A Deeper Look into the Weakly Damped Oscillator: Exploring the Properties of Galileo's Physical Pendulum

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In this lab we examined different properties of a physical pendulum. We tested the effect of a physical pendulum's initial displacement on its period, and found that there was a positive relationship at higher angles. For this reason, using smaller angles to find the period of our physical pendulum, we determined it to have a value of $T = (1.71 \pm 0.04)$ s. Our period and our pendulum's moment of inertia were used to find Earth's gravitational field: $g = (7.74 \pm 0.04)$ m/s^2. Since the expected value was $g = 9.801$ m/s^2, we have a percent error of 26.75 % and a percent difference of 23.60 %. Finally, we determined our decay parameter, characteristic time, and damping constant to be $\gamma = (8.25 \pm 3.86) \times 10^{-3}$ 1/s, $t_c = (1.21 \pm 0.01) \times 10^2$ s, and $b = (2.64 \pm 0.19) \times 10^{-3}$ kg/s, respectively.

I. INTRODUCTION

The work of Galileo Galilei was revolutionary, and some argue that his studies of the pendulum are what earned him the title of the "the father of modern science." In particular, Galileo was interested in discovering what were fundamental properties of matter. His claim in 1590 was that a balance could be used to treat "heaviness" as a property of all matter. [1] But, what about pendulums in particular? Between the years of 1603 and 1604, Galileo would work with pendulums to study another property of matter and motion: acceleration. [1] In this, he would argue that time itself is a variable in problems of motion. This unifying theory of matter, a theory stating that matter can be described by different properties and equations of motion, is what would later be used to shift the understanding of the solar system from geocentric to heliocentric, Galileo himself using his findings to eventually come out in support of the Copernican theory. [1]

One of his most important takeaways was that the period of the pendulum was independent of his amplitude. [2] We now know this to be half-true, as the equations for the period of a physical pendulum apply much better to smaller angles, where the amplitude is roughly equal to its sine. [3]

In this lab we will be analyzing the motion of a physical pendulum, rather than a simple pendulum. While a simple pendulum can be idealized as a point mass on a string, the physical pendulum is a little bit more complicated. Unlike the simple pendulum, the shape of the physical pendulum must be taken into account in its equation of motion. [4]

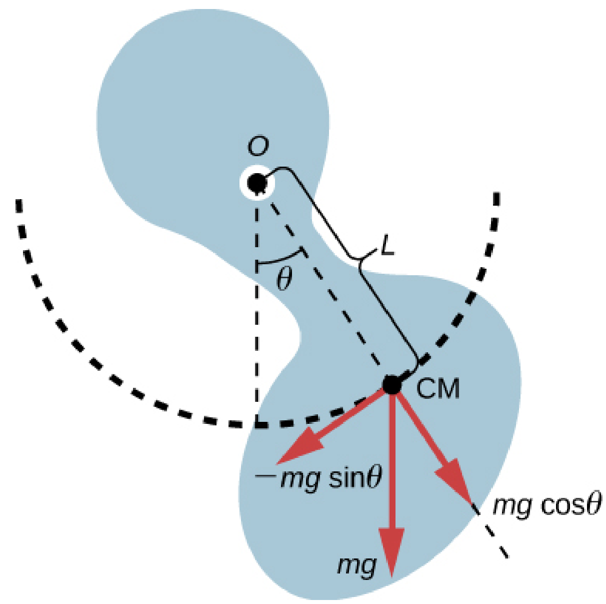

FIG. 1: A physical pendulum oscillating under the restoring force, gravity, acting on its center of mass (from [4]).

II. THEORY

Just how different is a physical pendulum from a simple pendulum? It is essential to understand that in order for a system to exhibit oscillatory motion, it needs to have a restoring force working to return the system to its equilibrium position. In the case of both, the simple pendulum and the physical pendulum, this restoring force is the force gravity. In a simple pendulum, this restoring force acts on the center of a spherical bob, since the mass of the string can be assumed negligible. However, when it comes to a physical pendulum, the placement of the object's center of mass (on which the restoring force acts) is not nearly as obvious. [4] A diagram of a physical pendulum can be seen in Fig. 1.

Consider a hanging object that is free to oscillate about a fixed point. We already know that the force restoring the object back to its equilibrium position is the force of gravity, so what is actually keeping the object in motion? Why does gravity have to work so hard to continuously put the object back in its equilibrium position? This is because there is a torque being applied about the object's center of mass! [4]

The equation for torque $\vec{\tau}$ is given by

$$\vec{\tau} = \vec{r} \times \vec{F} \ , \tag{1}$$

with $\vec{r}$ as the radius (the distance between the axis of rotation and the center of mass) and $\vec{F}$ as the component of the force tangental to the motion. The torque has a magnitude $|\tau|$ given by

$$|\tau| = rF \sin\theta \ , \tag{2}$$

wherein θ represents the angle of displacement from equilibrium. In Fig. 1, it is easy to see how the tangential component of the force is $-mg\sin\theta$, with g denoting the gravitational field 9.801 m/s^2. This value g was chosen using a local gravity calculator. [5]

