Simulating a Die to explore Chaos Theory

Daniel Cohen-Cobos
Department of Physics, The College of Wooster, Wooster, Ohio 44691, USA
(Dated: May 22, 2022)

In this article, we looked into the motion of a die being dropped to a surface to analyze the chaotic motion and relate the findings to chaos theory/nonlinear dynamics. We first used a physical die to understand the motion of the bounces and then we took it to *Autodesk 3ds Max*, using the *Mass FX* plugin to conduct a physics simulation. We used this to construct a mathematical model to build our own testable simulation in *Wolfram Mathematica*. The simulation was built as a 2D system to simplify the math and reproduce the final results in 2D density plots, where a color scale represented a third variable. It took the angular position in one dimension and height as initial parameters and returned the time it would take for the die to stop moving. We first tested these initial parameters to make sure they exerted nonlinear behavior to use in the final simulation. The outputted graphs reproduced chaotic behavior and nonlinear patterns such as periodicity and bifurcations. This showed that even a simplified model of a die is a perfect system to study chaos theory.

I. INTRODUCTION

Chaos theory is considered by many physicists one of the three biggest scientific revolutions of the twentieth century along with relativity and quantum mechanics. Chaos, also known as **Nonlinear Dynamics**, was developed by **Henri Poincaré** [1] at the beginning of the century, later to be developed by researchers such as **Edward Lorentz** [2] or **Mitchell Feigenbaum** [3] once computational physics came into existence. It establishes that most physical systems may appear simple, but exhibit nonperiodic motion extremely sensitive to initial conditions which become non-trivial and very hard to guess. [4]

To test this concept, we were motivated by the concept of dice to analyze its motion and evaluate this revolutionary theory. Dice have been used since the very beginnings of human civilization. If you have played Monopoly, Dungeons and Dragons, or any board game that requires the user to move a piece in a random number of places, you have encountered a die (singular of dice) before. Dice have appeared in numerous mythologies and religious texts symbolizing a choice made by the divine, where the output of a die is completely unpredictable and non-deterministic, and therefore it is the divine that chooses what the resulting number will be. The oldest precursor of a die has been found to be knucklebones from different animals such as sheep or buffalo dating back to prehistory to the first human civilizations. These bones were cast, and the resulting 'face' of the bone that would result on top would appear to be completely random. [5]

II. THEORY

A. Background research

The standard die consists of a cube-shaped object marked with a number from 1 to 6 on each face, where the numbers on opposite sides of the die add up to 7. As proven, it is a chaotic system that cannot be entirely deterministic from the naked human eye, as it is very sensitive to initial conditions [6]. Such sensitivity can be seen in Fig. 1, where a small angle variation causes the final result to change, not showing any clear apparent patterns in the results. This results were done on a computer simulation taking into account the dynamics of the die and error estimation of its simulation.

B. Physics of a die

To make a simple model, in this paper we looked at a die where one of its axes is unchanged, which significantly reduces the mathematics as we look at the model as if it only had two spatial coordinates. The reason for this was to reduce the model to a simple set of equations that allowed 2D representation, and 3D representation using the third parameter as a color scale in a 2D density plot. The coordinates established where x for horizontal displacement, and y for the height of the die, adding an angular parameter θ to determine the orientation of the die.

When the die was in free fall, it would experience regular Newtonian/Hamiltonian motion and energy conservation. For this part of the die's motion, we derived the following set of equations:

$$x = x_i + v_x \ \Delta t \tag{1a}$$

$$y = y_i + v_y \ \Delta t - \frac{9.8}{2}\Delta t^2 \tag{1b}$$

$$\theta = \theta_i + \omega \ \Delta t \tag{1c}$$

$$v_y = v_{y_i} - 9.8 \ \Delta t \ , \tag{1d}$$

where Δt is the time increment, and v_{x_i},v_{y_i} and ω_i are the initial velocities of x, y and θ respectively at the start of the iteration. Here we are considering the perfect acceleration of $9.8\,\mathrm{m/s^2}$, without air drag, as

