Small-Scale Percolation in a Physical Experiment

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The phenomena of percolation is explored in a physical experiment to determine a 2-dimensional critical point. Keys, or strips of acrylic with a known amount of material randomly removed are created and randomly placed together to create a grid through which a route from one end to the other may exist. The amount of material removed from the total grid, the total vacancy, is calculated as an average of the individual vacancies of the keys, and the ability of each random configuration to allow the passage of fluid is recorded. In this system, vacancies are allowed to overlap, which may more closely represent an organic system where particle sizes are not constant. A two-dimensional critical probability of 0.42 ± 0.03 is found, which is 29% from the expected value of 0.59.

I. INTRODUCTION

A. Model

To one, the term percolation may bring to mind the idea of coffee. This is not an incorrect thought; as water flows through coffee grounds, the liquid is finding a path from the top of the filter to the bottom. If the coffee grounds were too compacted or too fine, the water would not have a route to travel through the system, and the coffee would not percolate.

In more scientific terms, percolation is a model of adjacent or linked points. The mechanisms by which points are linked can be sorted into two types: points with fixed positions and random linkages (bond percolation); and points with random positions and linkages determined by a rule dependent upon position (site percolation). Both shorthand names originate from their applications in solid-state physics, and while there are situations that require new sources of randomness or a combination of types, these two are the most prevalent [2].

Consider two points, a start and a goal. If there exists any series of points, a *path*, between the start and the goal, the system is said to percolate. It is possible that many paths exist between the points; but only one is required for the system to be percolating. To illustrate this, consider the example of a simple platform-style game, where a player can move and jump from one solid block to another and must reach a goal at the top of the stage, as in FIG. 1. The entire field that the player can move across is a grid where each block is either an empty space or a solid block the player can land on and jump from, but not pass through. The probability that each block is either empty or solid is a variable, which ranges from zero to one, which I will call *vacancy*. Given that a player can jump up to two blocks high and six blocks across, it is clear that with an empty grid, there is no chance that the player will be able to jump to a goal located more than two blocks high. As this vacancy increases, however, there are blocks a player can land on to reach higher layers, and it becomes possible that a route exists from the starting position at the bottom layer to the goal at the top. In any situation where the player can finish at the goal, this system is percolating.

It should be clear that given the presence of blocks is a random probability, not all possible fields can be finished. A second probability can be associated with the original of vacancy: the probability of the entire grid to produce a system that percolates, known as the percolation probability. From a set of fields at a given vacancy value, one can produce a ratio of systems that percolate to systems that do not percolate, resulting in a value for percolation probability. Bear in mind, a number of other factors determine the percolation probability; but they can be ignored for this basic definition.

The question becomes, at which vacancy does the percolation probability become high enough that most of the systems have a route from the start to the finish?

B. Application

Most real-world applications of percolation theory contain such a large number of particles that the grid may be considered infinite and contained within an unbounded volume. In such a system, probabilities tend to disappear into an average, and thus percolation acts as a phase-

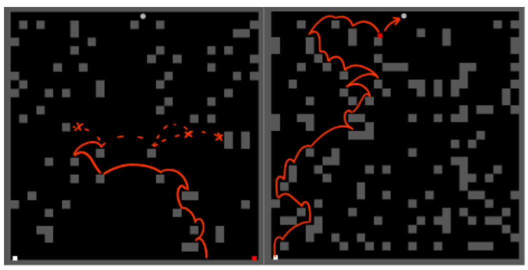
(a) (b)

FIG. 1: A simple java game I created in 2015 using Khan Academy's JavaScript course. Simple demonstration of percolation. In (a) the player fails to reach the goal; the system does not percolate. In (b), a route exists to the goal, therefore the system percolates. Game can be played at https://www.khanacademy.org/computer-programming/2pr-randomizer/5317211975712768.

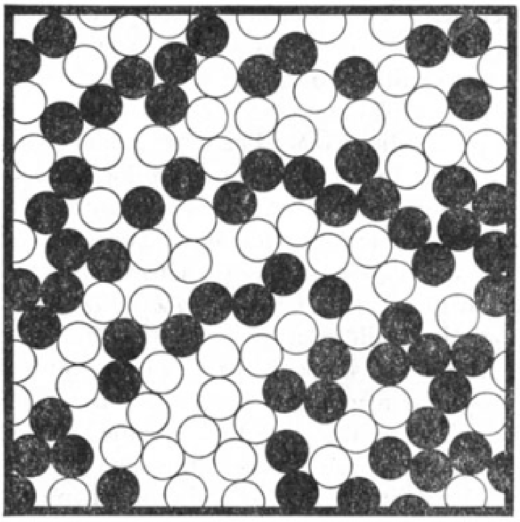

FIG. 2: Conductive (black) and non-conductive (white) grains in a conductor. For the medium to conduct, a route must exist from one side to the other. Borrowed from J. W. Essam [2].

change transition. In other words, in a graph of percolation probability to vacancy, percolation probability will remain at zero until a specific critical point is reached, beyond which the percolation probability will abruptly become one. The condensation of water vapor into liquid water is thought to be an example of this phase-change relationship of percolation [2].