The small angle theorem states that $\sin\theta \approx \theta$, meaning that at small angles, the sine of an angle is approximately equal to the angle itself. So, it is for this reason that we can simplify $-mg\sin\theta$ into $-mg\theta$. Furthermore, the net torque τ_{net} is given by

$$\tau_{\text{net}} = I\ddot{\theta} \ , \tag{3}$$

and

$$\tau_{\text{net}} = -mgL\theta \ , \tag{4}$$

where we have equated the radius r with the length L between the object's point of rotation and its center of mass. $\ddot{\theta}$ is the second derivative of the angle θ of displacement with respect to time. This is simply a fancier and more convenient way of writing angular acceleration. I denotes the moment of inertia about a pendulum's axis of rotation.

We can rewrite Eqs. (3) and (4) as

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{mgL}{I}\theta \ . \tag{5}$$

We can take note that the second derivative of the angle θ of displacement is equal to itself times a negative constant, just like the cosine and sine functions. It follows that the general solution is

$$\theta(t) = \theta_0 \cos(\omega t + \phi) \ . \tag{6}$$

The angle θ_0 represents the amplitude, or the initial angle from displacement, ω is the angular frequency, and ϕ is a phase shift. [4]

Since the angular frequency ω is given by

$$\omega = \sqrt{\frac{mgL}{I}} \ , \tag{7}$$

and the period T is given by

$$T = \frac{2\pi}{\omega} \ , \tag{8}$$

we can rewrite the period as

$$T = 2\pi\sqrt{\frac{I}{mgL}} \ . \tag{9}$$

This is why, in a simple pendulum, with $I = mL^2$, the period simplifies to [4]

$$T = 2\pi\sqrt{\frac{L}{g}} \ . \tag{10}$$

Unfortunately, the case is not quite as simple for the physical pendulum. Furthermore, we also have to take into account other factors such as friction and air resistance. These are called damping forces.

As one might guess, the introduction of damping forces makes things a little more complicated, but it is nothing we cannot handle. When an object is moving through a resistive fluid, such as air, the resistance is dependent on the object's velocity. [6] In our case, we can reasonably approximate that this force of resistance $\vec{f}$ is proportional to our velocity $\vec{v}$. Though, it is important to note that there are other instances where the resistance force is instead proportional to v^2. [6] Luckily, we are dealing with the first case. The equation for our resistive force is given by

$$\vec{f} = -b\vec{v} \ . \tag{11}$$

An easy way to derive the damped oscillator equation is by considering another form of oscillatory motion: spring motion. If we consider an oscillating mass attached to a spring, the mass is acting under two forces: the spring force $-kx$ and the resistive force $-b\dot{x}$. Note that $\dot{x}$ is just another way to denote velocity. The equation of the net force $F_{\text{net}} = m\ddot{x}$ is then

$$m\ddot{x} = -b\dot{x} - kx \ , \tag{12}$$

which can be rearranged into the form

$$m\ddot{x} + b\dot{x} + kx = 0 \ . \tag{13}$$

To get our equation of motion, we can rewrite Eq. (13) as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \ , \tag{14}$$

with $2\beta = (b/m)$ and $\omega_0^2 = (k/m)$.

Fascinatingly, even though this equation comes from Hooke's law, the relationship still holds for other types of linear, second-order, homogeneous equations. [6] This means that we are able to apply this equation to our pendulum! It also means that if we have two independent solutions $x_1(t)$ and $x_2(t)$, then any solution must come in the form of $C_1x(t) + C_2x(t)$. [6]

I ask that one takes our word for it, that these two independent solutions are the functions $\mathrm{e}^{r_1 t}$ and $\mathrm{e}^{r_2 t}$, with a general solution of

$$x(t) = C_1\mathrm{e}^{r_1 t} + C_2\mathrm{e}^{r_2 t} \ , \tag{15}$$

wherein

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2} \tag{16}$$

and

$$r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2} \ . \tag{17}$$

.

In this lab, the type of damping we are working with is weak damping, also called underdamping. In this case, our decay parameter β is small and therefore less than ω_0^2. This would mean that our square roots in Eqs. (16) and (17) become imaginary. It is to our benefit to rename the square root portions of these equations to

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} \ . \tag{18}$$

We can now write Eq. (15) as

$$x(t) = e^{-\beta t}(C_1\mathrm{e}^{i\omega_1 t} + C_2\mathrm{e}^{-i\omega_1 t}) \ . \tag{19}$$

Even better, we can write Eq. (19) as

$$x(t) = A\mathrm{e}^{-\beta t}\cos{(\omega_1 t - \phi)} \ . \tag{20}$$

Eq. (20) is a simple harmonic motion equation, but with an amplitude $A\mathrm{e}^{-\beta t}$ that decreases exponentially. [6] The frequency is denoted by ω_1 and ϕ is a phase shift.