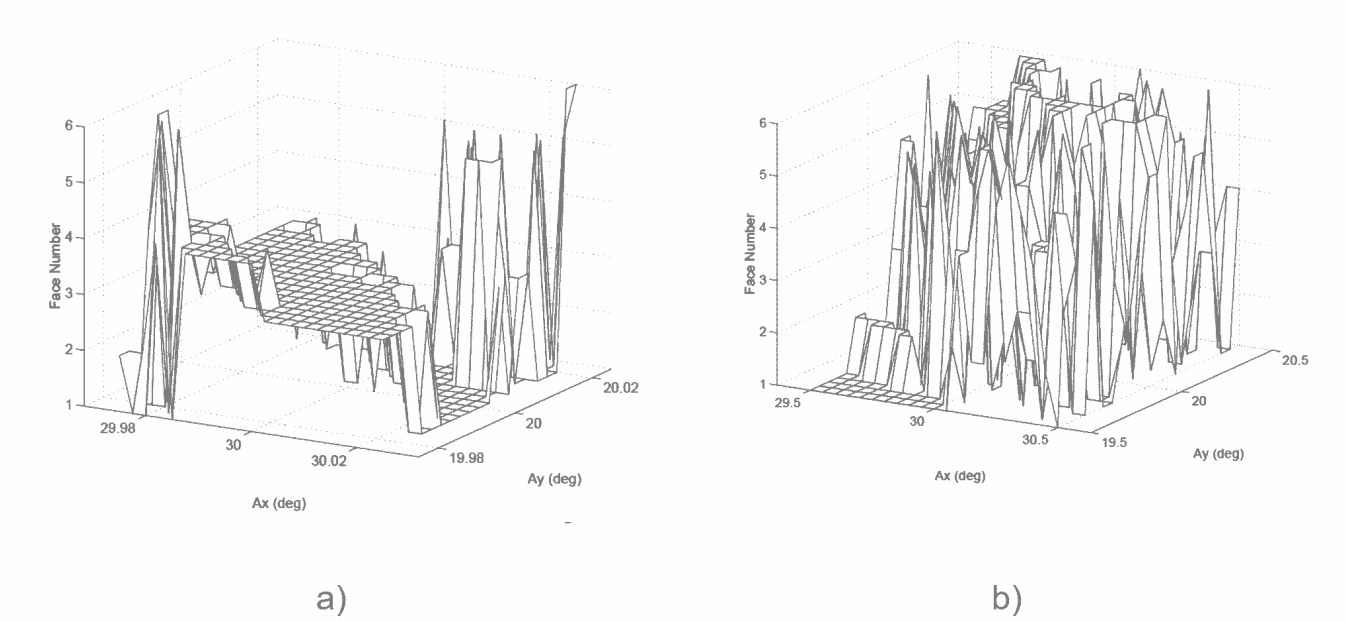

FIG. 1: Face of die resulting after being released at a height of 20 cm varying the angle at which it is thrown, showing no apparent pattern in the results. Very small angle in a) and a bigger angle at b). We can see in a) that the simulation is zoomed in by observing the shape created at dice face number 3, which does not show a clear pattern. Interpreting this results as chaotic behavior. Image taken from [6].

if the object was released in a void. This assumption was taken throughout the whole experiment to get a simplified version of the die's motion. If the die was a nonlinear model, it would remain chaotic in any setup. [7]

On to the more complicated motion of the die, which caused the chaotic behavior. We used Newtonian mechanics to create a diagram of a die bouncing on a surface which can be seen in Fig. 2. The die impacts with an initial velocity v_i that will produce the bouncing velocity v_0 in the figure, where we established the relationship $v_i = v_0\beta$, where β is the bouncing parameter from 0 to 1, making $\beta \to 0$ no bounciness at all, and $\beta \to 1$ making v_0 equal to the incoming velocity as if it was a frictionless bounce.

Using Fig. 2 as the base model, we noticed that bounces would mostly occur on one of its corners. This is because whatever angle the die was rotated, one of the sides would have to be perfectly parallel to the table when bouncing, which was very improbable. So we assumed bouncing only occurred on the corners. The equations derived were the following:

$$v_x = v\sin\phi + v_{x_i} = v_0\sin\phi\cos\phi + v_{x_i} \tag{2a}$$

$$v_y = v\cos\phi = v_0\cos^2\phi \tag{2b}$$

$$\omega = \frac{v_0\sin\phi}{r}\ , \tag{2c}$$

where r is the distance from the center of the die to any of its corners. In the simulation, because the corner that hits the surface may not be the ones closest to $\theta = 0$ or $\theta = \pi$, the angle ϕ used to calculate the velocities used in Eqs. (2), will be set to θ adding or subtracting $\pi/2$ until $-\pi/4 \leq \phi \leq \pi/4$.

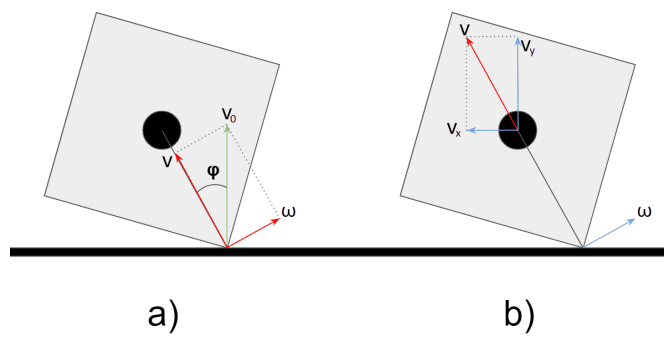

FIG. 2: Diagram of a die bouncing when it interacts with a surface. The angle from the center of mass to the impacted point is denoted with ϕ, and bouncing at a perpendicular velocity v_0 to the surface. This velocity can be seen in a) colored green. It produces a linear velocity v in the direction of the center of mass of the die (center of die) bouncing in the opposite direction to the contact point of the die and the surface denoted in color red in b). It also produces a tangential angular velocity ω. The linear velocity can be divided into their respective coordinate velocities v_x and v_y.

