =\hat{x} \hat{x}-\hat{x} \hat{y}+2 \hat{y} \hat{x}-2 \hat{y} \hat{y} \\
& =-1-3 \hat{x} \hat{y} \tag{5.13}
\end{align*}
$$

is a scalar plus a bivector.
A special element of the algebra is the trivector

$$
\begin{equation*}
\mathcal{I}=\hat{x} \hat{y} \hat{z} \tag{5.14}
\end{equation*}
$$

Repeated applications of the Eq. 5.11 antisymmetries and the Eq. 5.12 normalizations imply

$$
\begin{equation*}
\mathcal{I}^{2}=\hat{x} \hat{y} \hat{z} \hat{x} \hat{y} \hat{z}=-\hat{y} \hat{x} \hat{z} \hat{x} \hat{y} \hat{z}=+\hat{y} \hat{z} \hat{x} \hat{x} \hat{y} \hat{z}=+\hat{y} \hat{z} \hat{y} \hat{z}=-\hat{z} \hat{y} \hat{y} \hat{z}=-\hat{z} \hat{z}=-1 \tag{5.15}
\end{equation*}
$$

so $\mathcal{I}$ squares to negative unity like the imaginary number $i=\sqrt{-1}$. In fact, $\mathcal{I}$ relates vectors and bivectors via the duality transformations

$$
\begin{align*}
& \mathcal{I} \hat{x}=\hat{x} \hat{y} \hat{z} \hat{x}=-\hat{x} \hat{y} \hat{x} \hat{z}=+\hat{x} \hat{x} \hat{y} \hat{z}=+\hat{y} \hat{z},  \tag{5.16a}\\
& \mathcal{I} \hat{y}=\hat{x} \hat{y} \hat{z} \hat{y}=-\hat{x} \hat{y} \hat{y} \hat{z}=-\hat{x} \hat{z},  \tag{5.16b}\\
& \mathcal{I} \hat{z}=\hat{x} \hat{y} \hat{z} \hat{z}=+\hat{x} \hat{y} . \tag{5.16c}
\end{align*}
$$

Further permutations imply

$$
\begin{align*}
& \hat{x} \mathcal{I}=\mathcal{I} \hat{x}=\hat{y} \hat{z}  \tag{5.17a}\\
& \hat{y} \mathcal{I}=\mathcal{I} \hat{y}=\hat{z} \hat{x}  \tag{5.17b}\\
& \hat{z} \mathcal{I}=\mathcal{I} \hat{z}=\hat{x} \hat{y} \tag{5.17c}
\end{align*}
$$

Thus the trivector $\mathcal{I}$ commutes with all vectors, $\vec{v} \mathcal{I}=\mathcal{I} \vec{v}$.

### 5.3.2 Dot, Wedge, \& Cross Products

Expand the geometric product of two generic vectors

$$
\begin{equation*}
\vec{u} \vec{v}=\left(\hat{x} u_{x}+\hat{y} u_{y}+\hat{z} u_{z}\right)\left(\hat{x} v_{x}+\hat{y} v_{y}+\hat{z} v_{z}\right) \tag{5.18}
\end{equation*}
$$

to get

$$
\begin{align*}
\vec{u} \vec{v}= & +\hat{x} \hat{x} u_{x} v_{x}+\hat{x} \hat{y} u_{x} v_{y}+\hat{x} \hat{z} u_{x} v_{z} \\
& +\hat{y} \hat{x} u_{y} v_{x}+\hat{y} \hat{y} u_{y} v_{y}+\hat{y} \hat{z} u_{y} v_{z} \\
& +\hat{z} \hat{x} u_{z} v_{x}+\hat{z} \hat{y} u_{z} v_{y}+\hat{z} \hat{z} u_{z} v_{z} \tag{5.19}
\end{align*}
$$

Invoke the antisymmetries and normalizations of the basis vectors to segregate the symmetric and antisymmetric parts and write

$$
\begin{align*}
\vec{u} \vec{v}= & +u_{x} v_{x}+u_{y} v_{y}+u_{z} v_{z} \\
& +\hat{y} \hat{z}\left(u_{y} v_{z}-u_{z} v_{y}\right)+\hat{z} \hat{x}\left(u_{z} v_{x}-u_{x} v_{z}\right)+\hat{x} \hat{y}\left(u_{x} v_{y}-u_{y} v_{x}\right) \tag{5.20}
\end{align*}
$$

which is a scalar plus a bivector

$$
\begin{equation*}
\vec{u} \vec{v}=\vec{u} \cdot \vec{v}+\vec{u} \wedge \vec{v} . \tag{5.21}
\end{equation*}
$$

The symmetric, scalar part is the interior or inner or dot or scalar product

$$
\begin{equation*}
\vec{u} \cdot \vec{v}=u_{x} v_{x}+u_{y} v_{y}+u_{z} v_{z} \tag{5.22}
\end{equation*}
$$

and the antisymmetric, bivector part is the exterior or outer or wedge or bivector product

$$
\begin{equation*}
\vec{u} \wedge \vec{v}=\hat{y} \hat{z}\left(u_{y} v_{z}-u_{z} v_{y}\right)+\hat{z} \hat{x}\left(u_{z} v_{x}-u_{x} v_{z}\right)+\hat{x} \hat{y}\left(u_{x} v_{y}-u_{y} v_{x}\right) \tag{5.23}
\end{equation*}
$$

Use the Eq. 5.17 dualities to write the wedge product as

$$
\begin{equation*}
\vec{u} \wedge \vec{v}=\mathcal{I} \hat{x}\left(u_{y} v_{z}-u_{z} v_{y}\right)+\mathcal{I} \hat{y}\left(u_{z} v_{x}-u_{x} v_{z}\right)+\mathcal{I} \hat{z}\left(u_{x} v_{y}-u_{y} v_{x}\right) \tag{5.24}
\end{equation*}
$$

or

$$
\begin{equation*}
\vec{u} \wedge \vec{v}=\mathcal{I} \vec{u} \times \vec{v} \tag{5.25}
\end{equation*}
$$

where the traditional cross or vector product

$$
\begin{equation*}
\vec{u} \times \vec{v}=\hat{x}\left(u_{y} v_{z}-u_{z} v_{y}\right)+\hat{y}\left(u_{z} v_{x}-u_{x} v_{z}\right)+\hat{z}\left(u_{x} v_{y}-u_{y} v_{x}\right) \tag{5.26}
\end{equation*}
$$

To remember the cross product formula, begin with $x y z$,

$$
\begin{equation*}
\vec{u} \times \vec{v}=\hat{x} u_{y} v_{z}+\cdots, \tag{5.27}
\end{equation*}
$$

antisymmetrize the components by subtraction,

$$
\begin{equation*}
\vec{u} \times \vec{v}=\hat{x}\left(u_{y} v_{z}-u_{z} v_{y}\right)+\cdots \tag{5.28}
\end{equation*}
$$

and symmetrize the indices by cyclic permutation $x \rightarrow y \rightarrow z \rightarrow x$,

$$
\begin{align*}
\vec{u} \times \vec{v} & =\hat{x}\left(u_{y} v_{z}-u_{z} v_{y}\right) \\
& +\hat{y}\left(u_{z} v_{x}-u_{x} v_{z}\right) \\
& +\hat{z}\left(u_{x} v_{y}-u_{y} v_{x}\right) . \tag{5.29}
\end{align*}
$$

The Eq. 5.22 , Eq. 5.23 , and Eq. 5.26 product formulas imply multiplication rules for the dot product

$$
\begin{array}{ll}
\hat{x} \cdot \hat{x}=1, & \hat{y} \cdot \hat{x}=0, \\
\hat{z} \cdot \hat{x}=0  \tag{5.30}\\
\hat{x} \cdot \hat{y}=0, & \hat{y} \cdot \hat{y}=1, \\
\hat{z} \cdot \hat{y}=0 \\
\hat{x} \cdot \hat{z}=0, & \hat{y} \cdot \hat{z}=0, \\
\hat{z} \cdot \hat{z}=1
\end{array}
$$

and the wedge product

$$
\begin{array}{lll}
\hat{x} \wedge \hat{x}=0, & \hat{y} \wedge \hat{x}=\hat{y} \hat{x}, & \hat{z} \wedge \hat{x}=\hat{z} \hat{x} \\
\hat{x} \wedge \hat{y}=\hat{x} \hat{y}, & \hat{y} \wedge \hat{y}=0, & \hat{z} \wedge \hat{y}=\hat{z} \hat{y}  \tag{5.31}\\
\hat{x} \wedge \hat{z}=\hat{x} \hat{z}, & \hat{y} \wedge \hat{z}=\hat{y} \hat{z}, & \hat{z} \wedge \hat{z}=0
\end{array}
$$

and the cross product

$$
\begin{array}{lll}
\hat{x} \times \hat{x}=\overrightarrow{0}, & \hat{y} \times \hat{x}=-\hat{z}, & \hat{z} \times \hat{x}=+\hat{y} \\
\hat{x} \times \hat{y}=+\hat{z} & \hat{y} \times \hat{y}=\overrightarrow{0}, & \hat{z} \times \hat{y}=-\hat{x}  \tag{5.32}\\
\hat{x} \times \hat{z}=-\hat{y}, & \hat{y} \times \hat{z}=+\hat{x}, & \hat{z} \times \hat{z}=\overrightarrow{0}
\end{array}
$$

Thus the dot product vanishes for perpendicular vectors and the wedge and cross products vanish for parallel vectors. Furthermore, by Eq. 5.21, if two vectors are parallel, their wedge product vanishes and their geometric product reduces to their dot product; if they are perpendicular, their dot product vanishes and their geometric product reduces to their wedge product. One consequence is that a vector's geometric square is also its length squared,

$$
\begin{equation*}
\vec{v}^{2}=\vec{v} \cdot \vec{v}+\vec{v} \wedge \vec{v}=\vec{v} \cdot \vec{v}=|\vec{v}|^{2}=v^{2} \tag{5.33}
\end{equation*}
$$

|  | $\hat{x}$ | $\hat{y}$ | $\hat{z}$ | $\wedge$ | $\hat{x}$ | $\hat{y}$ | $\hat{z}$ | $\times$ | $\hat{x}$ | $\hat{y}$ | $\hat{z}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\hat{x}$ | 1 | 0 | 0 | $\hat{x}$ | 0 | $\hat{x} \hat{y}$ |  | $\hat{x}$ | 0 | $+\hat{z}$ | $-\hat{y}$ |
| $\hat{y}$ | 0 | 1 | 0 | $\hat{y}$ | $\hat{y} \hat{x}$ | 0 | $\hat{y} \hat{z}$ | $\hat{y}$ |  | 0 | $+\hat{x}$ |
| $\hat{z}$ | 0 | 0 | 1 | $\hat{z}$ | $\hat{z} \hat{x}$ | $\hat{z} \hat{y}$ | 0 | $\hat{z}$ | $+\hat{y}$ | $-\hat{x}$ |  |

Figure 5.5: Multiplication tables, left times top, for the dot, wedge, and cross products.

The Fig. 5.5 multiplication tables enable dotting, wedging, or crossing any two vectors. For example, if $\vec{u}=3 \hat{x}-2 \hat{y}$ and $\vec{v}=2 \hat{x}+\hat{z}$, then

$$
\begin{align*}
\vec{u} \cdot \vec{v} & =(3 \hat{x}-2 \hat{y}) \cdot(2 \hat{x}+\hat{z}) \\
& =6 \hat{x} \cdot \hat{x}+3 \hat{x} \cdot \hat{z}-4 \hat{y} \cdot \hat{x}-2 \hat{y} \cdot \hat{z} \\
& =6+0-0-0 \\
& =6 \tag{5.34}
\end{align*}
$$

and

$$
\begin{align*}
\vec{u} \times \vec{v} & =(3 \hat{x}-2 \hat{y}) \times(2 \hat{x}+\hat{z}) \\
& =6 \hat{x} \times \hat{x}+3 \hat{x} \times \hat{z}-4 \hat{y} \times \hat{x}-2 \hat{y} \times \hat{z} \\
& =\overrightarrow{0}-3 \hat{y}+4 \hat{z}-2 \hat{x} \\
& =-2 \hat{x}-3 \hat{y}+4 \hat{z} \tag{5.35}
\end{align*}
$$

### 5.3.3 Geometric Intepretation

Given any two vectors $\vec{u}$ and $\vec{v}$, without loss of generality rotate and scale the axes so that

$$
\begin{equation*}
\vec{u}=\hat{x} \tag{5.36}
\end{equation*}
$$

and

$$
\begin{equation*}
\vec{v}=\hat{x} v_{x}+\hat{y} v_{y} \tag{5.37}
\end{equation*}
$$

From the geometry of Fig. 5.6, the dot product

$$
\begin{equation*}
\vec{u} \cdot \vec{v}=\hat{x} \cdot\left(\hat{x} v_{x}+\hat{y} v_{y}\right)=v_{x}=v \cos \theta=u v \cos \theta \tag{5.38}
\end{equation*}
$$

and the cross product

$$
\begin{equation*}
\vec{u} \times \vec{v}=\hat{x} \times\left(\hat{x} v_{x}+\hat{y} v_{y}\right)=\hat{z} v_{y}=\hat{z} v \sin \theta=\hat{n} u v \sin \theta \tag{5.39}
\end{equation*}
$$

where $\hat{n}$ is a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$ pointing in the direction a right-handed screw would advance when rotated from $\vec{u}$ to $\vec{v}$. The dot product


Figure 5.6: The dot product generates the projected length (left) and the wedge and cross products generate the shaded area (right).
is proportional to the cosine of the angle between the vectors and, complementarily, the cross product is proportional to the sine of the angle between the vectors. The dot product generates a scalar and, complementarily, the cross product generates a vector.

If $\hat{n}$ is a unit vector and $\vec{v}$ is a general vector, then decompose any vector

$$
\begin{equation*}
\vec{v}=\hat{n}^{2} \vec{v}=(\hat{n} \hat{n}) \vec{v}=\hat{n} \hat{n} \vec{v}=\hat{n}(\hat{n} \vec{v})=\hat{n}(\hat{n} \cdot \vec{v}+\hat{n} \wedge \vec{v})=\vec{v}_{\|}+\vec{v}_{\perp} \tag{5.40}
\end{equation*}
$$

where

$$
\begin{equation*}
\vec{v}_{\|}=\hat{n}(\hat{n} \cdot \vec{v}) \tag{5.41}
\end{equation*}
$$

is the projection and

$$
\begin{equation*}
\vec{v}_{\perp}=\hat{n}(\hat{n} \wedge \vec{v}) \tag{5.42}
\end{equation*}
$$

is the rejection of $\vec{v}$, parallel and perpendicular to $\hat{n}$.