We can quantify this rate of decay by finding the characteristic time t_c, which is the time it takes for the amplitude to decay to (1/e) of the initial amplitude A. From Eq. (20), we can see that this characteristic time is going to occur at

$$t_\mathrm{c} = \frac{1}{\beta} \ . \tag{21}$$

A visualization of this can be seen in Fig. 2.

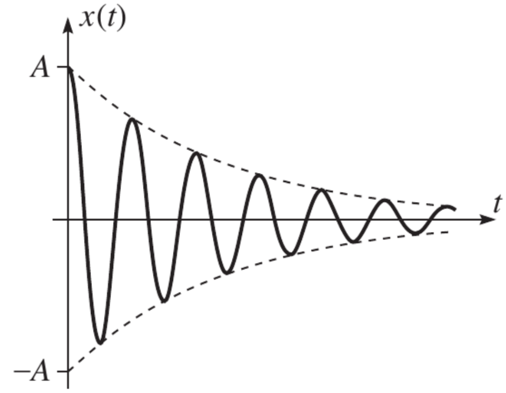

FIG. 2: A graph of position versus time of a weakly damped oscillator, with the dashed lines representing the envelope function (from [6]).

To apply all of this to our pendulum situation, there is one final tweak to make. We need to convert this linear equation to fit our radial situation. Keeping in mind that mass is linear moment of inertia, Eqs. (14) and (20) become our newer, radial versions

$$I\ddot{\theta} + c\dot{\theta} + k\sin\theta = 0 \tag{22}$$

and

$$\theta = \theta_0\mathrm{e}^{-\gamma t}\cos{(w_1' t - \phi)} \ , \tag{23}$$

wherein

$$\omega_1' = \sqrt{\omega_0'^2 - \gamma^2} \tag{24}$$

and

$$\omega_0' = \sqrt{\frac{k}{I}} \ . \tag{25}$$

Our new constant c has a value bh^2 and k has a value of mgh. Our radial decay parameter is defined as

$$\gamma = \frac{c}{2I} \ . \tag{26}$$

One way we can test the validity of our data is by using Eq. (9) and rearranging it as

$$g = \frac{4\pi^2 I}{T^2 mL} \ . \tag{27}$$

Once we have arranged Eq. (9) into this form, we can plug in our experimental values and constants to solve for the gravitational constant g. We can compare our result to the actual, accepted value of $g = 9.801$ m/s^2. However, before we can do this, we must solve for our moment of inertia.

For a slab rotating about a perpendicular axis through its center of mass, the moment of inertia about its center of mass I_{CM} is given by [7]

$$I_{\mathrm{CM}} = \frac{m(y^2+z^2)}{12} \; . \tag{28}$$

The values y and z are the length and width of the meter stick, respectively.

Since Eq. (28) gives us the moment of inertia I_{CM} about the center of mass, we need to use the parallel axis theorem to find the moment of inertia $I_{||}$ about the axis of rotation.

The parallel axis theorem states that

$$I_{||} = I_{\mathrm{CM}} + mL^2 \; , \tag{29}$$

with L as the distance between the two axes.

Once we have checked the validity of our results, we can move on to finding the value of our damping constant b. The envelope function, which is represented by the dashed line in Fig. 2, is given by

$$\theta(t) = \theta_0 e^{-\gamma t} \; . \tag{30}$$

If we take the natural logarithm of both sides we get the following:

$$\ln(\theta(t)) = -\gamma t + \ln(\theta_0) \; . \tag{31}$$

Using our collected data, we can plot the natural logarithms of our pendulum's maximum and minimum deviations from equilibrium versus time. [2] This allows us to quantify our rate of decay γ, as it will be the negative slope of our semi-log plot from Eq. (31).

The relationship between characteristic time t_{c} and the linear decay parameter β is outlined in Eq. (21). Similarly, the relationship between the characteristic time t_{c} and the radial decay rate is

$$t_{\mathrm{c}} = \frac{1}{\gamma} \; , \tag{32}$$

as evidenced in Eq.(23).

Finally, by recalling Eq. (26), we can find our damping constant b tp be

$$b = \frac{2I\gamma}{L^2} \; . \tag{33}$$

III. PROCEDURE

In this lab, we used a rotary motion sensor to measure the oscillations of a physical pendulum, collecting the angle of displacement as a function of time and recording it on PASCO Capstone, our data-recording software. The physical pendulum we used in this lab was a meter stick with a (60.0 ± 0.1) g sliding mass attached, as pictured in Fig. 3.

FIG. 3: Our physical pendulum, composed of a meter stick with a (60.0 ± 0.1) g sliding mass attached at (90.0 ± 0.1) cm.