III. PROCEDURE

We used a computer simulation which required us to analyze the motion of a physical die and understand its behavior. To do this, we calculated its density and volume to find its characteristic initial parameters and took a closer look at the bouncing motion to construct an accurate simulation. We used a scientific scale to find the mass and a graduated cylinder to measure the volume.

A. Physical setup

We used Open Source Physics' *Tracker* [8] to monitor the height of a released die on a table frame by frame to find how the y-component of the die changed over time to look for any linearity or periodicity in its movement. An *Apple iPhone* was used to record the videos.

The graphs that *Tracker* produced were used for the final simulation, and helped determining the vary parameters for this experiment. The reason for using a computer instead of a physical model was the chaotic behavior system of the die and the unpredictability of any physical model. Its initial conditions were so sensitive that reproducing the same physical behavior twice was extremely improbable. A computer simulation added constraints that would simplify the model enough to receive data that could be visualized and analyzed much more easily.

B. Simulation

Using the data gathered from the physical die, we used *Autodesk 3ds Max*'s physics simulation using the *Mass FX* plugin to test out Eqs. (1) and (2). We used a 3D model of a die and released it at different initial angles and heights 'baking' the physics simulation to view the position and rotation of the die at all times for a closer look if its dynamics.

Because the die appeared to exhibit nonlinear behavior in all its parameters, we chose time as the analyzed parameter, evaluating the time it took for the die to stop moving after being dropped to a plain surface on specific initial conditions. When testing the behavior of the die, we noticed that the time it would take to stand still could be a wide range, which was an indication that the behavior was chaotic, which was what motivated this experiment to analyze the time parameter in the model.

To do this, we programmed a simulation of the physics of a die being released from a determined height (first initial parameter) positioned at a certain angle (second initial parameter) in *Wolfram Mathematica* [9], using Eqs. (1) and Eqs. (2).

C. Data gathering

To verify that the initial angle and height pushed the model to produce chaos, we first look over these parameters to find the sensitivity and how they vary when the other parameter is constant. This helped us determine how to evaluate data using both initial parameters as the axis in a 3D plot.

Afterward, we used Mathematica's *DensityPlot* to plot over different ranges of height and angular position as initial parameters, using a color scale to find patterns and shapes in the resulting graph. This final plot would then be used as a reference to produce more density plots zooming in on interesting areas of looking for indications of linearity, periodicity, or even bifurcations.

IV. RESULTS & ANALYSIS

A. Physical Die

We used a Plastic Die to measured its mass and volume. To calculate its density we used a scientific scale to find the die's mass, to be $5.565(1) \times 10^{-3}\,\mathrm{kg}$. We used a graduated cylinder as indicated in Fig. 3, since the die was not a perfect cube structure due to the hemispheric holes in each face indicating the number and the rounded edges. We measured the volume to be $4.5(5) \times 10^{-6}\,\mathrm{m}^3$. Calculating the density by dividing mass by volume and finding its error estimation we then got the total density of the die to be $1.236(1) \times 10^3\,\mathrm{kg/m}^3$

Using *Tracker*, we released a die from a height $10.0(5)\,\mathrm{cm}$, recording 20 different videos, using a frame rate of 60 frames per second. All graphs reproduced no sign of periodicity or linearity, but instead, they showed incentives of chaotic behavior in its bounces. The graphs produced showed no apparent relation between adjacent bounces.

An example of two experiments can be seen in Fig.4. The height achieved in a) after each bounce appears to be reduced subsequently. This could be seen as a linear behavior where the maximum height decreases over time. This is not true, since we can see the opposite behavior in b), where the second bounce was not as strong as the third one. We noticed that after the second bounce the height achieved was close to half as much as the height achieved after the third bounce. This did not only show that two similar throws produced completely different data, but also that the bounces are nonlinear and show no periodicity, making them unpredictable to the human eye.