Figure 5.7: The projection and rejection of the vector $\vec{v}$ on the vector $\hat{n}$.

For an example, consider the vectors

$$
\begin{align*}
\hat{n} & =\hat{x}  \tag{5.43}\\
\vec{v} & =\hat{x}+2 \hat{y} \tag{5.44}
\end{align*}
$$

as in Fig. 5.7. The dot and wedge products are

$$
\begin{align*}
\hat{n} \cdot \vec{v} & =1  \tag{5.45}\\
\hat{n} \wedge \vec{v} & =2 \hat{x} \hat{y} \tag{5.46}
\end{align*}
$$

and so the projection and rejection are

$$
\begin{align*}
\vec{v}_{\|} & =\hat{x}  \tag{5.47}\\
\vec{v}_{\perp} & =2 \hat{y} \tag{5.48}
\end{align*}
$$

## Problems

1. Given the vectors $\vec{u}=\hat{x}-2 \hat{y}+3 \hat{z}, \vec{v}=\hat{x}-\hat{y}+\hat{z}, \vec{w}=2 \hat{x}+\hat{y}$, evaluate the following sums.
(a) $\vec{u}+\vec{v}$.
(b) $\vec{u}-\vec{w}$.
(c) $\vec{u}+2 \vec{v}-3 \vec{w}$.
2. Repeatedly apply the Eq. 5.11 antisymmetries and the Eq. 5.12 normalizations to prove the Eq. 5.17 geometric product dualities.
3. Using the vectors of Problem 51, evaluate the following dot products.
(a) $\hat{x} \cdot \hat{y}$.
(b) $\hat{x} \cdot \hat{x}$.
(c) $\vec{u} \cdot \vec{v}$.
(d) $\vec{u} \cdot \vec{w}$.
(e) $\vec{u} \cdot(\vec{v}+\vec{w})$.
(f) $\vec{u} \cdot \vec{u}$.
(g) What is the angle between $\vec{u}$ and $\vec{v}$ ?
(h) Can you see the top of Mount Everest from a boat floating in the Bay of Bengal? Hint: Project the vector from Earth's center to the top of Everest on the line joining Earth's center to the boat.
(i) Use the dot product to find the angles between the body diagonals of a cube.
4. Using the vectors of Problem 51, evaluate the following cross products.
(a) $\hat{x} \times \hat{y}$.
(b) $\hat{x} \times \hat{x}$.
(c) $\vec{u} \times \vec{v}$.
(d) $\vec{u} \times \vec{w}$.
(e) $\vec{u} \times(\vec{v}+\vec{w})$.
(f) Construct a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$.
(g) Prove that the magnitudes of $\vec{r} \times \vec{s}$ is the area of the parallelogram whose sides are $\vec{r}$ and $\vec{s}$.
(h) Prove that the "box" product $\vec{q} \cdot \vec{r} \times \vec{s}$ is the volume of the parallelepiped whose sides are $\vec{q}, \vec{r}$, and $\vec{s}$.
5. Use the dot and wedge products to find the projection and rejection of the vector $\vec{v}=3 \hat{x}-2 \hat{y}$ parallel and perpendicular to the direction $\hat{n}=$ $(\hat{x}+\hat{y}) / \sqrt{2}$. Draw a picture like Fig. 5.7 illustrating the construction.

## Chapter 6

## Newton's Laws

Newton's laws organize mechanics by vector forces rather than scalar energies.


Figure 6.1: Isaac Newton's personal first edition copy of his Mathematical Principles of Natural Philosophy with his own handwritten corrections for the second edition [16].

### 6.1 Translation

Newton postulated three laws of motion: constant velocity inertial motion is natural; forces vary directly with accelerations and inversely with masses; forces come in action-reaction pairs. These laws follow from the principle of stationary
action and its corollaries of momentum and energy conservation.
For translational motion, where the Lagrangian $\mathcal{L}\left[s, v_{s}\right]=K\left[v_{s}\right]-U[s]$ depends on positions and linear velocities, the Eq. 4.8 force is the rate of change of momentum with time,

$$
\begin{equation*}
f_{s}=\frac{d p_{s}}{d t} \tag{6.1}
\end{equation*}
$$

which is Newton's second law. In modern vector notation,

$$
\begin{equation*}
\vec{f}=\frac{d \vec{p}}{d t} \tag{6.2}
\end{equation*}
$$

An everyday push or pull is the force that changes the momentum of a mass. This form of the second law, which is found in the Principia Mathematica of Fig. 6.1. applies even to systems whose mass is changing, like a rocket exhausting burned fuel - or a horse-drawn cart losing hay as it rolls along a bumpy road.

If the mass $m$ is constant, then

$$
\begin{equation*}
\vec{f}=\frac{d \vec{p}}{d t}=\frac{d}{d t}(m \vec{v})=m \frac{d \vec{v}}{d t}=m \vec{a} \tag{6.3}
\end{equation*}
$$

or

$$
\begin{equation*}
\vec{a}=\frac{\vec{f}}{m} \tag{6.4}
\end{equation*}
$$

where the force is implicitly the vector sum of all the forces acting on the mass,

$$
\begin{equation*}
\vec{f}=\vec{f}_{\text {net }}=\vec{f}_{\text {tot }}=\vec{f}_{1}+\vec{f}_{2}+\cdots+\vec{f}_{N}=\sum_{n=1}^{N} \vec{f}_{n} \tag{6.5}
\end{equation*}
$$

A mass's acceleration, or the rate of change of velocity with time, is the net or total force divided by the mass. This suggests the causality: force causes acceleration, force changes motion (not force causes motion). Imagine the constant downward force of gravity changing the motion of tossed ball from up to down (or changing its velocity from positive to negative).

Solve many practical problems by decomposing the force law into components along a particular direction. For example, the sum of the forces in the $s$ direction is the mass time the acceleration in the $s$ direction,

$$
\begin{equation*}
\sum f_{s}=m a_{s} \tag{6.6}
\end{equation*}
$$

Since the momentum of an isolated system is conserved, the momentum $\vec{p}=m \vec{v}$ is constant. Further, if the system's mass $m$ does not change, then its velocity is constant,

$$
\begin{equation*}
\vec{v}=\vec{v}_{0} \tag{6.7}
\end{equation*}
$$

which is Newton's first law. As Galileo had already concluded, in the absence of external interactions, mass moves at constant velocity. Constant velocity means unchanging speed in a straight line. The first law is a special case of the second law where force $\vec{f}=\overrightarrow{0}$ and so acceleration $\vec{a}=\vec{f} / m=\overrightarrow{0}$. It can be
counterintuitive or difficult to observe in everyday life because of the practical problem of reducing or eliminating external interactions. Figure skating is one familiar activity where the law of "inertia" might be somewhat intuitive.

In order to conserve total momentum, when two masses interact, they must exchange momenta that are equal in magnitude but opposite in direction. During a small time $d t$, the change in the momentum of mass $m_{i}$ due to its interaction with mass $m_{j}$ is $d \vec{p}_{i j}$. The total momentum of the isolated system of both masses

$$
\begin{equation*}
\overrightarrow{0}=d \vec{p}=d \vec{p}_{12}+d \vec{p}_{21} \tag{6.8}
\end{equation*}
$$

does not change, so

$$
\begin{equation*}
d \vec{p}_{12}=-d \vec{p}_{21} \tag{6.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\vec{f}_{12}=\frac{d \vec{p}_{12}}{d t}=-\frac{d \vec{p}_{21}}{d t}=-\vec{f}_{21} \tag{6.10}
\end{equation*}
$$

which is Newton's third law. When two masses interact, the force on one is equal in magnitude and opposite in direction to the force on the other.

However, "equal but opposite" momentum transfers can have very different results if the masses are very different. For example, when you jump to dunk a basketball, Earth barely recoils. Note that the Eq. 6.10 "action-reaction" forces always apply to different masses. In the basketball jump, you push on Earth and Earth pushes on you, but your smaller mass results in a far larger acceleration, as acceleration is inverse to mass by Newton's second law.

### 6.1.1 Contact Acceleration

Newton's laws build good intuition about forces and accelerations. Suppose an external force $\vec{f}$ accelerates a heavier block of mass $m_{1}$ in contact with a lighter block of mass $m_{2}<m_{1}$, as on the left of Fig. 6.2. By Newton's second law, the acceleration $\vec{a}$ of both blocks together or the lighter block alone

$$
\begin{equation*}
\vec{a}=\frac{\overrightarrow{f_{21}}}{m_{2}}=\frac{\vec{f}}{m_{1}+m_{2}} \tag{6.11}
\end{equation*}
$$

Hence the contact force

$$
\begin{equation*}
\overrightarrow{f_{c}}=\vec{f}_{21}=\frac{m_{2}}{m_{1}+m_{2}} \vec{f}=\frac{1}{1+m_{2} / m_{1}} \vec{f} \tag{6.12}
\end{equation*}
$$

As a check, using Newton's third law, the acceleration of the heavier block alone

$$
\begin{equation*}
\vec{a}=\frac{\vec{f}+\vec{f}_{12}}{m_{1}}=\frac{\vec{f}-\vec{f}_{21}}{m_{1}}=\frac{1}{m_{1}}\left(1-\frac{m_{2}}{m_{1}+m_{2}}\right) \vec{f}=\frac{\vec{f}}{m_{1}+m_{2}} \tag{6.13}
\end{equation*}
$$

as expected.
If the external force accelerates the lighter block in contact with the heavier block, as on the right of Fig. 6.2, the new contact force

$$
\begin{equation*}
\vec{f}_{c}^{\prime}=\vec{f}_{12}^{\prime}=\frac{m_{1}}{m_{2}+m_{1}} \vec{f}=\frac{1}{1+m_{1} / m_{2}} \vec{f} \tag{6.14}
\end{equation*}
$$



Figure 6.2: An external force $\vec{f}$ accelerates two blocks in contact. A larger contact force $f_{12}^{\prime}>f_{21}$ accelerates the larger mass.
and the ratio of the contact forces magnitudes

$$
\begin{equation*}
\frac{f_{21}}{f_{12}^{\prime}}=\frac{f_{c}}{f_{c}^{\prime}}=\frac{m_{2}}{m_{1}}<1 \tag{6.15}
\end{equation*}
$$

so the larger contact force accelerates the larger mass.

### 6.1.2 Train

A locomotive accelerates a train of four cars of mass $m$ with a force $\vec{f}=\overrightarrow{f_{4}}$, as in Fig. 6.3, where the tension in the coupling between the cars is $\vec{f}_{n}$.


Figure 6.3: A locomotive accelerates a train of four cars. The tension coupling the cars decreases toward the rear of the train. Each dashed subgroup separately obeys Newton's second law.

By Newton's second law, the acceleration of the train as a whole and of any sequence of cars is

$$
\begin{equation*}
\vec{a}=\frac{\vec{f}}{4 m}=\frac{\overrightarrow{f_{3}}}{3 m}=\frac{\overrightarrow{f_{2}}}{2 m}=\frac{\overrightarrow{f_{1}}}{m} \tag{6.16}
\end{equation*}
$$

Hence the tensions are

$$
\begin{align*}
\overrightarrow{f_{1}} & =m \vec{a}  \tag{6.17a}\\
\overrightarrow{f_{2}} & =2 m \vec{a}  \tag{6.17b}\\
\overrightarrow{f_{3}} & =3 m \vec{a}  \tag{6.17c}\\
\vec{f}=\overrightarrow{f_{4}} & =4 m \vec{a} \tag{6.17~d}
\end{align*}
$$

While accelerating, the greatest tension is between the locomotive and the first car, and the least tension is between the penultimate car and the caboose.

### 6.1.3 Atwood Machine

An Atwood machine demonstrates constant acceleration. It consists of two masses $M$ and $m \leq M$ hanging from either side of an ideal pulley by a single massless string, as in Fig. 6.4. If the string's tension force magnitude is $f_{t}>0$ and the gravity force magnitude or weight of the masses is $F_{g}=M g$ and $f_{g}=m g$, then Newton's second law applied to both masses requires

$$
\begin{align*}
\vec{f}_{t}+\vec{f}_{g} & =m \vec{a}  \tag{6.18a}\\
\vec{F}_{g}+\vec{f}_{t} & =M \vec{A} \tag{6.18b}
\end{align*}
$$

If the string is inextensible, the accelerations of both masses are the same, $\vec{a}=\vec{A}$. Projecting onto the $s$ direction gives

$$
\begin{align*}
f_{t}-m g & =m a_{s}  \tag{6.19a}\\
M g-f_{t} & =M a_{s} \tag{6.19b}
\end{align*}
$$

Adding implies

$$
\begin{equation*}
(M-m) g=(M+m) a_{s} \tag{6.20}
\end{equation*}
$$

so the constant acceleration

$$
\begin{equation*}
a_{s}=\frac{M-m}{M+m} g \geq 0 \tag{6.21}
\end{equation*}
$$



Figure 6.4: An Atwood machine exhibits constant acceleration (left) and may be conceptually "unwrapped" to a straight line (right).

Substituting the acceleration into Eq. 6.19a gives the tension magnitude

$$
\begin{equation*}
f_{t}=m\left(a_{s}+g\right)=\frac{2 M m}{M+m} g=2 m_{r} g \tag{6.22}
\end{equation*}
$$

where the reduced mass

$$
\begin{equation*}
m_{r}=\frac{M m}{M+m}=\frac{1}{M / m+1} M \leq M \tag{6.23}
\end{equation*}
$$

and

$$
\begin{equation*}
m_{r}=\frac{M m}{M+m}=\frac{1}{1+m / M} m \leq m \tag{6.24}
\end{equation*}
$$

is less than either mass and is the reciprocal of the sum of the reciprocals of the masses,

$$
\begin{equation*}
\frac{1}{m_{r}}=\frac{1}{M}+\frac{1}{m} \tag{6.25}
\end{equation*}
$$

Thus the tension magnitude is less than twice either weight, and by addition and halving, the tension magnitude is less than the total weight,

$$
\begin{equation*}
f_{t} \leq(M+m) g \tag{6.26}
\end{equation*}
$$

When the masses are equal, $M=m$, the acceleration vanishes, $a_{s}=0$, the masses move at constant (possibly zero) velocity, and the tension magnitude is the weight of either mass, $f_{t}=m g=M g$.