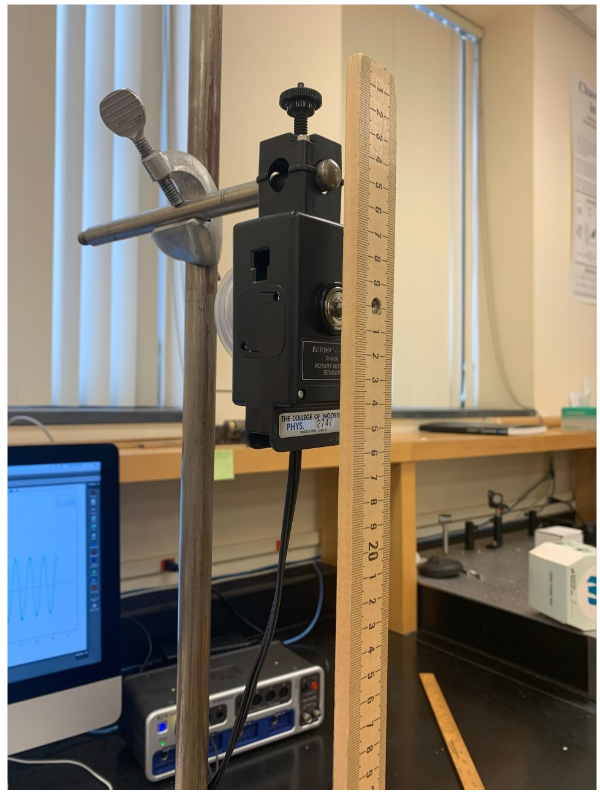

FIG. 4: Rotary motion sensor with the physical pendulum attached. The physical pendulum is in its equilibrium position $\theta = 0$ rad.

Our rotary motion sensor can be seen in Fig. 4. It connects to a computer, which collects the recorded data. The rotary motion sensor has a rotating knob (the potentiometer shaft) on which an object can be attached. The attached object is free to rotate clockwise and counterclockwise on the knob. The rotary motion sensor sensor tracks it as it does so, recording the changes in radial displacement from the position it was in at the start of a recording.

We attached our physical pendulum to the potentiometer shaft through a hole in the meter stick at

(90.0 ± 0.1) cm. Before we started each new recording, we first made sure that the pendulum was resting at its equilibrium position, $\theta = 0$ rad, in order to get accurate measurements.

Once the recording started, we raised our physical pendulum to our desired initial displacement θ_0 from equilibrium and let it go. The PASCO Capstone software recorded and graphed the displacement θ versus time t.

The number of measurements that our radial motion sensor took was customizable. When we were finding values of our period T as a function of different initial displacements θ_0, we had our counts set to 20 Hz. This means that our radial motion sensor took twenty measurements for every one second. We manually recorded the first two peaks in each trial and calculated the time in between them, obtaining the period T and enabling us to graph the period as a function of initial displacement in IgorPro.

Later, when we wanted more data to find the envelope function of our damped pendulum, we changed the counts to 50 Hz. During this process, we needed more than just the first two data peaks. After we lifted the pendulum to our desired initial displacement and released it, we would let it run for a minute or two and manually collected as many peaks as the data provided. Once we had the values of these different peaks we made a semi-log plot in IgorPro so that we could quantify the decay parameter γ outlined in Eq. (31).

The meter stick and the sliding mass had a combined mass of The meter stick with the sliding mass attached has a combined mass of $(1.563 \pm 0.014) \times 10^{-1}$ kg. To find the center of mass, we balanced the meter stick on a knife-edge and found the spot where it was balanced. The center of mass was found to be located at $(6.590 \pm 0.005) \times 10^{-1}$ m. This means that the distance L between the point of rotation and the center of mass was $(5.600 \pm 0.015) \times 10^{-1}$ m.

IV. RESULTS & ANALYSIS

A. The Effect of Initial Amplitude on Period

Galileo Galilei would have argued that the period of a pendulum is the same regardless of the initial amplitude. He would be wrong, of course, but in all fairness he was from the 1500s. Still, it is a good idea to put this to the test! What happens if we record different periods for different initial amplitudes.

We measured the period T as a function of initial displacement θ_0 from the pendulum's equilibrium position, and the results can be seen in Fig. 5. The error bars in the vertical direction were determined to be 0.05 s and the error bars in the horizontal direction were determined to be 0.005 rad.

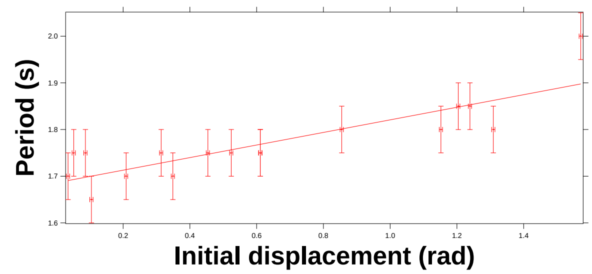

FIG. 5: Period T vs. initial displacement θ_0 from the equilibrium position. The slope has a value of (0.135 ± 0.021) s/rad. This means that the period of a physical pendulum is not independent of its initial displacement.

Fig. 5 shows that the period T does indeed seem to change as the initial displacement θ_0 from the pendulum's equilibrium position gets larger. It is also for this reason that we elected to use only initial displacements θ_0 with a value under $(\pi/9)$ rad (20°) to determine our period T. This is because when the angle of initial displacement θ_0 is small (the sine of the angle is roughly equal to the angle itself), the period stays relatively constant. Working with initial displacements in this small-angle-range is also more relevant to finding the damping constant b.