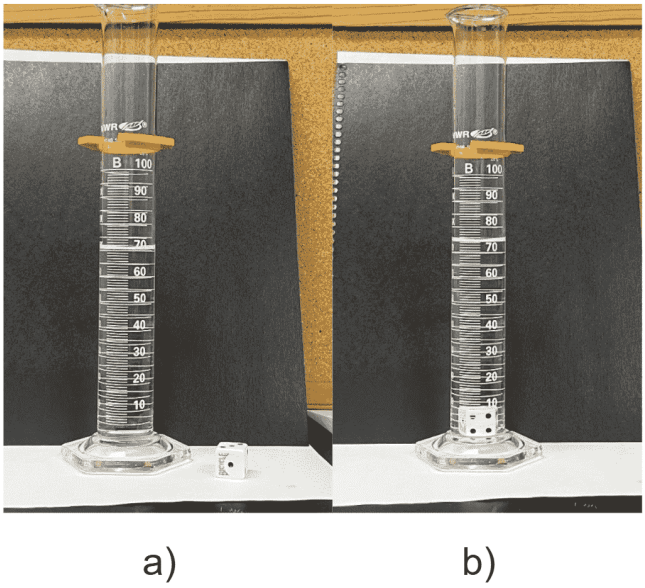

FIG. 3: Measure of the volume of a die using a graduated cylinder. Where in a) the water height measures 70(1) ml, and b) once the die goes in the cylinder, the water rises to 74.5(5), indicating the volume of the die to be $4.5(5) \times 10^{-6}\,\mathrm{m}^3$

B. 3ds Max

Using *Autodesk 3ds Max* [11], we created a computer simulation to compare the physical movement of the die captured with *Tracker* and the Physics simulation produced by the *Mass FX* plugin. The result was visually the same, which we used to 'bake' a simulation and evaluate the movement to find if Eqs. (1) and Eqs. (2) suited the model, which they both did. Because of data gathering limitations, we decided to continue the experiment with a *Wolfram Mathematica* simulation, which allowed better control over the mathematics behind each step integration and manage the initial parameters better. Another reason for the change in software for the experiment was that *3ds Max* was a software made for modeling, which was useful to understand the volume, dynamics, and shape of the die; but not for simulating purposes.

C. Simulation

Because the new *Mathematica* simulation uses time steps, there is a slight error deviation from every bounce due to the possibility of the die touching the surface between time steps. We first ran the simulation with parameters $\theta = 0$, $\Delta t = 0.001$ s and initial height 20 cm to find how many bounces occurred. The resulting number was 50, which multiplied by the time steps gives $\pm$ 0.05 s as **error estimation** for the time results of the simulation.

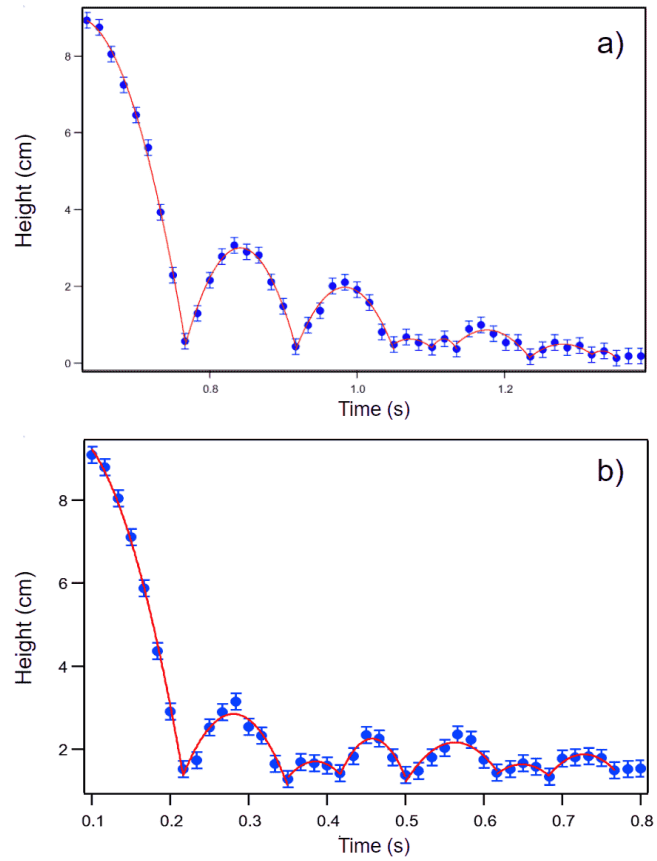

FIG. 4: Plots of height vs. time, where in both cases a die was released at an initial height 10cm, with polynomial curve fitting to connect the data points and see the bounces of the die. Data gathered in *Tracker*, locating the object frame by frame adding an error estimation of $\delta t = \pm$ 0.2 cm due to the deformation of the die going at a fast velocity in each picture (this was the biggest value measured); curve fitting done in *Igor Pro 8* [10]. Both a) and b) where released at the same height and similar angle.

1. *Analysis of initial Parameters*

We plotted the two analyzed initial conditions angle and height to determine how sensitive these conditions were to the final simulation. The first plot corresponding to the height can be seen in Fig. 5, where we can observe the chaotic patterns of the die. The upper points of the function appear to follow a clear path, while the lower points appear to be chaotic. In the lower right corner, we can see a zoomed image of the graph showing how the time reported by the simulation is extremely sensitive to the height parameter and exerted no apparent periodicity.