Check the acceleration using the Lagrangian approach. The kinetic energy

$$
\begin{equation*}
K=\frac{1}{2} M v_{s}^{2}+\frac{1}{2} m v_{s}^{2} \tag{6.27}
\end{equation*}
$$

and the potential energy

$$
\begin{equation*}
U=-M g s+m g s \tag{6.28}
\end{equation*}
$$

so the Lagrangian

$$
\begin{equation*}
\mathcal{L}=K-U=\frac{1}{2}(M+m) v_{s}^{2}+(M-m) g s \tag{6.29}
\end{equation*}
$$

The Lagrange equation

$$
\begin{equation*}
(M-m) g=\frac{\partial \mathcal{L}}{\partial s}=\frac{d}{d t} \frac{\partial \mathcal{L}}{\partial v_{s}}=(M+m) a_{s} \tag{6.30}
\end{equation*}
$$

so the acceleration

$$
\begin{equation*}
a_{s}=\frac{M-m}{M+m} g \tag{6.31}
\end{equation*}
$$

as before.

### 6.1.4 Incline

A box rests on a plank. How far can the plank tilt before the box slides, as in Fig. 6.5?

Assume a maximal frictional force magnitude proportional to the normal or perpendicular force between the box and the plank,

$$
\begin{equation*}
f_{f}=\mu f_{n} \tag{6.32}
\end{equation*}
$$

where the proportionality constant $\mu$ is the static friction coefficient. By Newton's second law, no slipping means zero acceleration and

$$
\begin{equation*}
\overrightarrow{f_{f}}+\vec{f}_{n}+\overrightarrow{f_{g}}=m \vec{a}=\overrightarrow{0} \tag{6.33}
\end{equation*}
$$



Figure 6.5: The steepest incline $\theta$ for which the box does not slide down an incline with friction coefficient $\mu$.

Choose a coordinate system parallel and perpendicular to the incline and decompose this vector equation into two scalar equations

$$
\begin{align*}
\mu f_{n}+0-m g \sin \theta & =m a_{x}=0  \tag{6.34a}\\
0+f_{n}-m g \cos \theta & =m a_{y}=0 \tag{6.34b}
\end{align*}
$$

or

$$
\begin{align*}
\mu f_{n} & =m g \sin \theta  \tag{6.35a}\\
f_{n} & =m g \cos \theta \tag{6.35b}
\end{align*}
$$

and by division

$$
\begin{equation*}
\mu=\tan \theta \tag{6.36}
\end{equation*}
$$

The friction coefficient $\mu$ sets the maximum angle $\theta$, and measuring this angle determines the coefficient.

### 6.1.5 Movable Incline

A box of mass $m$ slides down a movable wedge of mass $M$ and inclination angle $\theta$, as in Fig. 6.6. Neglecting friction, what are the accelerations of the box and wedge?

The acceleration of the box relative to the ground $\vec{a}$ is the acceleration of the box relative to the wedge $\vec{a}_{r}$ plus the acceleration of the wedge relative to the ground $\vec{A}$,

$$
\begin{equation*}
\vec{a}=\vec{a}_{r}+\vec{A} \tag{6.37}
\end{equation*}
$$

or in components parallel and perpendicular to the ground

$$
\begin{align*}
& a_{X}=+a_{r} \cos \theta-A  \tag{6.38a}\\
& a_{Y}=-a_{r} \sin \theta \tag{6.38b}
\end{align*}
$$

Newton's second law applied to the box implies

$$
\begin{equation*}
\vec{f}_{g}+\vec{f}_{n}=m \vec{a} \tag{6.39}
\end{equation*}
$$



Figure 6.6: The wedge recoils as the box slides down it. The relative acceleration $\vec{a}_{r}$ between the box and the wedge is parallel to the wedge's slope.
and applied to the wedge implies

$$
\begin{equation*}
\vec{F}_{g}-\vec{f}_{n}+\vec{F}_{n}=M \vec{A} \tag{6.40}
\end{equation*}
$$

or in components parallel and perpendicular to the ground

$$
\begin{align*}
& f_{n} \sin \theta=m a_{X}=+m a_{r} \cos \theta-m A,  \tag{6.41a}\\
& -m g+f_{n} \cos \theta=m a_{Y}=-m a_{r} \sin \theta, \tag{6.41b}
\end{align*}
$$

and

$$
\begin{align*}
& -f_{n} \sin \theta=M A_{X}=-M A,  \tag{6.42a}\\
& -M g+F_{n}-f_{n} \cos \theta=M A_{Y}=0 . \tag{6.42b}
\end{align*}
$$

Solve these four equations in four unknowns to find the accelerations

$$
\begin{align*}
A & =\frac{m \sin \theta \cos \theta}{M+m \sin ^{2} \theta} g  \tag{6.43}\\
a_{r} & =\frac{(M+m) \sin \theta}{M+m \sin ^{2} \theta} g \tag{6.44}
\end{align*}
$$

and the normal forces

$$
\begin{align*}
f_{n} & =\frac{m \cos \theta}{M+m \sin ^{2} \theta} M g  \tag{6.45}\\
F_{n} & =\frac{M+m}{M+m \sin ^{2} \theta} M g \tag{6.46}
\end{align*}
$$

As a check, if the wedge is very massive so that $M \rightarrow \infty$, then

$$
\begin{align*}
A & \rightarrow 0  \tag{6.47}\\
a_{r} & \rightarrow g \sin \theta  \tag{6.48}\\
f_{n} & \rightarrow m g \cos \theta  \tag{6.49}\\
F_{n} & \rightarrow \infty \tag{6.50}
\end{align*}
$$

as expected.
Further check the acceleration using the Lagrangian approach. The velocity of the box relative to the ground $\vec{v}$ is the velocity of the box relative to the wedge $\vec{v}_{r}=d \vec{r} / d t$ plus the velocity of the wedge relative to the ground $\vec{V}=d \vec{R} / d t$,

$$
\begin{equation*}
\vec{v}=\vec{v}_{r}+\vec{V} \tag{6.51}
\end{equation*}
$$

Hence, the kinetic energy

$$
\begin{equation*}
K=\frac{1}{2} M V^{2}+\frac{1}{2} m v^{2} \tag{6.52}
\end{equation*}
$$

where

$$
\begin{equation*}
v^{2}=\vec{v} \cdot \vec{v}=v_{r}^{2}+2 \vec{v}_{r} \cdot \vec{V}+V^{2}=v_{r}^{2}+2 v_{r} V \cos [\pi-\theta]+V^{2}=v_{r}^{2}-2 v_{r} V \cos [\theta]+V^{2} . \tag{6.53}
\end{equation*}
$$

The potential energy

$$
\begin{equation*}
U=-M g r \sin \theta \tag{6.54}
\end{equation*}
$$

The Lagrangian

$$
\begin{align*}
\mathcal{L} & =K-U \\
& =\frac{1}{2} M V^{2}+\frac{1}{2} m\left(v_{r}^{2}-2 v_{r} V \cos [\theta]+V^{2}\right)+M g r \sin \theta \\
& =\frac{1}{2}(M+m) V^{2}+\frac{1}{2} m v_{r}^{2}-m v_{r} V \cos [\theta]+M g r \sin \theta \tag{6.55}
\end{align*}
$$

There are two Lagrange equations, one for the wedge coordinate $R$ and the other for the box coordinate $r$,

$$
\begin{align*}
0 & =\frac{\partial \mathcal{L}}{\partial R}=\frac{d}{d t} \frac{\partial \mathcal{L}}{\partial V}=(M+m) A-m a_{r} \cos \theta  \tag{6.56a}\\
m g \sin \theta & =\frac{\partial \mathcal{L}}{\partial r} \tag{6.56b}
\end{align*}=\frac{d}{d t} \frac{\partial \mathcal{L}}{\partial v_{r}}=m a_{r}-m A \cos \theta .
$$

Multiply the second equation by $\cos \theta$ and add to the first to get

$$
\begin{equation*}
m g \sin \theta \cos \theta=(M+m) A-m A \cos ^{2} \theta \tag{6.57}
\end{equation*}
$$

and so

$$
\begin{equation*}
A=\frac{m g \sin \theta \cos \theta}{M+m-m \cos ^{2} \theta}=\frac{m \sin \theta \cos \theta}{M+m \sin ^{2} \theta} g \tag{6.58}
\end{equation*}
$$

as before.

### 6.2 Rotation

An extended mass of pieces $d m$ at distances $r$ from a fixed axis rotates with angular velocity $\omega_{\theta}=d \theta / d t$, as on the left of Fig. 6.7. At a distance $r$, rotation through an angle $\theta$ sweeps out an arc length $\ell=r \theta$, which implies a tangential
speed $v_{\theta}=r \omega_{\theta}$ (and a tangential acceleration $a_{\theta}=r \alpha_{\theta}$ ). The total kinetic energy is the sum

$$
\begin{equation*}
K=\int d K=\int \frac{1}{2} d m v_{\theta}^{2}=\int \frac{1}{2} d m r^{2} \omega_{\theta}^{2}=\frac{1}{2}\left(\int d m r^{2}\right) \omega_{\theta}^{2}=\frac{1}{2} I \omega_{\theta}^{2} \tag{6.59}
\end{equation*}
$$

where the rotational inertia

$$
\begin{equation*}
I=\int d m r^{2}=\int r^{2} d m \tag{6.60}
\end{equation*}
$$

depends on the mass and its distribution about the rotation axis.


Figure 6.7: Rotation about a fixed axis (left) and motion past a fixed axis (right).

For rotational motion, where the Lagrangian $\mathcal{L}\left[\theta, \omega_{\theta}\right]=K\left[\omega_{\theta}\right]-U[\theta]$ depends on angles and angular velocities rather than positions and linear velocities, the generalized force is the torque

$$
\begin{equation*}
\tau_{\theta}=f_{\theta}=\frac{\partial \mathcal{L}}{\partial \theta}=-\frac{\partial U}{\partial \theta} \tag{6.61}
\end{equation*}
$$

and the generalized momentum is the angular momentum

$$
\begin{equation*}
L_{\theta}=p_{\theta}=\frac{\partial \mathcal{L}}{\partial \omega_{\theta}}=I \omega_{\theta} \tag{6.62}
\end{equation*}
$$

where $\tau$ and $L$ are conventional symbols for torque and angular momentum. The resulting Lagrange equation

$$
\begin{equation*}
f_{\theta}=\frac{d p_{\theta}}{d t} \tag{6.63}
\end{equation*}
$$

or

$$
\begin{equation*}
\tau_{\theta}=\frac{d L_{\theta}}{d t} \tag{6.64}
\end{equation*}
$$

is the rotational form of Newton's second law. In vector notation,

$$
\begin{equation*}
\vec{\tau}=\frac{d \vec{L}}{d t} \tag{6.65}
\end{equation*}
$$

If the mass and its distribution are fixed, then

$$
\begin{equation*}
\vec{\tau}=I \frac{d \vec{\omega}}{d t}=I \vec{\alpha}, \tag{6.66}
\end{equation*}
$$

where $\vec{\alpha}=d \vec{\omega} / d t$ is the mass's angular acceleration.
A point mass $m$ moving with velocity $\vec{v}$ at a displacement $\vec{r}$ from a fixed axis, as on the right of Fig. 6.7, has angular velocity

$$
\begin{equation*}
\vec{\omega}=\omega \hat{\omega}=\left|\frac{v_{\theta}}{r}\right| \hat{\omega}=\frac{v \sin \varphi}{r} \hat{\omega}=\frac{\hat{\omega} r v \sin \varphi}{r^{2}}=\frac{\vec{r} \times \vec{v}}{r^{2}} \tag{6.67}
\end{equation*}
$$

and hence angular momentum

$$
\begin{equation*}
\vec{L}=I \vec{\omega}=\int d m r^{2} \vec{\omega}=\left(\int d m\right)\left(r^{2} \vec{\omega}\right)=m \vec{r} \times \vec{v}=\vec{r} \times \vec{p} \tag{6.68}
\end{equation*}
$$

and torque

$$
\begin{equation*}
\vec{\tau}=\frac{d \vec{L}}{d t}=\frac{d}{d t}(\vec{r} \times \vec{p})=\frac{d \vec{r}}{d t} \times \vec{p}+\vec{r} \times \frac{d \vec{p}}{d t}=\vec{v} \times m \vec{v}+\vec{r} \times \vec{f}=\vec{r} \times \vec{f} \tag{6.69}
\end{equation*}
$$

In general, torque is the moment of force $\vec{\tau}=\vec{r} \times \vec{f}$, and angular momentum is the moment of momentum $\vec{L}=\vec{r} \times \vec{p}$.

Table 6.1: Translation and rotation formulas in popular notation.

| Translation | Rotation |
| :--- | :--- |
| $d \vec{r}$ | $d \vec{\theta}$ |
| $\vec{v}=\frac{d \vec{r}}{d t}$ | $\vec{\omega}=\frac{d \vec{\theta}}{d t}$ |
| $\vec{a}=\frac{d \vec{v}}{d t}=\frac{d^{2} \vec{r}}{d t^{2}}$ | $\vec{\alpha}=\frac{d \vec{\omega}}{d t}=\frac{d^{2} \vec{\theta}}{d t^{2}}$ |
| $m$ | $I=\int d m r^{2}$ |
| $K=\frac{1}{2} m v^{2}$ | $K=\frac{1}{2} I \omega^{2}$ |
| $\vec{p}=m \vec{v}$ | $\vec{L}=I \overrightarrow{=}=\vec{\omega} \times \vec{p}$ |
| $\vec{f}$ | $\vec{\tau}=\vec{r} \times \vec{f}$ |
| $\sum \vec{f}=\frac{d \vec{p}}{d t}$ | $\sum \vec{\tau}=\frac{d \vec{L}}{d t}$ |
| $\sum \vec{f}=m \vec{a}$ | $\sum \vec{\tau}=I \vec{\alpha}$ |
| $\sum$ |  |

Table 6.1 compares the key translational and rotational formulas (not all of which are valid in every situation). One disanalogy between rotation and translations is that while vectors can represent translations, they cannot represent finite rotations, which do not commute. (For example, rotating a book $90^{\circ}$
about its cover and then $90^{\circ}$ about its spine orients it differently than rotating it $90^{\circ}$ about its spine and then $90^{\circ}$ about its cover.) Fortunately, infinitesimal rotations do commute, allowing vectors to represent rotational or angular velocity and acceleration.