The period T of our physical pendulum was determined to be (1.71 ± 0.04) s by taking the mean of our seven measurements. We found the absolute error ΔT by calculating the standard deviation:

$$\Delta T = \sqrt{\frac{1}{N-1}\sum_{n-1}^{N}(T_n - \bar{T})^2}\,. \tag{34}$$

The value N represents the number of trials and $\bar{T}$ is the mean of our different period measurements. This gives us an absolute error ΔT of 0.04 s.

We can also obtain a value for the relative error, which is simply the uncertainty of a measurement divided by said measurement. In our case, our period T has a relative error $(\Delta T/T)$ of 2.16×10^{-2}.

B. Finding the Gravitational Constant g Using Moment of Inertia and the Period

In order to try and find a value for Earth's gravitational field g, we first need a value for our moment of inertia I about the pendulum's rotation point. However, before we can do that, we must first find the moment of inertia about our pendulum's center of mass.

The mass of our physical pendulum was found to be $(1.56 \pm 0.02) \times 10^{-1}$ kg. The pendulum measured at (1.00 ± 0.01) m long and $(2.50 \pm 0.10) \times 10^{-2}$ m wide. Using Eq. (28), we find that our physical pendulum's moment of inertia about its center of mass I_{CM} is $(1.30 \pm 0.13) \times 10^{-3}$ kg $\cdot$ m^2.

Since all three of our variables have their own respective uncertainties, in order to find the uncertainty of the moment of inertia I, we must take into account how all of those errors propagate, and this gives us our absolute error ΔI_{CM}. This is given by the equation

$$\Delta I_{\text{CM}} = \sqrt{\left(\frac{\partial\left(\frac{m(y^2+z^2)}{12}\right)}{\partial m}\Delta m\right)^2 + \left(\frac{\partial\left(\frac{m(y^2+z^2)}{12}\right)}{\partial y}\Delta y\right)^2 + \left(\frac{\partial\left(\frac{m({}^{y}2+z^2)}{12}\right)}{\partial z}\Delta z\right)^2} \tag{35}$$

$$= \sqrt{\left(\frac{y^2+z^2}{12}\Delta m\right)^2 + \left(\frac{my}{6}\Delta y\right)^2 + \left(\frac{mz}{6}\Delta z\right)^2} \tag{36}$$

$$= \frac{m(y^2+z^2)}{12}\sqrt{\left(\frac{\Delta m}{m}\right)^2 + \left(\frac{2y}{y^2+z^2}\Delta y\right)^2 + \left(\frac{2z}{y^2+z^2}\Delta z\right)^2}, \tag{37}$$

which outputs an absolute error of 0.13×10^{-3} km $\cdot$ m^2 and a relative error of 1.02×10^{-1}.

Once we have a value for the moment of inertia I_{CM} about the pendulum's center of mass, we can use the parallel axis theorem as defined in Eq. (29) to find the moment of inertia I about our physical pendulum's axis of rotation.

Since the moment of inertia I_{CM} about the pendulum's center of mass is $(1.30 \pm 0.13) \times 10^{-3}$ kg $\cdot$ m^2, the pendulum has a mass m of $(1.56 \pm 0.07) \times 10^{-1}$ kg, and the distance L between the axis of rotation and the center of mass is $(5.60 \pm 0.02) \times 10^{-1}$ m, Eq. (29) tells us that the moment of inertia I becomes $(5.03 \pm 0.06) \times 10^{-2}$ kg $\cdot$ m^2 at the pendulum's axis of rotation.

The absolute error of this measurement is given by

$$\Delta I = \sqrt{\left(\frac{\partial\left(I_{\text{CM}}+mL^2\right)}{\partial I_{\text{CM}}}\Delta I_{\text{CM}}\right)^2 + \left(\frac{\partial\left(I_{\text{CM}}+mL^2\right)}{\partial m}\Delta m\right)^2 + \left(\frac{\partial\left(I_{\text{CM}}+mL^2\right)}{\partial L}\Delta L\right)^2} \tag{38}$$

$$= \sqrt{(\Delta I_{\text{cm}})^2 + (L^2\Delta m)^2 + (2mL\Delta L)^2} \tag{39}$$

$$= (I_{\text{CM}}+mL^2)\sqrt{\left(\frac{\Delta I_{\text{CM}}}{I_{\text{CM}}+mL^2}\right)^2 + \left(\frac{L^2}{I_{\text{CM}}+mL^2}\Delta m\right)^2 + \left(\frac{2mL}{I_{\text{CM}}+mL^2}\Delta L\right)^2}, \tag{40}$$

and it yields a value of 0.06×10^{-2} kg $\cdot$ m^2. Consequentially, the relative error $(\Delta I/I)$ has a value of 1.15×10^{-2}.

Now that we have a value for our pendulum's moment of inertia about its rotation axis, we are finally able to use Eq. (27) to find a calculated value of Earth's gravitational field g and compare it to the accepted value $g = 9.807$ m/s^2.