Likewise, we analyzed the angle parameter by plotting the angle versus the time reported, to also evaluate the sensitivity of the initial position of the angle. The resulting plot can be seen in Fig. 6. Here we can see at first

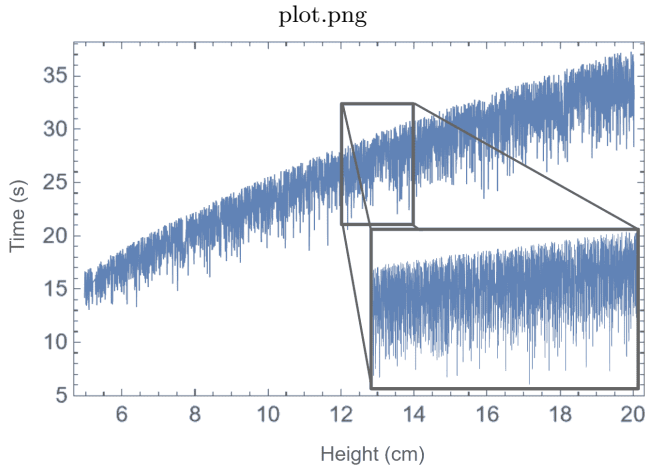

FIG. 5: Time in seconds to stop moving vs. initial height at which the die was dropped. The die starts at an angle $\theta = 0$. The error estimation due to the time step of the simulation is 0.05 s. The integration time step is $\Delta t = 0.001$.

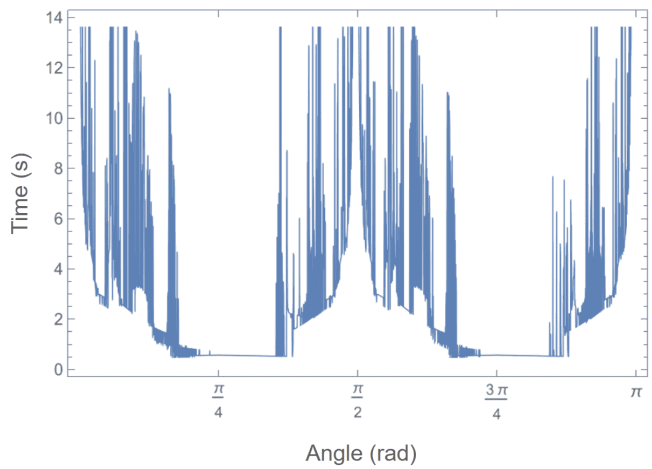

FIG. 6: Time in seconds to stop moving vs. initial angle at which the die was dropped. The die starts at a height 10 cm. The error estimation due to the time step of the simulation is 0.05 s. The integration time step is $\Delta t = 0.001$.

glance the symmetry of the time reported with periodicity from 0 to $\pi/2$, repeating the pattern afterward.

We can also observe two horizontal lines at $\theta \to \pi/4$ and $\theta \to 3\pi/4$. This is because the simulation detects the die 'stopped' when the resulting velocity after a bounce is low, and at $\theta = \pi/4$ and $\theta = 3\pi/4$ the die bounces off transferring the maximum linear velocity to angular velocity, with a small bounce in the y-component and a high rotation. Making the die rotate fast, but not lifting the die high enough for the next bounce to be sufficient to continue the motion of the die.

We can conclude by Figs. 5 and 6 that both initial conditions of the simulator are extremely sensitive, which fulfills the principal characteristic of a chaotic system.

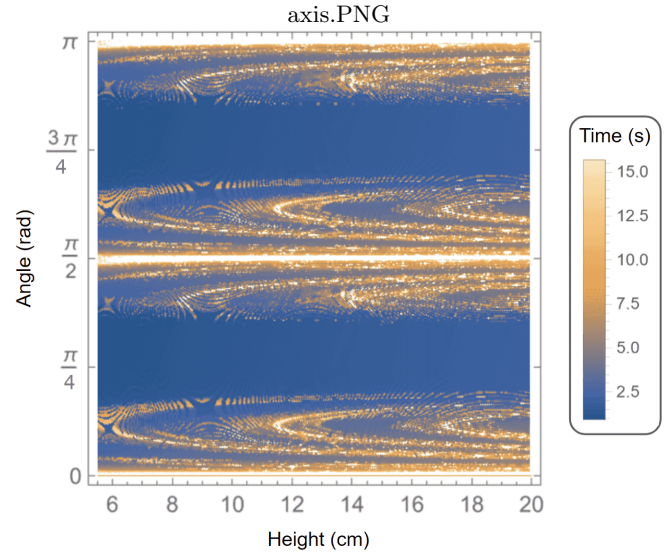

FIG. 7: Density plot of the simulation, where the axes represent the height at which the die is released and the initial angle θ at which the die is positioned. The color scale corresponds to the time the die takes to stop moving.