### 6.2.1 Massive Pulley

As an example, an ideal cable draped over a pulley of mass $M_{p}$, radius $R$, and rotational inertia $I=\frac{1}{2} M_{p} R^{2}$, ties a box of mass $m$ sliding horizontally to a box of mass $M$ falling vertically, as in Fig. 6.8.


Figure 6.8: An ideal cable draped over a pulley of rotational inertia $I$ ties a box of mass $m$ sliding horizontally to a box of mass $M$ falling vertically.

The translational and rotational versions of Newton's second law applied to each mass imply

$$
\begin{align*}
\vec{f}_{n}+\vec{f}_{g}+\vec{f}_{t} & =m \vec{a}  \tag{6.70}\\
\vec{r} \times \vec{f}_{t}+\vec{R} \times \vec{F}_{t} & =I \vec{\alpha}  \tag{6.71}\\
\vec{F}_{t}+\vec{F}_{g} & =M \vec{A} \tag{6.72}
\end{align*}
$$

where $r=R$ and $a=A$ (even though $\vec{r} \neq \vec{R}$ and $\vec{a} \neq \vec{A}$ as $\hat{r} \perp \hat{R}$ and $\hat{a} \perp \hat{A}$ ). The angular acceleration $\vec{\alpha}$ is parallel to the pulley's axis and inward, so that if the fingers of a right hand curl with the rotation of the pulley, then the thumb points in the direction of the angular acceleration. Decompose along the acceleration to find

$$
\begin{align*}
0+0+f_{t} & =m a  \tag{6.73}\\
-\not R f_{t}+\not R F_{t} & =\left(\frac{1}{2} M_{p} R^{\not 又}\right)\left(\frac{a}{R}\right),  \tag{6.74}\\
-F_{t}+M g & =M a \tag{6.75}
\end{align*}
$$

Add the equations to get

$$
\begin{equation*}
M g=m a+\frac{1}{2} M_{p} a+M a \tag{6.76}
\end{equation*}
$$

and so the acceleration

$$
\begin{equation*}
a=\frac{M}{m+M_{p} / 2+M} g \leq \frac{M}{m+M} g \tag{6.77}
\end{equation*}
$$

Check the acceleration using the Lagrangian approach. If the velocity $\vec{v}=$ $d \vec{s} / d t$, where $s$ locates the positions of the boxes, then the kinetic energy

$$
\begin{align*}
K & =\frac{1}{2} m v^{2}+\frac{1}{2} I \omega^{2}+\frac{1}{2} M v^{2} \\
& =\frac{1}{2} m v^{2}+\frac{1}{2}\left(\frac{1}{2} M_{p} R^{2}\right)\left(\frac{v}{R}\right)^{2}+\frac{1}{2} M v^{2} \\
& =\frac{1}{2}\left(m+\frac{M_{p}}{2}+M\right) v^{2} \tag{6.78}
\end{align*}
$$

and the potential energy

$$
\begin{equation*}
U=-M g s \tag{6.79}
\end{equation*}
$$

as the potential energy decreases as $s$ increases. Hence the Lagrangian

$$
\begin{equation*}
\mathcal{L}=K-U=\frac{1}{2}\left(m+\frac{M_{p}}{2}+M\right) v^{2}+M g s \tag{6.80}
\end{equation*}
$$

The Lagrange equation

$$
\begin{equation*}
M g=\frac{\partial \mathcal{L}}{\partial s}=\frac{d}{d t} \frac{\partial \mathcal{L}}{\partial v_{s}}=\left(m+\frac{M_{p}}{2}+M\right) a \tag{6.81}
\end{equation*}
$$

so the acceleration

$$
\begin{equation*}
a=\frac{M}{m+M_{p} / 2+M} g \tag{6.82}
\end{equation*}
$$

as before.

### 6.3 Circular Motion

Newton's second law relates forces to accelerations or velocity changes. Accelerations can change both the magnitudes and directions of velocities. Importantly, objects in constant circular motion have velocities of constant magnitude but changing direction.

Consider an object moving in a circle of radius $r$ at speed $v$, as in Fig. 6.9. In a short time $d t$, it moves an arc length

$$
\begin{equation*}
d \ell=r d \varphi \tag{6.83}
\end{equation*}
$$

at a constant speed

$$
\begin{equation*}
v=\frac{d \ell}{d t}=r \frac{d \varphi}{d t}=r \omega \tag{6.84}
\end{equation*}
$$

yet from the geometry, its velocity changes by

$$
\begin{equation*}
d \vec{v}=-\hat{r} v d \varphi \tag{6.85}
\end{equation*}
$$

with an acceleration

$$
\begin{equation*}
\vec{a}=\frac{d \vec{v}}{d t}=-\hat{r} v \frac{d \varphi}{d t}=-\hat{r} v \omega \tag{6.86}
\end{equation*}
$$

or

$$
\begin{equation*}
\vec{a}=-\hat{r} r \omega^{2}=-\hat{r} \frac{v^{2}}{r} \tag{6.87}
\end{equation*}
$$

which is $a=r \omega^{2}=v^{2} / r$ towards the circle's center.


Figure 6.9: An object in constant circular motion at times $t$ and $t+d t$.

For a more algebraic and less geometric derivation, if the object's position

$$
\begin{equation*}
\vec{r}=\hat{x} R \cos \varphi+\hat{y} R \sin \varphi \tag{6.88}
\end{equation*}
$$

then its velocity

$$
\begin{equation*}
\vec{v}=\frac{d \vec{r}}{d t}=-\hat{x} R \sin \varphi \frac{d \varphi}{d t}+\hat{y} R \cos \varphi \frac{d \varphi}{d t}=(-\hat{x} R \sin \varphi+\hat{y} R \cos \varphi) \omega \tag{6.89}
\end{equation*}
$$

and its acceleration

$$
\begin{equation*}
\vec{a}=\frac{d \vec{v}}{d t}=(-\hat{x} R \cos \varphi-\hat{y} R \sin \varphi) \omega^{2}=-\vec{r} \omega^{2}=-\hat{r} r \omega^{2} \tag{6.90}
\end{equation*}
$$

as before.

### 6.3.1 Hill

As an example, consider a car moving at speed $v$ over a hill of radius $R$, as in Fig. 6.10. How fast can the car drive and still stay in contact with the road?


Figure 6.10: A car cresting a hill is instantaneously moving in a circle.

At the peak of the hill, the car is instantaneously moving in a circle of radius $R$ at speed $v$, and so Newton's second law

$$
\begin{equation*}
\vec{f}_{n}+\vec{f}_{g}=m \vec{a} \tag{6.91}
\end{equation*}
$$

implies

$$
\begin{equation*}
-f_{n}+m g=m a_{\mathrm{down}}=+m \frac{v^{2}}{R} \tag{6.92}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
+f_{n}-m g=m a_{\mathrm{up}}=-m \frac{v^{2}}{R} \tag{6.93}
\end{equation*}
$$

Hence the normal force of the hill on the car is less than the car's weight,

$$
\begin{equation*}
f_{n}=m g-m \frac{v^{2}}{R}=m\left(g-\frac{v^{2}}{R}\right)<m g \tag{6.94}
\end{equation*}
$$

For the car to remain in contact with the road, the normal force must not vanish, and $0<f_{n}$ implies $v^{2} / R<g$ or

$$
\begin{equation*}
v<\sqrt{g R} \tag{6.95}
\end{equation*}
$$

### 6.3.2 Slingshot

Consider a mass $m$ in a sling moving in a vertical circle of radius $R$ at speed $v$, as in Fig. 6.11. At the circle's top, Newton's second law

$$
\begin{equation*}
\overrightarrow{f_{t}}+\overrightarrow{f_{g}}=m \vec{a} \tag{6.96}
\end{equation*}
$$

implies

$$
\begin{equation*}
f_{t}+m g=m a_{\text {down }}=m a_{\mathrm{inward}}=+m \frac{v^{2}}{R} . \tag{6.97}
\end{equation*}
$$

Hence the top tension

$$
\begin{equation*}
f_{t}=m \frac{v^{2}}{R}-m g=m\left(\frac{v^{2}}{R}-g\right) \tag{6.98}
\end{equation*}
$$

To remain taut, the sling tension must be positive, and $0<f_{t}$ implies $v^{2} / R>g$ or

$$
\begin{equation*}
v>\sqrt{g R} \tag{6.99}
\end{equation*}
$$



Figure 6.11: A sling moves a mass $m$ in a vertical circle of radius $R$ at speed $v$.

At the circle's bottom, Newton's second law

$$
\begin{equation*}
\vec{F}_{t}+\vec{f}_{g}=m \vec{a} \tag{6.100}
\end{equation*}
$$

implies

$$
\begin{equation*}
+F_{t}-m g=m a_{\mathrm{up}}=m a_{\mathrm{inward}}=+m \frac{v^{2}}{R} \tag{6.101}
\end{equation*}
$$

Hence the bottom tension

$$
\begin{equation*}
F_{t}=m \frac{v^{2}}{R}+m g=m\left(\frac{v^{2}}{R}+g\right)>m g \tag{6.102}
\end{equation*}
$$

At the bottom, the sling tension must support the mass and accelerate it in a circle. Half the difference between the bottom and top tensions is the mass's weight,

$$
\begin{equation*}
\frac{F_{t}-f_{t}}{2}=m g \tag{6.103}
\end{equation*}
$$

### 6.3.3 Inertial Frames

Newton's laws are only valid in non accelerating or inertial reference frames. Only in such frames do isolated objects move in straight lines, for example. Newton's laws are invalid in accelerating or non inertial reference frames, such as rotating reference frames. However, despite its rotation, Earth's surface is an approximate inertial reference frame. Earth's equator moves at a speed

$$
\begin{equation*}
v=\frac{2 \pi R_{\oplus}}{T} \approx \frac{25000 \mathrm{mi}}{24 \mathrm{hr}} \approx 1000 \frac{\mathrm{mi}}{\mathrm{hr}} \tag{6.104}
\end{equation*}
$$

relative to the poles. Hence the circular acceleration at the equator is

$$
\begin{equation*}
a=\frac{v^{2}}{R_{\oplus}} \approx \frac{(1000 \mathrm{mi} / \mathrm{hr})^{2}}{4000 \mathrm{mi}}=250 \frac{\mathrm{mi}}{\mathrm{hr}^{2}}=\frac{250}{22 \times 3600} \frac{22 \mathrm{mph}}{\mathrm{~s}} \approx \frac{1}{300} g \ll g . \tag{6.105}
\end{equation*}
$$

### 6.4 Work \& Impulse

In everyday language, a force is a push or a pull. In mechanics, a force $\vec{f}$ mediates energy and momentum transfers. A mass $m$ with velocity $\vec{v}=d \vec{s} / d t$ has kinetic energy

$$
\begin{equation*}
K=\frac{1}{2} m v^{2}=\frac{1}{2} m \vec{v} \cdot \vec{v} \tag{6.106}
\end{equation*}
$$

which changes with time at the rate

$$
\begin{equation*}
\frac{d K}{d t}=\frac{1}{2} m \frac{d \vec{v}}{d t} \cdot \vec{v}+\frac{1}{2} m \vec{v} \cdot \frac{d \vec{v}}{d t}=m \frac{d \vec{v}}{d t} \cdot \vec{v}=\vec{f} \cdot \vec{v}=\vec{f} \cdot \frac{d \vec{s}}{d t} \tag{6.107}
\end{equation*}
$$

In a small time $d t$ the mass moves a small displacement $d \vec{s}$, and the kinetic energy changes by

$$
\begin{equation*}
d K=\vec{f} \cdot d \vec{s} \tag{6.108}
\end{equation*}
$$

In the same time, the momentum changes by

$$
\begin{equation*}
d \vec{p}=\vec{f} d t \tag{6.109}
\end{equation*}
$$

Sum over space to form the work

$$
\begin{equation*}
W=\Delta K=\int \vec{f} \cdot d \vec{s} \tag{6.110}
\end{equation*}
$$

and sum over time to form the impulse

$$
\begin{equation*}
J=\Delta \vec{p}=\int \vec{f} d t \tag{6.111}
\end{equation*}
$$

(Since the symbol $I$ is overused, the next letter in the alphabet $J$ is often used for impulse.) Force integrated over space is the work done, while force integrated over time is the impulse given. Kinetic energy change is force integrated over space, while momentum change is force integrated over time.

### 6.4.1 Hockey Puck

As an example, consider a hockey puck of mass $m$ slapped with initial speed $v_{0}$ that slides a length $\ell$ before stopping. The friction coefficient between the puck and the ice follows from the Eq. 6.110 work energy relation,

$$
\begin{equation*}
0-\frac{1}{2} m v_{0}^{2}=\Delta K=W=\int \vec{f}_{f} \cdot d \vec{s}=-f_{f} \ell=-\mu f_{n} \ell=-\mu m g \ell \tag{6.112}
\end{equation*}
$$

or

$$
\begin{equation*}
\mu=\frac{v_{0}^{2}}{2 g \ell} \tag{6.113}
\end{equation*}
$$

### 6.5 Variable Mass Rockets

A rocket ejects burned fuel rearward to recoil forward, as in Fig. 6.12. The forward force on the rocket is the reaction to the rearward force on the exhaust. Because the rocket's mass is not conserved, analyze the dynamics with the more general version of Newton's second law, $\vec{f}=d \vec{p} / d t$.


Figure 6.12: A rocket ejects mass in one direction to recoil in the othe r , even in the vacuum of empty space. Exhaust velocity is negative relative to the rocket but may be negative or positive relative to the stars (if the rocket moves sufficiently slow or fast).