Given that our moment of inertia I is $(5.03 \pm 0.06) \times 10^{-2}$ kg $\cdot$ m^2, our period T is 1.71 ± 0.04 s, our mass m is $(1.56 \pm 0.02) \times 10^{-1}$ kg, and our distance L between the center of mass and the axis of rotation is $(5.60 \pm 0.02) \times 10^{-1}$ m, Eq. (27) outputs a value of $g = 7.74 \pm 0.37$ m/s^2. This is, of course, a little off, but there are definitely a few factors that could explain this. Namely, the fact that the sliding mass on our physical pendulum was idealized as a point so that we could still treat its shape as a rectangular slab. Furthermore, the relationship we tested was ideal, and had no room to take into account real-world setbacks such as friction and air resistance.

We found the absolute error Δg for our measured value g to be 0.37 m/s^2 with the equation

$$\Delta g = \sqrt{\left(\frac{\partial\left(\frac{4\pi^2 I}{T^2 mL}\right)}{\partial I}\Delta I\right)^2 + \left(\frac{\partial\left(\frac{4\pi^2 I}{T^2 mL}\right)}{\partial T}\Delta T\right)^2 + \left(\frac{\partial\left(\frac{4\pi^2 I}{T^2 mL}\right)}{\partial m}\Delta m\right)^2 + \left(\frac{\partial\left(\frac{4\pi^2 I}{T^2 mL}\right)}{\partial L}\Delta L\right)^2} \tag{41}$$

$$= \sqrt{\left(\frac{4\pi^2}{T^2 mL}\Delta I\right)^2 + \left(-\frac{8\pi^2 I}{T^3 mL}\Delta T\right)^2 + \left(-\frac{4\pi^2 I}{T^2 m^2 L}\Delta m\right)^2 + \left(-\frac{4\pi^2 I}{T^2 mL^2}\Delta L\right)^2} \tag{42}$$

$$= \frac{4\pi^2 I}{T^2 mL}\sqrt{\left(\frac{\Delta I}{I}\right)^2 + \left(-\frac{2}{T}\Delta T\right)^2 + \left(-\frac{\Delta m}{m}\right)^2 + \left(-\frac{\Delta L}{L}\right)^2}\ . \tag{43}$$

It follows that we also get a relative error ($\Delta g/g$) of 4.71×10^{-2}.

One of the better ways to get an idea of how far off our measured value is from the accepted value is to take the percent error and the percent difference, which are given by the equations

$$\%_{\text{error}} = \frac{x_{\text{measured}} - x_{\text{accepted}}}{x_{\text{accepted}}} \times 100\%\ , \tag{44}$$

and

$$\%_{\text{difference}} \frac{|x_{\text{exp}} - x_{\text{theor}}|}{\frac{(x_{\text{exp}}+x_{\text{theor}})}{2}} \times 100\%\ . \tag{45}$$

Using this method, we get a percent error of 26.75 % and a percent difference of 23.60 %. While this is not terrible, and it shows that our results are still within the same ballpark of the actual, accepted value of $g = 9.807$ m/s^2, percentages in the twenty-percentage area still indicate plenty of room for improvement. So, while this value gives us insight into the relationship between the period of a pendulum T, Earth's gravitational field g, the moment of inertia I, the mass m, and the length L of the pendulum arm, I would not trust anything made by an engineer who used $g = 7.74$ m/s^2 in her calculations.

C. Finding the Damping Constant b

In order to find the damping constant b, there are a few steps we must take beforehand. The first order of business is taking different trials for different small-angle initial displacements θ_0 from the pendulum's equilibrium position. In these trials, we collected the different local maxima of displacement θ from equilibrium versus time t. We then took the natural logarithm of the local maxima θ to create a linearized, semi-log plot of these different trials, which can be seen in Fig. 6.

Fig. 6 shows a plot of Eq. (31) for different starting displacements θ_0. The slopes are the negative decay parameter γ, which we can use to find the damping constant b.

The negative slope of the green, top-most, left-most plot is $\gamma_{\text{green}} = (1.04 \pm 0.02) \times 10^{-2}$ 1/s. The green plot had an initial displacement of $\theta_0 = (0.56 \pm 0.01)$ rad. The negative slope of the purple plot, right below the previous, is $\gamma_{\text{purple}} = (6.29 \pm 0.19) \times 10^{-3}$ 1/s. This plot had an initial displacement of $\theta_0 = (0.31 \pm 0.01)$ rad. The negative slope of the blue plot, which is directly to the right of the purple plot, was $\gamma_{\text{blue}} = (5.50 \pm 0.22) \times 10^{-3}$ 1/s and it had an initial displacement of $\theta_0 = (0.26 \pm 0.01)$ rad. Finally, the pink plot, which is the longest and bottom-most plot, had a negative slope of $\gamma_{\text{pink}} = (7.20 \pm 0.22) \times 10^{-3}$ 1/s. It had an initial displacement of $\theta_0 = (0.09 \pm 0.01)$ rad.