2. *3D Density plots*

Varying these parameters in the simulation, we can finally create 3-dimensional plots to find patterns in the chaotic behavior. To represent the output of the graph, we created density plots using a color scale to express the third dimension in a 2D plot, where the parameters act as the axis of the graphs. Creating an initial plot to find the overall shape that is produced, we made Fig. 7, where we used a uniform color scale from dark blue to gold to remark the most notable shapes.

We first noticed the horizontal line that is significantly brighter at $\theta \to \pi/2$, since the die starts with an angle that will bounce off with a small rotation, due to its position. At the exact value $\theta = \pi/2$ we find that the die bounces perfectly on its corner never adding rotation and therefore bouncing the most time in the simulation. No velocity gets transformed into velocity in the x-axis or angular rotation. This is the highest the die can last after many bounces. This horizontal line can also be seen at the top and bottom of the graph due to a $\pi/2$ shift of the die, which corresponds to a symmetrical rotation to another corner.

We also noticed that if the graph was divided at $\pi/2$, both upper and lower parts of the graph seem identical, indicating periodicity from $0 \leq \theta \leq \pi/2$, repeating that pattern. This is because of the symmetry of the die. Once this symmetry was found, we analyzed only one of these sections, or what was equivalent, the middle section of the graph which included the upper and lower golden shapes of the graph. This new graph can be seen in Fig. 8

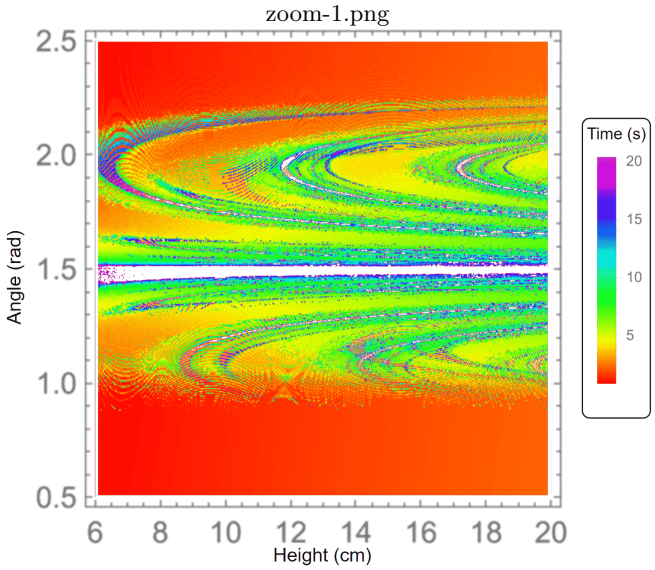

FIG. 8: Zoomed in of Fig. 7 for $0.5 < \phi < 2.5$. The axes represent the height at which the die is released and the initial angle θ at which the die is positioned. The Hue color scale corresponds to the time the die takes to stop moving.

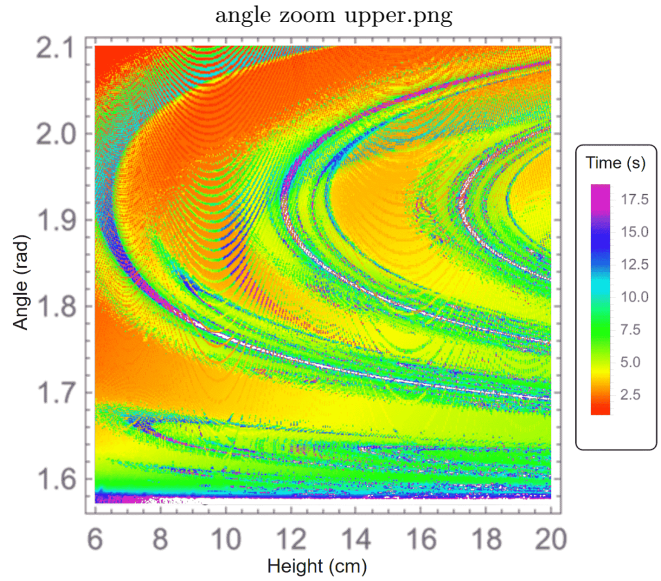

FIG. 9: Zoomed in of Fig. 8 for $\pi/2 < \phi < 2.1$. The axes represent the height at which the die is released and the initial angle θ at which the die is positioned. The Hue color scale corresponds to the time the die takes to stop moving.