Because no external forces act on the rocket and its exhaust, their momentum is conserved. If the rocket of mass $M$ moves at velocity $V_{s}>0$ and in time $\delta t>0$ ejects a mass $-\delta M>0$ at velocity $v_{s} \lessgtr 0$ relative to the distant stars, then

$$
\begin{align*}
0 & =\Delta p_{s}=p_{s}^{\prime}-p_{s} \\
& =\left((M+\delta M)\left(V_{s}+\delta V_{s}\right)+(-\delta M) v_{s}\right)-M V_{s} \\
& =M V_{s}+M \delta V_{s}+\delta M V_{s}+\delta M \delta V_{s}-\delta M v_{s}-M V_{s} \tag{6.114}
\end{align*}
$$

and so

$$
\begin{equation*}
\frac{\delta V_{s}}{\delta M}=\frac{v_{s}-V_{s}-\delta V_{s}}{M} \tag{6.115}
\end{equation*}
$$

In the limit $\delta t \rightarrow 0$, both $\delta V_{s} \rightarrow 0$ and $\delta M \rightarrow 0$ such that the rate of change of the rocket's velocity with respect to mass

$$
\begin{equation*}
\frac{d V_{s}}{d M}=\frac{v_{s}-V_{s}}{M}=\frac{v_{r}}{M}=-\frac{v}{M}, \tag{6.116}
\end{equation*}
$$

where $v_{r}=v_{s}-V_{s}$ is velocity of the exhaust relative to the rocket, and $v=$ $-v_{r}>0$ is the speed of the exhaust relative to the rocket. Assuming the relative exhaust speed $v$ is constant, integrate

$$
\begin{equation*}
\int_{V_{i}}^{V_{f}} d V_{s}=-v \int_{M_{i}}^{M_{f}} \frac{d M}{M} \tag{6.117}
\end{equation*}
$$

to find

$$
\begin{equation*}
V_{f}-V_{i}=-v\left(\log M_{f}-\log M_{i}\right) \tag{6.118}
\end{equation*}
$$

which is the rocket equation

$$
\begin{equation*}
\Delta V=v \log \left[\frac{M_{i}}{M_{f}}\right]>0 \tag{6.119}
\end{equation*}
$$

The $\Delta V$ of a rocket is proportional to the relative exhaust speed and the logarithm of the ratio of the initial to final masses. Typical rockets maximize their $\Delta V$ by minimizing their payload fraction

$$
\begin{equation*}
\frac{M_{f}}{M_{i}}=e^{-\Delta V / v} \tag{6.120}
\end{equation*}
$$

By Eq. 6.119, from rest an ideal rocket can exceed its exhaust speed $\Delta V>v$ provided its payload fraction $M_{f} / M_{i}<e \approx 2.7$. Actual rocket payload fractions are sometimes less than $1 \%=0.01 \ll e$, as rockets are mainly fuel.

## Problems

1. You ask a mule to pull a plow. The mule resists, explaining that "If I tug on the plow, Newton's third law asserts that the plow will tug on me with an equal but opposite force. Since these forces will cancel each other out, it is obvious that we're not going anywhere. Therefore, there is no point in trying." Carefully (but gently) explain to the mule the error in its reasoning and, using appropriate diagrams, explain why it is possible for the mule to accelerate the plow.
2. A crane hauls a crate of mass $m$ upward at a constant acceleration $a$. What is the magnitude of the tension force on the crate by the crane's cable?
3. You are standing on a bathroom scale in an elevator moving downward. If your mass is 60 kg and the scale reads 750 N , what is the magnitude and direction of the elevator's acceleration?
4. A box rests on a slab. If the static friction coefficient between the box and the slab is $\mu$, what is the maximum acceleration of the slab beyond which the box will slide?
5. You a pilot a spaceship of mass $m$ hauling a cargo of mass $M$ with a cable with a breaking tension of $f_{t}$.
(a) What is the maximum acceleration you can give the cargo?
(b) What thrust must your engines exert to provide this acceleration?
6. An ideal cable draped over an ideal pulley ties a box of mass $m$ sliding horizontally to a box of mass $M$ falling vertically.
(a) Find the boxes' acceleration $a$ and the cable's tension $f_{t}$.
(b) Check that your formulas for acceleration and tension make sense in the limits $M=0$ and $m=0$.
7. Solve the Section 6.1.4 incline problem using a coordinate system parallel and perpendicular to the ground.
8. Find the acceleration and both tensions in an Atwood's machine with suspended masses $M$ and $m$ and a massive pulley of rotational inertia $I=\frac{1}{2} M_{p} R^{2}$.
9. As you drive a car around a curve at a constant speed of 50 mph , an accelerometer in the car measures its sideways acceleration to be 0.1 g . What is the radius of the curve?
10. You ride a Ferris wheel that rotates at a constant rate. At the highest point, the seat exerts a normal force $f_{n}$ on you; at the lowest point, the seat exerts a normal force of $F_{n}$ on you. How much do you weigh?
11. "Rotor" is an amusement park ride that consists of a hollow cylindrical room that rotates around a central vertical axis. You enter the room and stand against the curved wall. The room begins to rotate, and when a certain speed is reached, the floor drops away, revealing a deep pit. You do not fall, though, because a friction force exerted by your contact interaction with the wall supports you. Assuming a radius $R$ and a static friction coefficient $\mu$, derive a formula for the minimum rotation period $T$ that will pin you to the wall.
12. You drive a car through a hairpin curve of radius $R$ that banks at an angle $\theta$. If the static friction coefficient between your tires and the road is $\mu$, what is the maximum speed with which you can go around the curve? Hint: Sketch the curve both from above and in cross section.
13. An unpowered roller-coaster car starts at rest at the top of a hill of height $H$, rolls down the hill, and then goes around a vertical loop of radius $R$. How high should the hill be so that the car does not lose contact with the track? Hint: Combine conservation of energy with Newton's Laws.
14. A spring of stiffness $\kappa$ connects two otherwise free masses $M$ and $m$ located at positions $X$ and $x$.
(a) Use Newton's second law and Hooke's Eq. 2.35 to write two differential equations for the motion of the masses. Hint: Choose the algebraic signs of the spring forces carefully.
(b) Show that the center-of-mass $x_{\mathrm{cm}}=(m x+M X) /(m+M)$ does not accelerate. Hint: Add the equations to eliminate the force terms.
(c) Show that the coordinate difference $\delta=X-x$ obeys the simple harmonic oscillator Eq. 4.26. Hint: Linearly combine the equations.
(d) Express the resulting temporal frequency $\omega$ as a function of the reduced mass $m_{r}=m M /(m+M)$. What happens as $M \rightarrow \infty$ ?
15. A wheel of mass $m$, radius $r$, and rotational inertia $I$ rolls without slipping down an incline at an angle $\varphi$ from the horizontal. As the wheel rotates through an angle $\theta$, its axis translates through a length $\ell=r \theta$.
(a) Write the rotational and translational kinetic energies of the wheel.
(b) Write the gravitational potential energy of the wheel.
(c) Construct the Lagrangian in terms of $\theta$ and $\omega_{\theta}=d \theta / d t$.
(d) Substitute into Lagrange's equations and solve for the linear acceleration $a_{\ell}=r \alpha_{\theta}$, where $\alpha_{\theta}=d \omega_{\theta} / d t$.
(e) Compute the accelerations of a hoop of rotational inertia $I=m r^{2}$, a disk or cylinder of rotational inertia $I=\frac{1}{2} m r^{2}$, and a wheel or box that slides without friction. In a race, which wins?
(f) Create a two-column and three-row table listing the fractions of kinetic energy in rotation and translation for the hoop, disk, and box. Which is the best translator?
16. Consider a multi-stage rocket with relative exhaust speed $v$.
(a) What is the $\Delta V$ of the first stage if $80 \%$ of the rocket's mass is the first stage fuel or wet mass and $10 \%$ is the first stage dry mass (with the remaining $10 \%$ being the rest of the rocket)?
(b) What is the total $\Delta V$ if the rocket has two additional but similar (and subsequently smaller) stages, where each stage is discarded after it exhausts its fuel?
(c) What is the rocket's payload fraction?

## Chapter 7

## Gravity

Newton's laws unite the fall of an apple with the orbit of Luna.


Figure 7.1: Newton's famous drawing of projectiles shot from a mountain top with increasing speeds until they beginning falling around Earth; terrestrial falls and celestial orbits are extremes of the same phenomenon [7].

### 7.1 Universal Gravity

Isaac Newton realized that the moon is falling - falling towards Earth's center like an apple falling from a tree, but with enough tangential speed to miss Earth's surface, as in Fig. 7.1. This insight united the work of Newton's predecessors in the previous generation, Galileo Galilei and Johannes Kepler, on terrestrial fall and celestial orbits.

Newton postulated a universal gravitational interaction between any two masses. In modern language, if a distance $r$ separates two masses $m$ and $M$, then their gravitational potential energy

$$
\begin{equation*}
U=-G \frac{M m}{r} \tag{7.1}
\end{equation*}
$$

By Eq. 2.34 , the radial force between them

$$
\begin{equation*}
f_{r}=-\frac{d U}{d r}=-G \frac{M m}{r^{2}} \tag{7.2}
\end{equation*}
$$

so the vector force

$$
\begin{equation*}
\vec{f}=-\hat{r} G \frac{M m}{r^{2}} \tag{7.3}
\end{equation*}
$$

and force magnitude

$$
\begin{equation*}
f=G \frac{M m}{r^{2}} \tag{7.4}
\end{equation*}
$$

where the universal gravitational constant

$$
\begin{equation*}
G=6.67 \times 10^{-11} \frac{\mathrm{~N} \mathrm{~m}^{2}}{\mathrm{~kg}^{2}}=66.7 \frac{\mathrm{pN} \mathrm{~m}^{2}}{\mathrm{~kg}^{2}} \tag{7.5}
\end{equation*}
$$

so that two 1 kg masses separated by 1 m attract each other with the minuscule force of 66.7 pN . If the masses floated in the vacuum of deep space, the resulting feeble accelerations would require just over a day to collide them. The gravitational force is so weak that unless Earth is one of the masses, the force is typically overwhelmed by frictional forces (which are ultimately electromagnetic in origin).

The inverse-square nature of the gravitational force reflects the three dimensional nature of space: at a distance $r$ for a point mass, the force is "diluted" over a spherical surface of radius $r$ and area $4 \pi r^{2}$. The radial nature of the gravitational force reflects the isotropy of space: since a non-radial component point can't point in one direction without pointing in the other, it doesn't exist.

### 7.2 Newton's Shell Theorems

If the Eq. 7.2 gravitational force applied to only point masses, it would be of limited utility. However, Newton proved a couple of "superb" theorems (numbers XXX and XXXI of Book I of the Principia), which implies that the same force law applies both to hollow and solid spheres. This is tremendously useful in astronomy and astrophysics, as most stars and planets are well approximated as spheres.

Geometry and the idea of integrals as "ultimate sums" are all that is needed to prove the superb theorems. The geometry includes planar angles and solid angles, which are reviewed in Appendix B.

### 7.2.1 Interior Shell Theorem

Consider a point mass $m$ inside a spherical shell of mass $M$ and radius $\ell$, as in Fig. 7.2


Figure 7.2: Opposing small areas $\delta a$ and $\delta A$ of a spherical shell equally attract an interior point mass $m$.

At a distance $r$ subtending a small solid angle $\delta \Omega$ and at an angle $\theta$, the small area element

$$
\begin{equation*}
\delta a=\frac{r^{2} \delta \Omega}{\cos \theta} \tag{7.6}
\end{equation*}
$$

of mass

$$
\begin{equation*}
\delta m=M \frac{\delta a}{4 \pi \imath^{2}} \tag{7.7}
\end{equation*}
$$

attracts the interior point mass with a small force magnitude

$$
\begin{equation*}
\delta f=\frac{G m \delta m}{r^{2}}=\frac{G m}{r^{2}} M \frac{\delta a}{4 \pi \imath^{2}}=\frac{G m M}{y^{2}} \frac{1}{4 \pi \imath^{2}} \frac{p^{2} \delta \Omega}{\cos \theta}=\frac{G m M}{\imath^{2}} \frac{\delta \Omega}{4 \pi \cos \theta} . \tag{7.8}
\end{equation*}
$$

The inverse square nature of the force is essential to the cancellation of the distance $r$. The shared planar and solid angles $\theta$ and $\delta \Omega$ ensure that the force magnitude $\delta F$ on the opposite area element $\delta A$ is the same. Thus $\delta F=\delta f$ and $\delta \vec{F}=-\delta \vec{f}$, and these force pairs cancel. Divide the rest of the spherical shell into similar pairs to show that the net force on any interior point mass vanishes.

### 7.2.2 Exterior Shell Theorem

Consider a source shell of mass $M$ concentric with an observation shell of radius $r$, as in Fig. 7.3


Figure 7.3: Source element mass $\delta M$ causes gravitational acceleration $\delta \vec{g}$ at observation element area $\delta a$.

Let $\delta m$ be a small mass element of the source sphere, and let

$$
\begin{equation*}
\delta a=\frac{\ell^{2} \delta \Omega}{\cos \theta} \tag{7.9}
\end{equation*}
$$

be a small area element at a distance $\ell$ and an angle $\theta$ subtending a solid angle $\delta \Omega$ on the observation shell, where the point mass causes a radial gravitational acceleration

$$
\begin{equation*}
\delta g_{r}=-\delta g \cos \theta=-\frac{G \delta M}{\ell^{2}} \cos \theta \tag{7.10}
\end{equation*}
$$

which averaged over the observing sphere is

$$
\begin{equation*}
\left\langle\delta g_{r}\right\rangle=\frac{\int_{0} \delta g_{r} \delta a}{\int_{0} \delta a}=\frac{\left(-G \delta M \cos \theta / \ell^{2}\right)\left(\ell^{2} 4 \pi / \cos \theta\right)}{4 \pi r^{2}}=-\frac{G \delta M}{r^{2}} \tag{7.11}
\end{equation*}
$$

as $\int \delta \Omega=4 \pi$. By spherical symmetry, the total radial acceleration, due to all mass elements, is the same everywhere on the shell, and so

$$
\begin{equation*}
g_{r}=\left\langle g_{r}\right\rangle=\int_{s}\left\langle\delta g_{r}\right\rangle=-\frac{G M}{r^{2}} \tag{7.12}
\end{equation*}
$$

as $\int \delta M=M$. For an external point mass $m$, the force

$$
\begin{equation*}
f_{r}=m g_{r}=-\frac{G M m}{r^{2}} \tag{7.13}
\end{equation*}
$$

which is the same as the force between two point masses [17]. Apply the above argument to the point mass to generalize to the force between to two shells.

Decompose solid spheres into concentric shells to generalize to two solid spheres, even if the density varies with radius.

However, the force between two non-spheres is different. For example, in the slashdot (/.) body problem [18], a point of mass $m$ is displaced $\vec{r}$ from the center of a line of mass $M$, length $\ell$, and orientation $\hat{\ell}$. If the directions from the point mass to the line segment's ends are $\hat{r}_{ \pm}$, then the force on the point mass is

$$
\begin{equation*}
\vec{f}=-G M m \frac{\left(\hat{r}_{+}-\hat{r}_{-}\right)}{|\vec{\ell} \times \vec{r}|} \times \hat{n} \tag{7.14}
\end{equation*}
$$

where the unit vector $\hat{n}=\vec{\ell} \times \vec{r} /|\vec{\ell} \times \vec{r}|$ is normal to the plane of the motion. The slashdot force reduces to the Eq. 7.2 gravitational force for two point masses in the limit $\ell \rightarrow 0$.