Since the uncertainty in the time measurement was determined to be 0.01 s, the error bars in the horizontal direction have that value. As for the error bars in the vertical direction, these were determined using the equation

$$\Delta \ln(\theta) = \sqrt{\left(\frac{d(\ln(\theta))}{d\theta}\Delta\theta\right)^2}\ , \tag{46}$$

or, more simply,

$$\Delta \ln(\theta) = \sqrt{\left(\frac{\Delta\theta}{\theta}\right)^2}\ . \tag{47}$$

The value of our decay parameter γ was found by taking an average of the different slopes, and it was determined to be $\gamma = (8.25 \pm 3.86) \times 10^{-3}$ 1/s. Since the decay parameter γ is the inverse of the characteristic time t_c, the characteristic time is therefore $t_c = (1.21 \pm 0.01) \times 10^2$ s. This means that it would

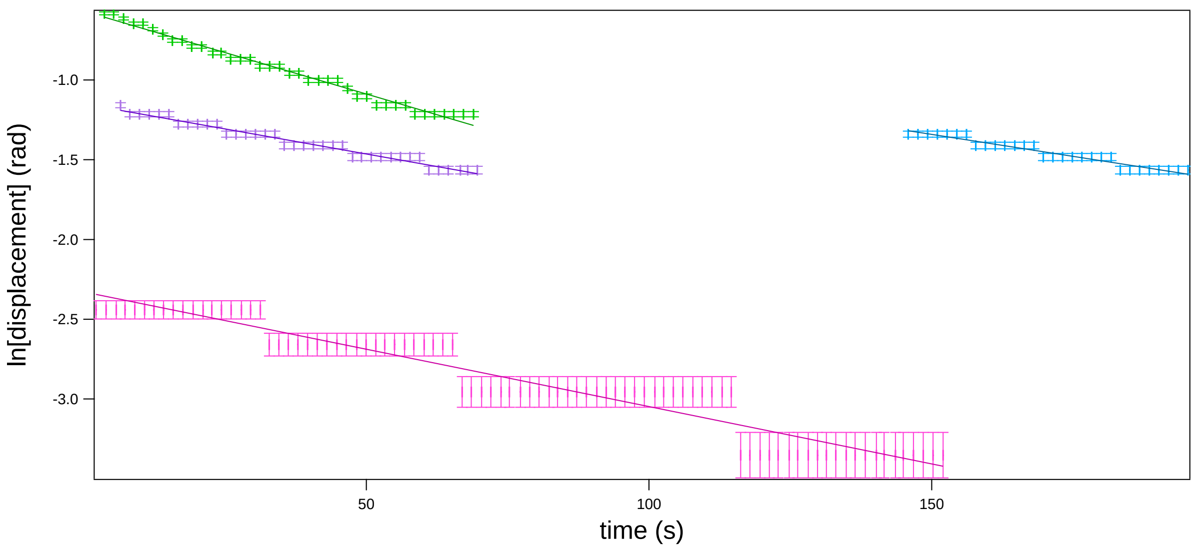

FIG. 6: Color coded semi-log plot of maxima in displacment θ vs. time t for different values of initial displacement θ_0. The initial displacements were $\theta_0 = 0.559$ rad for the green data set, $\theta_0 = 0.314$ rad for the purple data set, $\theta_0 = 0.087$ rad for the pink data set, and $\theta_0 = 0.262$ rad for the blue data set.

take approximately 121 s for the physical pendulum's amplitude θ to fall to (1/e) of its initial value θ_0

The absolute error $\Delta\gamma$ was found by taking the mean of the decay parameter values γ, in an equation similar to Eq. (34), so that

$$\Delta\gamma = \sqrt{\frac{1}{N-1}\sum_{n-1}^{N}(\gamma_n - \bar{\gamma})^2}\,. \tag{48}$$

In Eq. (48), $\bar{\gamma}$ denotes the mean of the different decay parameters γ. This gives us an absolute error of $\Delta\gamma = 3.86 \times 10^{-3}$ 1/s and a relative error of $(\Delta\gamma/\gamma) = 4.68 \times 10^{-1}$. This is a little too high for comfort, and it means that the variation between our decay parameter γ values is excessive.

Now that we have obtained the value of our decay parameter $\gamma = (8.25 \pm 3.86) \times 10^{-3}$ 1/s, we have everything we need to employ Eq. (33). Our moment of inertia I has a value of $(5.03 \pm 0.06) \times 10^{-2}$ kg $\cdot$ m^2 and the length L between our pendulum's axis of rotation and center of mass is $(5.60 \pm 0.02) \times 10^{-1}$ m. Putting it all together, we get a damping constant of $b = (2.65 \pm 0.19) \times 10^{-3}$ kg/s.