In this graph, we changed the color scale to *Hue* to notice the patterns better by increasing the range of colors that the graph could output. This graph gave more detail on the shapes and curves created with the simulation. We also noticed that the line at $\theta = \pi/2$ was not a perfect line since the lower bound of the line appears to curve towards the bottom at the left side (height between 6 cm and 8 cm. We also observed that the upper and lower part of the graph did not share any apparent symmetry, and would each need an individual analysis since they do not appear to be connected.

By further zooming into the upper side of the graph, we reproduced Fig. 9. We noticed an interference pattern making periodic curves along with the graph. This pattern can be specially noticed at heights between 8 cm and 10 cm and $1.8 \leq \theta \leq 2$ radians in the color green. This may be produced by the error estimation due to the time steps acting on the high sensitivity of the initial angle parameter, creating a shift in the results.

The most noticeable shape in the graph can be the biggest 'ring' that can be seen in the upper part of this graph at $\theta > 1.7$. We observed that inside this ringed shape was a smaller curve that appeared to be of similar form at a height close to 12 cm. This also happened with a smaller curve inside the small curve at a height of around 17 cm. The behavior led us to think that the ring pattern was periodic and that the rings would continue as the initial height went higher. This curved shape can also be seen in the lower part of the figure, at $\theta < 1.7$, finding that periodic behavior with a deformed geometry. The smaller curve can be seen around a height of 12 cm.

As well as the upper side of Fig. 8 was analyzed, we zoomed into the lower side to find more patterns to evaluate. This can be seen in Fig. 10. Here we saw with better detail the horizontal line from the density plot at $\theta \to \pi/2$ mentioned before with greater detail colored purple and white at the top of the figure. we also noticed that it had a bigger size than the portion of the horizontal line in the upper part of the graph from Fig. 9. The interference pattern is also visible in this portion of the plot.

We also observed a bifurcation shape close to {16 cm, 1.25 rad}, where we saw the same pattern of curves repeating as the previous graph, showing apparent periodicity, but this time the bifurcation interrupts the repeating shape. Because of this, the nonlinear pattern is hard to picture and any higher height than 20 cm becomes unguessable with the current portion of the graph. This led us to believe that the initial parameters are so sensitive that a much bigger graph is required to find periodicity if any at $\theta < \pi/2$ in the simulation.

Finally, we noticed that at the bottom of Fig. 9 at $\theta < 1.7$ a similar shape to the top of Fig. 10 at $\theta > 1.4$, where we saw the ring-like curves deformed in almost a mirrored way. This is better visualized in Fig. 8 at $\theta = (\pi/2) \pm\ 0.2$, where the closest deformed curves appear to share a mirrored symmetry. Showing that even if the top half indicates periodicity and repetition, it shares a mirrored portion with the lower half that has no ap-

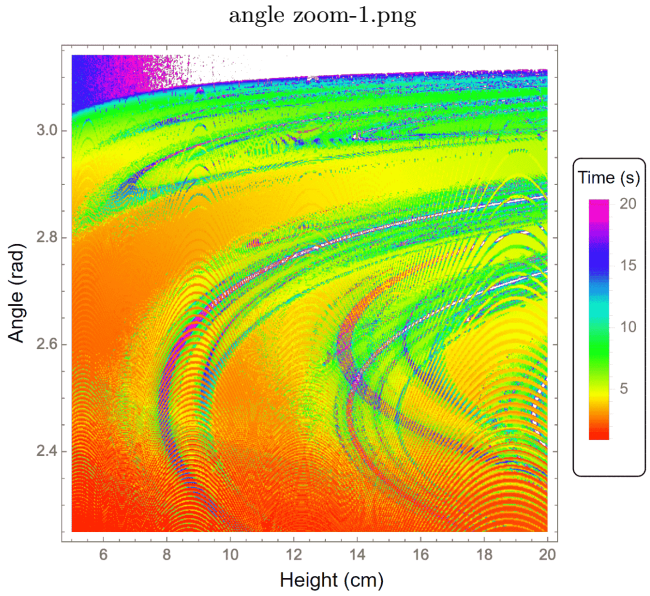

FIG. 10: Zoomed in of Fig. 8 for $1.1 < \phi < \pi/2$.Density plot of the simulation, where the axes represent the height at which the die is released and the initial angle θ at which the die is positioned. The Hue color scale corresponds to the time the die takes to stop moving.

parent periodicity, and has one bifurcation, proving that for every initial angle, the state of the system is highly chaotic.

V. CONCLUSION

In this paper, we created a mathematical model based on a die free falling and bouncing on a flat surface to find chaotic behavior. We used a physical die to evaluate its motion and compare it to a physics simulation using *Autodesk 3ds Max*. This software allowed us to record a simulation and find the kinematic equations of its motion using Newtonian mechanics. Using those sets of equations, we constructed a testable 2D simulation in *Wolfram Mathematica* that allowed us to gather data and evaluate the system at each integration time. We analyzed two initial conditions, which were angular position and height to find that the system was highly sensitive to both of these two parameters.