### 7.3 Trans Earth Tunnel

Consider a straight tunnel through Earth, as in Fig. 7.4. Drop a mass $m$ in one end. How long before it reaches the other end, and how long before it returns? Assume the tunnel is evacuated and ignore Earth's rotation.


Figure 7.4: Due to gravity, a point mass $m$ oscillates sinusoidally inside a trans Earth tunnel.

If the mass $m$ is at a distance $r$ from Earth's center, then by the interior shell theorem, the exterior mass doesn't contribute a force, and by the exterior shell theorem, the interior mass

$$
\begin{equation*}
M_{r}=M_{\oplus} \frac{\frac{4}{3} \pi r^{3}}{\frac{4}{3} \pi R_{\oplus}^{3}}=M_{\oplus} \frac{r^{3}}{R_{\oplus}^{3}} \tag{7.15}
\end{equation*}
$$

contributes a force as if all of it where at the sphere's center. Hence, if the mass is a distance $s$ from the tunnel's center, which is a distance $d$ from Earth's center, then Newton's second law

$$
\begin{equation*}
\overrightarrow{f_{n}}+\overrightarrow{f_{g}}=m \vec{a} \tag{7.16}
\end{equation*}
$$

implies
where $\sin \theta=s / d$. Hence,

$$
\begin{equation*}
-\frac{G M_{\oplus}}{p^{2}} \frac{p^{z}}{R_{\oplus}^{3}} \frac{s}{r^{\prime}}=\frac{d^{2} s}{d t^{2}} \tag{7.18}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{d^{2} s}{d t^{2}}+\frac{G M_{\oplus}}{R_{\oplus}^{3}} s=0 \tag{7.19}
\end{equation*}
$$

This simple harmonic oscillator differential equation, like Eq. 4.26, has the sinusoidal solution

$$
\begin{equation*}
s[t]=A \cos [\omega t+\delta] \tag{7.20}
\end{equation*}
$$

where the amplitude $A$ and the phase shift $\delta$ depend on the initial conditions, provided the angular frequency

$$
\begin{equation*}
\omega=\sqrt{\frac{G M_{\oplus}}{R_{\oplus}^{3}}}=\sqrt{\frac{g}{R_{\oplus}}} \tag{7.21}
\end{equation*}
$$

The mass $m$ executes sinusoidal motion with period

$$
\begin{equation*}
T=\frac{2 \pi}{\omega}=2 \pi \sqrt{\frac{R_{\oplus}}{g}} \approx 84 \mathrm{~min} \tag{7.22}
\end{equation*}
$$

Passengers could embark an unpowered vehicle at one end and fall through the trans Earth tunnel to disembark at the other end about $T / 2 \approx 42$ min later. The travel time is independent of the distance $d$, including $d=0$ for the longest possible trip through Earth's center to the antipodes!

### 7.4 Near-Earth Gravity

The Eq. 7.1 gravitational potential energy simplifies for motion in a terrestrial laboratory. According to the exterior shell theorem, a mass $m$ at a height $h$ above an Earth of mass $M_{\oplus}$ and radius $R_{\oplus}$ has potential energy

$$
\begin{equation*}
U\left[R_{\oplus}+h\right]=-G \frac{M_{\oplus} m}{R_{\oplus}+h}=-G \frac{M_{\oplus} m}{R_{\oplus}}\left(1+\frac{h}{R_{\oplus}}\right)^{-1} \tag{7.23}
\end{equation*}
$$

or by the binomial theorem and assuming $h \ll R_{\oplus}$,

$$
\begin{align*}
U\left[R_{\oplus}+h\right] & =-G \frac{M_{\oplus} m}{R_{\oplus}}\left(1-\left(\frac{h}{R_{\oplus}}\right)+\left(\frac{h}{R_{\oplus}}\right)^{2}-\cdots\right) \\
& \approx-G \frac{M_{\oplus} m}{R_{\oplus}}\left(1-\frac{h}{R_{\oplus}}\right) \\
& =-\frac{G M_{\oplus} m}{R_{\oplus}}+m \frac{G M_{\oplus}}{R_{\oplus}^{2}} h \\
& =U\left[R_{\oplus}\right]+m g h \tag{7.24}
\end{align*}
$$

where the terrestrial gravitational field (or equivalently the free-fall acceleration)

$$
\begin{equation*}
g=\frac{G M_{\oplus}}{R_{\oplus}^{2}}=9.8 \frac{\mathrm{~N}}{\mathrm{~kg}}=9.8 \frac{\mathrm{~m}}{\mathrm{~s}^{2}} \tag{7.25}
\end{equation*}
$$

"Big G" is a universal constant, presumably the same everywhere in the visible universe, but "small g" is the local gravitational field, which varies with altitude on Earth and is different on Luna or Mars. The first term on the right side of the Eq. 7.24 potential energy merely shifts the energy by a constant, as in Fig. 7.5 which does not affect either Lagrange's equations or the force, and may be omitted. The combination $m g h$ is a very useful linear approximation to the gravitational potential energy near Earth.


Figure 7.5: Newtonian gravitational potential energy (left) and linear approximation (right). In both cases, the potential energy increases with altitude.

### 7.5 Kepler's Laws

In the early 1600 s, while Galileo was describing terrestrial fall, Kepler was describing planetary motion. Using data of Tycho Brahe, Kepler inferred three laws of planetary motion: planet's orbit Sol in ellipses with Sol at one focus (not at the center); a line joining a planet to Sol sweeps out equal areas in equal
times; the square of a planet's orbital period is proportional to the cube of its larger orbital radius. The first law illustrates spontaneous symmetry breaking, because initial conditions can select an orbit that is not circularly symmetrical even though Newton's law of gravity is spherically symmetric. The second and third laws reflect the weakening of the gravitational force with distance, because the planets move slower when they are further from Sol.

Like Galileo's laws of fall, Kepler's laws of planetary motion follow from Newton's laws. Rapidly derive these laws for the special case of circular motion, where a small mass $m$ orbits a large mass $M \gg m$, so that the recoil of the large mass is negligible, as in Fig. 7.6.


Figure 7.6: A mass $m$ orbits a much larger mass $M \gg m$ in a circle centered on the larger mass.

Newton's second law implies

$$
\begin{equation*}
\vec{f}_{g}=m \vec{a} \tag{7.26}
\end{equation*}
$$

implies

$$
\begin{equation*}
\frac{G M \not n}{R^{2}}=\not x a_{\mathrm{in}}=\not \approx \frac{v^{2}}{R} \tag{7.27}
\end{equation*}
$$

so that the orbital speed

$$
\begin{equation*}
\frac{2 \pi R}{T}=v=\sqrt{\frac{G M}{R}} \tag{7.28}
\end{equation*}
$$

and the period squared

$$
\begin{equation*}
T^{2}=\frac{4 \pi^{2}}{G M} R^{3} \tag{7.29}
\end{equation*}
$$

which is a special case of Kepler's third law of periods. Substitute $T=2 \pi / \omega$ to produce the "1-2-3" form

$$
\begin{equation*}
G M^{1}=\omega^{2} R^{3} . \tag{7.30}
\end{equation*}
$$

For our solar system, a period of $T=1 \mathrm{yr}$ corresponds to an orbital radius of $R \approx 1 \mathrm{AU} \approx 150 \times 10^{6} \mathrm{~km}$, so that

$$
\begin{equation*}
\left(\frac{T}{\mathrm{yr}}\right)^{2}=\left(\frac{R}{\mathrm{AU}}\right)^{3} \tag{7.31}
\end{equation*}
$$

Furthermore, from Fig. 7.6, the area $\delta A$ swept out by the motion of the planet in a time $\delta t$ is

$$
\begin{equation*}
\delta A=\frac{\delta \theta}{2 \pi} A=\frac{1}{2 \pi}(\omega \delta \theta)\left(\pi R^{2}\right)=\frac{1}{2 \mathbb{\pi}}\left(\frac{L}{m R^{2}} \delta t\right)\left(\mathbb{T} R^{2}\right)=\frac{L}{2 m} \delta t . \tag{7.32}
\end{equation*}
$$

where $L=I \omega=m R^{2} \omega$ is the constant orbital angular momentum. In the limit $\delta t \rightarrow 0, \delta A \rightarrow 0$ such that

$$
\begin{equation*}
\frac{d A}{d t}=\frac{L}{2 m} \tag{7.33}
\end{equation*}
$$

which is constant in agreement with Kepler's second law of areas. The circular motion itself is a special case of Kepler's first law of orbits, as a circle is a special case of an ellipse.

### 7.6 Binary Orbits

Newton's laws generalize Kepler's laws to systems like the Fig. 7.7 binary, where two comparable masses $M \gtrsim m$ orbit their common center of mass at distances $R \lesssim r$.


Figure 7.7: Two comparable masses orbit their common center of mass, like Pluto and Charon.

If the center of mass is the coordinate origin, then

$$
\begin{equation*}
\overrightarrow{0}=\vec{r}_{\mathrm{cm}}=\frac{M \vec{R}+m \vec{r}}{M+m} \tag{7.34}
\end{equation*}
$$

so that

$$
\begin{equation*}
M \vec{R}=-m \vec{r} \tag{7.35}
\end{equation*}
$$

and by differentiation

$$
\begin{equation*}
M \vec{V}=-m \vec{v} \tag{7.36}
\end{equation*}
$$

Thus the orbital radii and speeds are inverse to the masses, $M R=m r$ and $M V=m v$. Newton's second law

$$
\begin{equation*}
\overrightarrow{f_{g}}=m \vec{a} \tag{7.37}
\end{equation*}
$$

implies

$$
\begin{equation*}
\frac{G \npreceq M}{(r+R)^{2}}=\not \check{a_{\mathrm{in}}}=\not \check{ } \frac{v^{2}}{r} \tag{7.38}
\end{equation*}
$$

so that the orbital frequency squared

$$
\begin{equation*}
\left(\frac{2 \pi}{t}\right)^{2}=\omega^{2}=\left(\frac{v}{r}\right)^{2}=\frac{G}{(r+R)^{2}} \frac{M}{r}\left(\frac{1+m / M}{1+R / r}\right)=\frac{G(M+m)}{(r+R)^{3}} \tag{7.39}
\end{equation*}
$$

as $R / r=m / M$. Hence, the period squared

$$
\begin{equation*}
t^{2}=\frac{4 \pi^{2}}{G(M+m)}(R+r)^{3}=\frac{4 \pi^{2}}{G(m+M)}(r+R)^{3}=T^{2} \tag{7.40}
\end{equation*}
$$

## Problems

1. Two spheres of radius $r$ and mass $m$ are touching each other. What is the gravitational force between them?
2. Black hole Schwarzschild radius.
(a) Use energy conservation to find the minimum speed $v$ to escape from the surface of a planet of radius $R$ and mass $M$.
(b) For a given mass $M$, what is the radius $R_{s}$ such that the escape speed is light speed $c$ ?
(c) Compute the Schwarzschild radius $R_{s}$ for Earth $M=M_{\oplus}$ in centimeters. If Earth were compressed to this radius, it would become a black hole.
3. Circular orbits.
(a) Use Newton's laws to find the period $T$ of a satellite orbiting a planet of mass $M$ in a circular orbit of radius $r$.
(b) Compute the period of an Earth grazing orbit of radius $r=R_{\oplus}$ in minutes. Neglect air drag.
(c) Compute the radius of Clarke orbit [19], where geosynchronous satellites orbit Earth once a day and appear to hover motionless above the surface, as a fraction of the distance between Earth and Luna.
4. Two 1 kg masses float in the vacuum of deep space separated by 1 m . How many days before their gravitational attraction brings them together? Hint: An exact solution requires numerical integration of the initial value problem. For an approximate solution, assume the masses' acceleration is constant at their initial values. Will the approximate solution be an overestimate or an underestimate?
5. Two equal but opposite masses $m>0$ and $-m<0$ float near each other in space.
(a) Assuming Newton's laws of motion and gravity still apply, describe the subsequent motion of the masses.
(b) Compute the momentum and kinetic energy of the system for all times. Are they conserved?
6. You stand on a roughly spherical asteroid of density $\rho$ and radius $R$.
(a) Derive a formula for the minimum orbital speed around the asteroid.
(b) If $R=22 \mathrm{~km}$ and $\rho=7.9 \mathrm{~g} / \mathrm{cm}^{3}$, can you run fast enough to enter orbit?
7. You plan to launch a satellite, called "Monday's Star", in a circular orbit so that it appears on the eastern horizon every Monday morning (at 6 AM) and never at any other time. How far to Luna should the satellite orbit? Hint: Use a version of the Eq. 7.31 form of Kepler's third law, but based on month and lunar distance rather than year and solar distance.
8. You observe a circular binary star system and measure its period $T$ and the orbital radii $r$ and $R$ of its two components. What are their masses?

## Chapter 8

## Kinetic Theory

Newton's laws elucidate air of twenty-four million trillion particles per milliliter.


Figure 8.1: Atoms in a solid (left) vibrate about fixed positions in a lattice, atoms in a liquid (right) slip and slide past each other, and atoms in a gas (right) move freely in all directions. While solids maintain their shapes, liquids and gases flow or diffuse to conform to their bounding volumes.

### 8.1 Ideal Gas Law

Near the start of his famous Lectures on Physics 4], Richard Feynman proposes that if all our scientific knowledge were destroyed except for one sentence, that sentence should be,
... all things are made of atoms - little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another.

Kinetic theory treats solids, liquid, and gases as collections of very many, very tiny atoms or molecules, envisioned as spherical masses in rapid motion, as
in Fig. 8.1. Classical mechanics applied to gas molecules predicts pressures and temperatures of macroscopic volumes. Macroscopic effective laws thereby emerge from microscopic fundamental laws.