The absolute error Δb of our damping constant can be found as follows:

$$\Delta b = \sqrt{\left(\frac{\partial\left(\frac{2I\gamma}{L^2}\right)}{\partial I}\Delta I\right)^2 + \left(\frac{\partial\left(\frac{2I\gamma}{L^2}\right)}{\partial \gamma}\Delta \gamma\right)^2 + \left(\frac{\partial\left(\frac{2I\gamma}{L^2}\right)}{\partial L}\Delta L\right)^2} \tag{49}$$

$$= \sqrt{\left(\frac{2\gamma}{L^2}\Delta I\right)^2 + \left(\frac{2I}{L^2}\Delta\gamma\right)^2 + \left(-\frac{4I\gamma}{L^3}\Delta L\right)^2} \tag{50}$$

$$= \frac{2I\gamma}{L^2}\sqrt{\left(\frac{\Delta I}{I}\right)^2 + \left(\frac{\Delta\gamma}{\gamma}\right)^2 + \left(-\frac{2}{L}\Delta L\right)^2}\,. \tag{51}$$

This means that the absolute error is $\Delta b = 0.19 \times 10^{-3}$ and the relative error is then $(\Delta b/b) = 7.14 \times 10^{-2}$.

V. CONCLUSION

In this lab, we studied the physical pendulum as a damped oscillator.

First, we examined the effect of initial amplitude on a physical pendulum's period. In theory, as seen in Eq. (10), the initial amplitude should have no bearing on the period. Unfortunately, in practice, this was found to not be the case. Rather, we found that there was a positive relationship between the initial amplitude and the period, and a graph of this relationship is pictured in Fig. 5. For this reason, when we calculated the mean value to use as our period T, we used only the data in which the initial displacement θ_0 was less than $(\pi/9)$ rad. We found our period to be (1.71 ± 0.04) s.

After demonstrating the effect of a physical pendulum's initial amplitude on its period, our next step was to quantify it. Using Eq. (27), we were able to test how well the theoretical relationship of Eq. (9) could hold up experimentally. Using the mean small-angle period of our physical pendulum and our moment of inertia, which we calculated to be $I = (5.03 \pm 0.06) \times 10^{-2}$ kg $\cdot$ m^2, we determined an experimental value of Earth's gravitational field; $g = 7.74 \pm 0.37$ m/s^2.

This experimental value is less than the actual value $g = 9.807$ m/s^2 by a factor of about 1.27, regardless of our efforts to keep our results as close to theoretical expectations as possible. Despite using only the smallest seven angles to calculate our mean period, we still had a percent error of 26.75 % and a percent difference of 23.60 %. While this could be larger than expected for a number of reasons, the first that comes to mind is the way in which the sliding mass fastened to the meter stick was idealized as a point. This assumption was made so that the physical pendulum's moment of inertia I could still be found using the formula for a rectangular plane.

The positive relationship between initial amplitude and the period of a physical pendulum is contrary to the theory put forward by physicist Galileo Galilei, who believed that there was no correlation between the two. Taking the credit for this discrepancy, the effect of damping factors, such as friction and air resistance, must be taken into account in order to properly describe real-world oscillatory motion. Naturally, it follows that our final study of the physical pendulum should be to treat our system as a weakly damped oscillator, and to quantify the damping constant b.

A semi-log plot of the maximum values of displacement θ vs. time t was plotted, yielding a slope of $(-8.25 \pm 3.86) \times 10^{-3}$ 1/s. This now-linear relationship, seen in Eq. (31), defines our radial decay parameter γ as the negative value of the slope. Eq. (32) then defines a characteristic time of $t_c = (1.21 \pm 0.01) \times 10^2$ s. Finally, Eq. (33) allowed us to solve for our damping constant, and we found it to be $b = (2.65 \pm 0.19) \times 10^{-3}$ kg/s.

From this lab, we were able to conclude that the period of a damped physical pendulum is not independent of the initial amplitude, and that the theoretical relationship outlined in Eq. (9) does not hold up past a certain threshold. Perhaps a future project could study this threshold more deeply. At what point is the angle of the initial amplitude large enough that the period of a pendulum is no longer consistent with that of smaller amplitudes? What degree of difference is there between a damped and undamped oscillator when it comes to consistency of the period of a pendulum? These unknowns could also potentially explain the error in the experimental value of Earth's gravitational field g. What might our percent error and percent difference look like if this same experiment was conducted in a vacuum? Unfortunately, we do not have access to a vacuum chamber, but it is nice to think about.

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[1] Machamer, P. & Hepburn, B. Galileo and the pendulum: Latching on to time. *Science & Education* **13**, 333–347 (2004). URL `https://doi.org/10.1023/B:SCED.0000041834.55645.1a`.

[2] Department of Physics, The College of Wooster, Wooster, OH. *Junior IS Lab Manual* (2022).

[3] Nave, R. Physical pendulum. http://hyperphysics.phy-astr.gsu.edu/hbase/pendp.html#c1 (1998). Accessed on 3 February 2022.

[4] Moebs, W., Ling, S. J. & Sanny, J. *University Physics Volume 1* (OpenStax, 2016).

[5] Nave, R. Local gravity calculator. https://www.sensorsone.com/local-gravity-calculator/ (2022). Accessed on 20 February 2022.

[6] Taylor, J. R. *Classical Mechanics* (University Science Books, 2005).

[7] Urone, P. P. & Hinrichs, R. *College Physics* (OpenStax, 2012).