The system evaluated using the time that the die would take to stop moving after being dropped. We plotted 2D density plots using time as a color function. The initial graph used a uniform color scale to find the most noticeable shapes. By zooming into the parameters and using a wider color palette as a color scale, we found nonlinearity, periodicity, and bifurcations in the system. In the angular parameter, there was repetition since the die was shaped like a cube, having a 90° symmetry in any rotation along with its faces. Initial height, on the other hand, did not show any sign of periodicity or symmetry in its results. This proved that the system did not only exert chaotic behavior but that the data was not deterministic or trivial. The portions of the graphs we used exerted such nonlinear characteristics, that they were not enough for us to imagine what the system could be at a higher initial height if it followed any kind of repetition.

Using these results, we can then conclude that a die being dropped is an excellent example to evaluate chaos theory. Furthermore, the model constructed in this article was a simplified version of the physical system.

VI. ACKNOWLEDGMENT

I'd like to thank The College of Wooster Physics Department for the tools needed for this report, Including Dr. Manz and the Teacher Assistant Lillian Miller for helping me with my setup and procedure. I would also like to thank the Koontz Fund for providing the funds for *Autodesk 3ds Max* and the computer requirements to use for this software.

[1] Poincaré, H. *On the three-body problem and the equations of dynamics* (Acta Mathematica 13, 1890).
[2] Lorenz, E. N. Deterministic nonperiodic flow. *Journal of Atmospheric Sciences* **20**, 130–141 (1963).
[3] Feigenbaum, M. J. Quantitative universality for a class of nonlinear transformations. *Journal of Statistical Physics* **19**, 25–52 (1978).
[4] Lindner, J. F. Nonlinear Dynamics. Tech. Rep., The College of Wooster, Wooster, OH, USA (Dec. 2021). Accesed on March 29th.
[5] Britannica, E. dice. `https://www.britannica.com/topic/dice` (2019). Accessed on April 5th 2022.
[6] Hale, L. C. Determinism in dice throwing and the transition to chaos. In *Proceedings of the American Society for Precision Engineering*, vol. 12, 272–275 (1995).
[7] Nayfeh, A. H. & Balachandran, B. *Applied nonlinear dynamics: analytical, computational, and experimental methods* (John Wiley & Sons, 2008).
[8] Tracker Video analysis and Modeling tool. `https://physlets.org/tracker/`. Accessed: May 22nd 2022.
[9] Wolfram mathematica. `https://www.wolfram.com/mathematica`. Accessed: May 5, 2022.
[10] Igor Pro 8. `https://www.wavemetrics.com/`. Accessed: May 5, 2022.
[11] Autodesk 3ds Max. `https://www.autodesk.com/products/3ds-max`. Accessed: May 5, 2022.
[12] Parallels Desktop. `https://www.parallels.com/`. Accessed: May 5, 2022.
[13] Chen, G. & Dong, X. From Chaos to Order, Perspectives and Methodologies in controlling Chaotic Nonlinear Dy-

namical Systems. *International Journal of Bifurcation and Chaos* **03**, 1363–1409 (1993).

[14] Abarbanel, H. D. I., Brown, R., Sidorowich, J. J. & Tsimring, L. S. The Analysis of Observed Chaotic Data in Physical Systems. *Rev. Mod. Phys.* **65**, 1331–1392 (1993). URL `https://link.aps.org/doi/10.1103/RevModPhys.65.1331`.

[15] Boccaletti, S., Kurths, J., Osipov, G., Valladares, D. & Zhou, C. The Synchronization of Chaotic Systems. *Physics Reports-Review Section of Physics Letters* **366**, 1–101 (2002).

[16] Curless, T. The application of idealized models to isolate dominant features of a physical double pendulum. Tech. Rep., The College of Wooster, Wooster, OH, USA (2021). Accesed March 29, 2022.

[17] Stewart, I. *Does God Play Dice? The New Mathematics of Chaos (Penguin Mathematics)* (Penguin UK, 1997), 2 edn.

[18] Bartlett, M. Chance or chaos? *Journal of the Royal Statistical Society: Series A (Statistics in Society)* **153**, 321–330 (1990).

Used Software and Funds

To study the dynamics of the die, we used **Autodesk 3ds Max** [11] adding a die model and applying MassFX properties to it to run physics simulations on it. Because the computer used was an **Apple Mac**, it could not run 3ds Max, so we used **Parallels** [12] to emulate a **Microsoft Windows** desktop and compute the simulations. We used a **3ds Max Student License** (free for a year) and a **Parallels Pro Edition License** that allowed us to use all the resources we needed from the computer lab. Useful links can be found here:
Parallels Licenses
Autodesk Licenses for Students