The absolute temperature $T>0$ of a gas is proportional to the average kinetic energy of its molecules, while heat is proportional to the total kinetic energy. For a monatomic gas, without rotational and vibrational degrees of freedom, and including a conventional factor of $3 / 2$,

$$
\begin{equation*}
\langle K\rangle=\left\langle\frac{1}{2} m v^{2}\right\rangle=\frac{3}{2} k_{B} T \tag{8.1}
\end{equation*}
$$

where the Boltzmann constant

$$
\begin{equation*}
k_{B}=1.38 \times 10^{-23} \frac{\mathrm{~J}}{\mathrm{~K}}=13.8 \frac{\mathrm{fJ}}{\mathrm{GK}} \tag{8.2}
\end{equation*}
$$

so that a rise in temperature of one gigakelvin increases the average kinetic energy by about 20 femtojoules. Earth's surface can get as hot as about $330 \mathrm{~K}\left(=56.7^{\circ} \mathrm{C}=134^{\circ} \mathrm{F}\right)$, while Pluto's surface can get as cold as about $33 \mathrm{~K}\left(=-240^{\circ} \mathrm{C}=-400^{\circ} \mathrm{F}\right)$. The classically allowable but quantumly unattainable absolute zero, $T=0 \mathrm{~K}$, would correspond to the complete absence of motion. Figure 8.2 compares four common temperature scales.


Figure 8.2: Temperatures $T$ versus mean molecular kinetic energy $\langle K\rangle$ in zeptojoules for two proportional (or absolute) temperature scales and two linear non-proportional temperature scales.

The pressure $P$ of a gas is the force $F$ it exerts an area $A$,

$$
\begin{equation*}
P=\frac{F}{A} \tag{8.3}
\end{equation*}
$$

The pressure of Earth's atmosphere at its surface is

$$
\begin{equation*}
P_{\oplus}=1.01 \times 10^{5} \mathrm{~Pa}=1.01 \times 10^{5} \frac{\mathrm{~N}}{\mathrm{~m}^{2}}=10.1 \frac{\mathrm{~N}}{\mathrm{~cm}^{2}}=14.7 \mathrm{psi} \tag{8.4}
\end{equation*}
$$

so a one square-centimeter air column, from sea level to infinity, weighs about ten newtons (and a one square-inch column weighs about fifteen pounds). Use "big" upper case $P$ for pressure and "small" lower case $p$ for momentum.


Figure 8.3: Molecules in a gas confined to a volume $V$ at at temperature $T$ collide elastically with a piston to create a pressure $P$. The vector $\vec{A}=\hat{x} A_{x}$ encodes the magnitude and orientation of the piston's cross sectional area.

Consider a gas confined by a piston moving in a cylinder, as in Fig. 8.3. Assume the piston moves in the $x$ direction, and encode its cross sectional area and orientation in the magnitude and direction of the vector $\vec{A}=\hat{x} A_{x}$. Each molecule of mass $m$ and velocity $\vec{v}$ collides elastically with the cylinder upon delivering the impulse

$$
\begin{equation*}
J_{x}=F_{x} \Delta t=\Delta p_{x}=2 m v_{x} \tag{8.5}
\end{equation*}
$$

If all the molecules move with the same speed toward the cylinder, then in a short time $\delta t$, all molecules within $v_{x} \delta t$ of the area $A_{x}$, and hence within the small volume $\delta V=v_{x} \delta t A_{x}$, collide with the cylinder. If the gas number density

$$
\begin{equation*}
n=\frac{N}{V}=\frac{\delta N}{\delta V} \tag{8.6}
\end{equation*}
$$

then

$$
\begin{equation*}
\delta N=n \delta V \tag{8.7}
\end{equation*}
$$

implies a collision rate

$$
\begin{equation*}
\frac{\delta N}{\delta t}=n \frac{\delta V}{\delta t}=n v_{x} A_{x} \tag{8.8}
\end{equation*}
$$

Hence the total force

$$
\begin{equation*}
f_{x}=\frac{\delta p_{x}}{\delta t}=\frac{\delta N \Delta p_{x}}{\delta t}=\left(n v_{x} A\right)\left(2 m v_{x}\right) \tag{8.9}
\end{equation*}
$$

and resulting pressure

$$
\begin{equation*}
\frac{f_{x}}{A_{x}}=2 n m v_{x}^{2} \tag{8.10}
\end{equation*}
$$

However, if only half the molecules move toward the piston, and not all move at the same speed, the actual force is half the average, and the pressure

$$
\begin{equation*}
P=\frac{F_{x}}{A_{x}}=\frac{\frac{1}{2}\left\langle f_{x}\right\rangle}{A_{x}}=\frac{1}{\not 2}\left(\not 2 n m\left\langle v_{x}^{2}\right\rangle\right)=n m\left\langle v_{x}^{2}\right\rangle . \tag{8.11}
\end{equation*}
$$

If the gas is isotropic, then the averages

$$
\begin{equation*}
\left\langle v_{x}^{2}\right\rangle=\left\langle v_{y}^{2}\right\rangle=\left\langle v_{z}^{2}\right\rangle \tag{8.12}
\end{equation*}
$$

and so

$$
\begin{equation*}
\left\langle v^{2}\right\rangle=\left\langle v_{x}^{2}+v_{y}^{2}+v_{z}^{2}\right\rangle=\left\langle v_{x}^{2}\right\rangle+\left\langle v_{y}^{2}\right\rangle+\left\langle v_{z}^{2}\right\rangle=3\left\langle v_{x}^{2}\right\rangle \tag{8.13}
\end{equation*}
$$

Hence the pressure

$$
\begin{equation*}
P=n m \frac{1}{3}\left\langle v^{2}\right\rangle=n \frac{2}{3}\left\langle\frac{1}{2} m v^{2}\right\rangle=n \frac{2}{3}\left(\frac{3}{2} k_{B} T\right)=n k_{B} T \tag{8.14}
\end{equation*}
$$

by the Eq. 8.1 temperature definition. Alternately,

$$
\begin{equation*}
P V=N k_{B} T \tag{8.15}
\end{equation*}
$$

which is the ideal gas law.
It follows that $N=P V / k_{B} T$, and equal volumes of different gases at the same pressure and temperature have the same number of molecules. At room temperature and pressure, a cubic centimeter contains about twenty-four million trillion molecules. Hence,

$$
\begin{equation*}
\left(\frac{P}{10^{5} \mathrm{~Pa}}\right)\left(\frac{V}{1 \mathrm{~cm}^{3}}\right) \approx\left(\frac{N}{2.4 \times 10^{19}}\right)\left(\frac{T}{300 \mathrm{~K}}\right) \tag{8.16}
\end{equation*}
$$

The product $N k_{B}=\left(N / N_{A}\right)\left(N_{A} k_{B}\right)=\mathcal{N} R$, where $N_{A}=6.02 \times 10^{23}$ is Avogadro's constant, $\mathcal{N}$ is the mole number, and $R$ is the gas constant. One mole of hydrogen atoms masses one gram.

### 8.2 Mean Free Path

The average distance a molecule moves between collisions is its mean free path $\ell$. To compute it, first consider the mean molecular speed. From the Eq. 8.1
temperature definition, the most useful measure of speed in kinetic theory is the root-mean-square speed

$$
\begin{equation*}
v_{\mathrm{rms}}=\sqrt{\left\langle v^{2}\right\rangle}=\sqrt{\frac{3 k_{B} T}{m}} \tag{8.17}
\end{equation*}
$$

Also relevant is the relative mean speed squared

$$
\begin{equation*}
\left\langle\delta v^{2}\right\rangle=\left\langle\left(\vec{v}-\vec{v}^{\prime}\right) \cdot\left(\vec{v}-\vec{v}^{\prime}\right)\right\rangle=\left\langle v^{2}\right\rangle-2\left\langle\vec{v} \cdot \vec{v}^{4}\right\rangle^{0}+\left\langle v^{\prime 2}\right\rangle=2\left\langle v^{2}\right\rangle \tag{8.18}
\end{equation*}
$$

as the velocities $\vec{v}$ and $\vec{v}^{\prime}$ are uncorrelated but share the same mean. Hence, the relative root-mean-square speed

$$
\begin{equation*}
\delta v_{\mathrm{rms}}=\sqrt{2} v_{\mathrm{rms}} \tag{8.19}
\end{equation*}
$$

is about $41 \%$ larger than the root-mean-square speed.
Two identical spheres will collide if they come within one diameter (or two radii) of each other. Thus, a sphere of diameter $d$ has interaction cross section $\sigma=\pi d^{2}$. More generally, cross section describes interaction probability in nuclear and elementary particle physics.


Figure 8.4: Graphical model of an ideal gas at room temperature with diameters, mean separations, and mean free path (red arrow) to scale.

If all but one molecules of a gas are at rest, in a time $t$, a molecule of diameter $d$ and cross section $\sigma=\pi d^{2}$ moving a distance $D=v_{\text {rms }} t$ sweeps out a cylindrical interaction volume $V_{i}=\sigma\left(v_{\mathrm{rms}} t\right)$ and interacts with $N_{c}=n V_{i}$ other molecules. Its mean free path is the distance travelled divided by the number of collisions,

$$
\begin{equation*}
\frac{D}{N_{c}}=\frac{D}{n V_{i}}=\frac{v_{\mathrm{rms}}}{n\left(\sigma y_{\mathrm{rms}}\right)}=\frac{1}{n \sigma} \tag{8.20}
\end{equation*}
$$

which is independent of the mean speed. However, if all the molecules are moving, then the interaction volume $V_{i}^{\prime}=\sigma\left(\delta v_{\mathrm{rms}} t\right)$ depends on the relative speed, and the mean free path

$$
\begin{equation*}
\ell=\frac{D}{N_{c}^{\prime}}=\frac{D}{n V_{i}^{\prime}}=\frac{v_{\mathrm{rms}} t}{n\left(\sigma \sqrt{2} v_{\mathrm{rms}} t\right)}=\frac{1}{n \sigma \sqrt{2}} \tag{8.21}
\end{equation*}
$$

decreases by about $29 \%$. Substituting the Eq. 8.14 number density implies

$$
\begin{equation*}
\ell=\frac{k_{B} T}{P \pi d^{2} \sqrt{2}} . \tag{8.22}
\end{equation*}
$$

At room temperature and pressure, air molecules with three hundred picometer diameters have about hundred nanometer mean free paths, as in Fig. 8.4 Hence,

$$
\begin{equation*}
\left(\frac{\ell}{100 \mathrm{~nm}}\right) \approx\left(\frac{T}{300 \mathrm{~K}}\right)\left(\frac{10^{5} \mathrm{~Pa}}{P}\right)\left(\frac{0.3 \mathrm{~nm}}{d}\right) . \tag{8.23}
\end{equation*}
$$

### 8.3 Compression \& Expansion

A gas can be compressed or expanded isothermally at constant temperature or adiabatically without heat exchange. If the temperature is constant, the Eq. 8.15 ideal gas implies that the product of pressure and volume $P V$ is constant, both initially and finally, so

$$
\begin{equation*}
P_{i} V_{i}=P_{f} V_{f}, \tag{8.24}
\end{equation*}
$$

and the pressure is inverse to the volume. Isothermal compression decreases the volume and increases the pressure, while isothermal expansion increases the volume and decreases the pressure.


Figure 8.5: Two isotherms (red) and one adiabat (blue). Bounded area is the work done during the compression and expansion of the gas.

If no heat is added or removed, a compression or expansion can change the temperature by increasing or decreasing the energy of the gas. Combine the Eq. 8.1 temperature definition with the Eq. 8.15 ideal gas law to write

$$
\begin{equation*}
P V=N k_{B} T=N \frac{2}{3}\langle K\rangle=\frac{2}{3} E, \tag{8.25}
\end{equation*}
$$

where $E=N\langle K\rangle$ is the total energy of the monatomic gas. Solve for the energy

$$
\begin{equation*}
E=\frac{3}{2} P V=\frac{P V}{\gamma-1} \tag{8.26}
\end{equation*}
$$

where $\gamma-1=2 / 3$ and the adiabatic index $\gamma=5 / 3$ for monatomic gases but is larger for diatomic and more complicated gases that can store energy in rotations and vibrations. Energy changes

$$
\begin{equation*}
\delta E=\frac{\delta P V+P \delta V}{\gamma-1} \tag{8.27}
\end{equation*}
$$

arise from pressure or volume changes.
From the Eq. 6.110 work-energy relation, the positive work done on the gas of initial pressure $P$ by the Fig. 8.3 piston during a small $\delta V<0$ compression increases its energy by

$$
\begin{equation*}
\delta E=N\langle\delta K\rangle=F_{x} \delta x=P A_{x} \delta x=-P \delta V>0 \tag{8.28}
\end{equation*}
$$

(with the same result to first order in small quantities using the final pressure $P+\delta P)$. Hence,

$$
\begin{equation*}
-P \delta V=\frac{\delta P V+P \delta V}{\gamma-1} \tag{8.29}
\end{equation*}
$$

Cross multiply

$$
\begin{equation*}
-\gamma P \delta V+P \delta V=\delta P V+P \delta V \tag{8.30}
\end{equation*}
$$

and separate variables to find

$$
\begin{equation*}
-\gamma \frac{\delta V}{V}=\frac{\delta P}{P} \tag{8.31}
\end{equation*}
$$

Integrate both sides

$$
\begin{equation*}
-\gamma \int_{V_{i}}^{V_{f}} \frac{d V}{V}=\int_{P_{i}}^{P_{f}} \frac{d P}{P} \tag{8.32}
\end{equation*}
$$

to get

$$
\begin{equation*}
-\gamma \log \frac{V_{f}}{V_{i}}=\log \frac{P_{f}}{P_{i}} \tag{8.33}
\end{equation*}
$$

Exponentiate both sides to show

$$
\begin{equation*}
\left(\frac{V_{i}}{V_{f}}\right)^{\gamma}=\left(\frac{V_{f}}{V_{i}}\right)^{-\gamma}=\frac{P_{f}}{P_{i}} \tag{8.34}
\end{equation*}
$$

or

$$
\begin{equation*}
P_{i} V_{i}^{\gamma}=P_{f} V_{f}^{\gamma} \tag{8.35}
\end{equation*}
$$

as in Fig. 8.5. Adiabatic compression increases the pressure faster than isothermal compression, while adiabatic expansion decreases the pressure faster than isothermal expansion.

### 8.4 Sound Speed

### 8.4.1 General Sound Speed

Sound waves are longitudinal waves of compression and expansion. Consider a pulse of elevated pressure and density moving at speed $v$ through a gas (or other elastic medium) of pressure $P$ and mass density $\rho$. In a coordinate system moving with the pulse, entry and exit pressure differences first decelerate and then accelerate packets of air, as in Fig. 8.6.


Figure 8.6: When moving with a compression wave, entry (yellow) pressure differences decelerate a packet of air, (gray).

Flow continuity requires that any mass that approaches entry must actually enter. If $A$ is the flow's cross sectional area, then

$$
\begin{equation*}
\frac{\delta m}{\delta t}=\frac{\delta m}{\delta x} \frac{\delta x}{\delta t}=\frac{\delta m}{A \delta x} A \frac{\delta x}{\delta t}=\rho A v \tag{8.36}
\end{equation*}
$$

is constant everywhere, so

$$
\begin{equation*}
\rho \AA \mathrm{A} v=\rho^{\prime} \mathrm{A} v^{\prime} \tag{8.37}
\end{equation*}
$$

which is the continuity equation. Hence,

$$
\begin{align*}
\rho \nsim & =(\rho+\delta \rho)(v+\delta v) \\
& =\rho \nsim+\rho \delta v+\delta \rho(v+\delta v) \tag{8.38}
\end{align*}
$$

and

$$
\begin{equation*}
-\rho \delta v=\delta \rho(v+\delta v) \approx \delta \rho v \tag{8.39}
\end{equation*}
$$

assuming the speed perturbation is small, $\delta v \ll v$. Thus, the fractional change in speed is opposite to the fractional change in density,

$$
\begin{equation*}
\frac{\delta v}{v}=-\frac{\delta \rho}{\rho} \tag{8.40}
\end{equation*}
$$

If a small mass $\delta m$ moves the small distance $\delta x$ in a small time $\delta t$, then Newton's second law at entry implies

$$
\begin{equation*}
\delta f_{x}=\delta m a_{x} \tag{8.41}
\end{equation*}
$$

or

$$
\begin{equation*}
(-\delta P \npreceq)=(\rho \delta x \notin)\left(\frac{\delta v}{\delta t}\right) \tag{8.42}
\end{equation*}
$$

where $\delta v<0$ and $a_{x}<0$. Hence the pressure difference

$$
\begin{equation*}
\delta P=-\rho \delta v \frac{\delta x}{\delta t} \approx-(-\delta \rho v) v=\delta \rho v^{2} \tag{8.43}
\end{equation*}
$$

by Eq. 8.40 . In the limit $\delta t \rightarrow 0$, the sound speed is the square root of the rate of change of pressure with density,

$$
\begin{equation*}
c_{s}=v=\sqrt{\frac{d P}{d \rho}} \tag{8.44}
\end{equation*}
$$

This result is proposition XLIX of Book II of Newton's Principia. The symbol $c_{s}$ for sound speed is in analogy with the symbol $c$ for the constant light speed. Sound speed depends on the gas's equation of state, which is its pressure as a function of its density $P[\rho]$.

### 8.4.2 Isothermal Sound Speed

Newton assumed that sound waves propagate isothermally. The Eq. 8.14 ideal gas law for molecules of mass $m$,

$$
\begin{equation*}
P=n k_{B} T=\frac{\rho}{m} k_{B} T \tag{8.45}
\end{equation*}
$$

implies the linear equation of state

$$
\begin{equation*}
P=C_{i} \rho \tag{8.46}
\end{equation*}
$$

where $C_{i}=k_{B} T / m$ is constant. Since $d P / d \rho=C_{i}=P / \rho$, the Eq. 8.44 sound speed

$$
\begin{equation*}
c_{s}=\sqrt{\frac{d P}{d \rho}}=\sqrt{\frac{P}{\rho}} \tag{8.47}
\end{equation*}
$$

which is about $15 \%$ low for normal temperature and pressure.

### 8.4.3 Adiabatic Sound Speed

A generation later, Pierre-Simon Laplace improved Newton's estimate by arguing that sound waves propagate adiabatically rather than isothermally. In practice, the rapidity of sound fluctuations leaves insufficient time for energy
flow to equalize temperatures. The Eq. 8.35 adiabatic gas law for molecules of mass $m=M / N$,

$$
\begin{equation*}
P V^{\gamma}=P_{0} V_{0}^{\gamma} \tag{8.48}
\end{equation*}
$$

implies the nonlinear equation of state

$$
\begin{equation*}
P=P_{0}\left(\frac{V_{0}}{V}\right)^{\gamma}=\frac{P_{0} V_{0}^{\gamma}}{M^{\gamma}} \frac{M^{\gamma}}{V^{\gamma}}=C_{a} \rho^{\gamma} \tag{8.49}
\end{equation*}
$$

where $C_{a}$ is constant. Since $d P / d \rho=\gamma C_{a} \rho^{\gamma-1}=\gamma P / \rho$, the Eq. 8.44 sound speed

$$
\begin{equation*}
c_{s}=\sqrt{\frac{d P}{d \rho}}=\sqrt{\frac{\gamma P}{\rho}} \tag{8.50}
\end{equation*}
$$

Insert the Eq. 8.14 ideal gas law and the Eq. 8.1 temperature definition to find

$$
\begin{equation*}
c_{s}=\sqrt{\gamma \frac{\not x k_{B} T}{\not n m}}=\sqrt{\gamma \frac{\frac{2}{3}\left\langle\frac{1}{2} m v^{2}\right\rangle}{\not n}}=\sqrt{\frac{\gamma}{3}} v_{\mathrm{rms}} \lesssim v_{\mathrm{rms}} \tag{8.51}
\end{equation*}
$$

as $\gamma=5 / 3$ for monatomic gasses and $\gamma \approx 1.4$ for the mix of gasses in air. Thus, sound speed is comparable to the root-mean-square speed of atoms or molecules in air. Under normal temperature and pressure, the $c_{s}=340 \mathrm{~m} / \mathrm{s}=$ 1200 kph sound speed is faster than driving but much slower than the $c=$ 1100000000 kph light speed.

## Problems

1. Assume an ideal gas at room temperature and pressure.
(a) Compute the number of molecules in a cubic millimeter.
(b) Compute the average distance between molecules as a multiple of their diameters, which are about 0.3 nm .
(c) Compute the average distance a molecule travels between collisions as a multiple of its diameter.
2. Adiabatically compressing a gas increases its temperature. By how much does the speed of a molecule in a cylinder increase in a direct "head-on" collision with a piston moving inward with speed $v_{p}$ ? Hint: Boost to the frame of reference of the moving piston, reflect the molecule, and boost back.

## Appendix A

## Notation

Table A.1 summarizes the symbols of this text. Some symbols are more universal then others.

Standard mathematics notation suffers from a serious ambiguity involving parentheses. In particular, parentheses can be used to denote multiplication, as in $a(b+c)=a b+a c$ and $f(g)=f g$, or they can be used to denote functions evaluated at arguments, as in $f(t)$ and $g(b+c)$. It can be a struggle to determine the intended meaning from context.

To avoid ambiguity, this text always uses round parentheses ( $\bullet$ ) to group for multiplication and square brackets $[\bullet]$ to list function arguments. Thus, $a(b)=a b$ denotes the product of two factors $a$ and $b$, while $f[x]$ denotes a function $f$ evaluated at an argument $x$. Mathematica [15] employs the same convention.

Table A.1: Symbols used in this text.

| Quantity | Symbol | Alternates | Units |
| :---: | :---: | :---: | :---: |
| total energy | E | $E_{T}, U$ | $\mathrm{J}=\mathrm{kg} \mathrm{m}^{2} / \mathrm{s}^{2}$ |
| kinetic energy | $K$ | $T, E_{K}$ |  |
| potential energy | $U$ | $V, E_{P}$ |  |
| Lagrangian | $\mathcal{L}$ |  |  |
| Action | $\mathcal{A}$ | $S, I, A$ | $\mathrm{J} \mathrm{s}=\mathrm{J} / \mathrm{Hz}$ |
| space or position | $s$ | $x$ | m |
| time | $t$ | $\tau$ | S |
| velocity | $v$ | $V$ | $\mathrm{m} / \mathrm{s}$ |
| acceleration | $a$ | $A$ | $\mathrm{m} / \mathrm{s}^{2}=\mathrm{N} / \mathrm{kg}$ |
| mass | $m, M$ | $\mu$ | kg |
| momentum | $p$ | $\pi$ | $\mathrm{kg} \mathrm{m} / \mathrm{s}$ |
| force | $f, F$ | $\varphi$ | $\mathrm{N}=\mathrm{kg} \mathrm{m} / \mathrm{s}^{2}$ |
| rotational inertia | $I$ |  | $\mathrm{kg} \mathrm{m}{ }^{2}$ |
| angular momentum | $L$ | $J$ | $\mathrm{kg} \mathrm{m}{ }^{2} / \mathrm{s}=\mathrm{J} \mathrm{s}$ |
| torque | $\tau$ | $M$ | $\mathrm{kg} \mathrm{m}{ }^{2} / \mathrm{s}^{2}=\mathrm{Nm}$ |
| work | W | $w$ | $\mathrm{kg} \mathrm{m}{ }^{2} / \mathrm{s}^{2}=\mathrm{J}$ |
| impulse | $J$ | $I$ | $\mathrm{kg} \mathrm{m} / \mathrm{s}$ |
| basis vectors | $\hat{x}, \hat{y}, \hat{z}$ | $\hat{x}_{1}, \hat{x}_{2}, \hat{x}_{3}$ |  |
| pseudoscalar | $\mathcal{I}=\hat{x} \hat{y} \hat{z}$ |  |  |
| spring stiffness | $\kappa$ | $k$ | N/m |
| length | $\ell$ | $l, L$ | m |
| friction coefficient | $\mu$ | $\mu_{s}, \mu_{k}$ |  |
| wavelength | $\lambda$ | $\Lambda$ | m |
| period | $T$ | $\tau, P$ |  |
| frequency | $f$ | $\nu$ | $\mathrm{Hz}=1 / \mathrm{s}$ |
| angular frequency | $\omega$ | $\Omega$ | $\mathrm{rad} / \mathrm{s}=1 / \mathrm{s}$ |
| spatial frequency | $k$ | $\kappa$ | $1 / \mathrm{m}$ |
| temperature | $T$ | $\tau$ | K |
| pressure | $P$ | $p$ | $\mathrm{Pa}=\mathrm{N} / \mathrm{m}^{2}$ |
| volume | $V$ | $v$ | $\mathrm{m}^{3}$ |
| number | $N$ | $n$ |  |
| number density | $n$ | $N$ | $1 / \mathrm{m}^{3}$ |
| mass density | $\rho$ | $D$ | $\mathrm{kg} / \mathrm{m}^{3}$ |
| cross section | $\sigma$ | $a$ |  |
| mean free path | $\ell$ | $\lambda$ | m |

## Appendix B

## Measure \& Angles

Figure B.1 summarizes geometric measures of a sphere, and Table B.1 compares planar and solid angles.


Figure B.1: Similar but distinct formulas for the CAVS - circumference $C$, equatorial area $A$, volume $V$, and surface area $S$ - of a sphere of radius $R$.

Table B.1: Planar and solid angles in popular notation.

| Angles | Solid Angles |
| :---: | :---: |
| $\begin{aligned} & 0 \leq \ell \leq 2 \pi r \\ & 0 \leq \theta=\frac{\ell}{r} \leq 2 \pi \end{aligned}$ | $\begin{aligned} & 0 \leq A \leq 4 \pi r^{2} \\ & 0 \leq \Omega=\frac{A}{r^{2}} \leq 4 \pi \end{aligned}$ |
| radians $1 \mathrm{rad}=\frac{180}{\pi} \operatorname{deg}$ | steradians $1 \mathrm{sr}=\left(\frac{180}{\pi}\right)^{2} \operatorname{deg}^{2}$ |

## Appendix C

## Bibliography

[1] The National Aeronautics and Space Administration.
[2] Galileo Galilei, Dialogo sopra i due massimi sistemi del mondo [A Dialogue Concerning Two Chief World Systems], Giovanni Battista Landini, Florence (1632).
[3] Roger Bach, Damian Pope, Sy-Hwang Liou, and Herman Batelaan, "Controlled double-slit electron diffraction", New J. Phys., 15033018 (2013).
[4] Richard P. Feynman, Robert B. Leighton, Matthew Sands, The Feynman Lectures on Physics (Addison-Wesley, 1963), Volume II, page 1-6.
[5] Tomás Tycrskaá, "The de Broglie hypothesis leading to path integrals, Eur. J. Phys. 17, 156-158 (1996).
[6] C. G. Gray, Edwin F. Taylor, "When action is not least", Am. J. Phys. 75, 434-458 (2007).
[7] Isaac Newton, Philosophice Naturalis Principia Mathematica [Mathematical Principles of Natural Philosophy], J. Societatis Regiae ac Typis J. Streater, London (1687).
[8] Jon Ogborn, Edwin F. Taylor, "Quantum physics explains Newtons laws of motion", Phys. Educ. 40, 26-34 (2005).
[9] Dwight E. Neuenschwander, Edwin F. Taylor, Slavomir Tuleja, "Action: Forcing Energy to Predict Motion", Phys. Teacher 44, 146-152 (2006).
[10] Jozef Hanca, Edwin F. Taylor, Slavomir Tulejac, "Deriving Lagranges equations using elementary calculus", Am. J. Phys. 72, 510-513 (2004).
[11] Leonard Susskind, George Hrabovsky, The Theoretical Minimum: What You Need to Know to Start Doing Physics (Basic Books, 2013).
[12] Alan Cromer, "Stable solutions using the Euler approximation", Am. J. Phys. 48, 455-459 (1981).
[13] Emmy Noether, "Invariante Variationsprobleme [Invariant Variation Problems]". Nachr. d. Knig. Gesellsch. d. Wiss. zu Gttingen, Math-phys. Klasse, 235257 (1918)..
[14] Chris Doran \& Anthony Lasenby, Geometric Algebra for Physicists. Cambridge University Press, 2003.
[15] Wolfram Research, Inc., Mathematica, Version 9.0, Champaign, IL (2010).
[16] Photograph by Andrew Dunn (2004 November 5). Published under the Creative Commons Attribution-Share Alike 2.0 Generic license.
[17] Christoph Schmid, "Newtons superb theorem: An elementary geometric proof", Am. J. Phys. 79, 536 (2011).
[18] John F. Lindner, Jacob Lynn, Frank W. King, and Amanda Logue, "Order and chaos in the rotation and revolution of a line segment and a point mass", Phys. Rev. E 81, 036208 (2010).
[19] Arthur C. Clarke, "Extra-terrestrial relays: Can rocket stations give worldwide radio coverage?", Wireless World, 305-308 (October 1945).