Another application of percolation theory is in conduction. A material created from a mix of conducting and non-conducting grains with a given vacancy for a grain to be conductive will only percolate above a critical threshold, beyond which there exists a route across the material. This model of percolation is an example of site percolation [4]. A diagram illustrating this model is presented in FIG. 2.

An application one might not expect is in brush fires. Site percolation is a model in which two adjacent points interact with one another, seeking a route from a start to a goal. Similarly, adjacent patches of dry brush would be able to ignite each other. In this way, the spread of a fire is a system that follows percolation theory.

C. History

Percolation has its origins in graph theory. Broadbent and Hammersly first posed this problem in 1957, connecting points in a linear graph, considering the vertices as the points of the model and a given pair of vertices that form an edge of the graph. The links between these vertices are determined by a vacancy independently of all others pairs, as in, pairs that do not form an edge are never linked, as in bond percolation. This problem related to telephone engineering, where large networks with many rapidly-changing linkages must find a route between two callers, even when other routes are in use [2]. An example diagram of the maps Broadbent and Hammersly created in their phone networks is given in FIG. 3

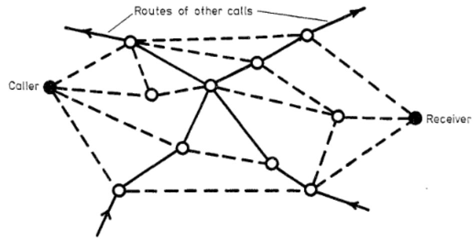

FIG. 3: Model of a phone network. Vertices can switch to connect to another vertex, but too many connections will block a call. Borrowed from J. W. Essam [2].

A later venture into a similar concept was done by Roach in 1968, attempting to diagram a radio communications network, where transmitters of fixed range are used. The fixed ranges of these transmitters must overlap to create a route of connected transmitters from one location to another [2].

The model of percolation in conductive grains was used by Goodman in 1975, although not to describe electrical conduction. In this similar model, the grains in FIG. 2 are replaced with two components of a glass, which was used to determine an optical absorption of the medium [2].

More recently, however, percolation has been extended far beyond the simple example of a two-dimensional grid. Systems of six dimensions and beyond, and even fractal dimensions have been examined, and interesting consequences and implications of higher dimensions have been discovered and theories have been created to predict their behaviour [6]. However, these are far beyond the scope of this experiment.

II. THEORY

As previously stated, in a system where the locations or connectivity of a particle is determined by a vacancy, represented with the variable p. In large systems, a critical probability p_c exists, below which, a system will never percolate, and beyond which, it will always do so. As such, in large systems, the relationship mirrors that of a phase-change transition. However, the incredibly large scale models required to perfectly demonstrate the pure phase-change relationship cannot be replicated with particle sizes as large as those in this experiment. Therefore, computations have been done on much smaller scales, on the range of ten to a hundred particles squared. What is found in these, though, is that grid size plays a large role in the behaviour of the percolation probability. The step function relationship will be smoothed out into a curve, the critical probability lying somewhere on that relationship. The randomness and probability associated with the percolation of a system resembles that of a thermodynamic system; so it is logical to begin with such a relationship [4]. Beginning with the Fermi-Dirac distribution noted in Daniel V. Schroeder's book (p. 267) [5],

$$\bar{n}_{FD} = \frac{1}{e^{(\epsilon-\mu)/kT}+1}, \tag{1}$$

where ϵ is the energy of a single-particle state occupied by a single particle, μ is the chemical potential, k is the Boltzmann constant, and T is the absolute temperature. This creates a curve which is a step function at low values of T, but becomes more gradual as T increases, as in FIG. 4. This is nearly the relationship we are looking for; however, it is mirrored compared to what we expect in the graph of p against percolation probability. This is easily resolved by substitution $-\epsilon$ for ϵ.

Now, there are a number of other substitutions that must be made to fit the function to the data obtained in this experiment. The process is outlined in M. Chinchilla's work [4]. These substitutions are as follows:

$$\frac{1}{kT} = 2a, \tag{2}$$

where a is twice the slope to be located at p_c,

$$\bar{\epsilon} = p, \tag{3}$$

equating the relationship to the percolation threshold, and

$$\mu = -p_c, \tag{4}$$

to find the percolation threshold itself. After these substitutions, the equation

$$f(p) = \frac{1}{e^{-2a(p-p_c)}+1} \tag{5}$$

is formed. Now, using a leap of insight to fit this to a simplified equation, we multiply the numerator and denominator of the fraction by a factor of $e^{2a(p-p_c)}$ to the result

$$f(p) = \frac{e^{2a(p-p_c)}}{1+e^{2a(p-p_c)}}. \tag{6}$$

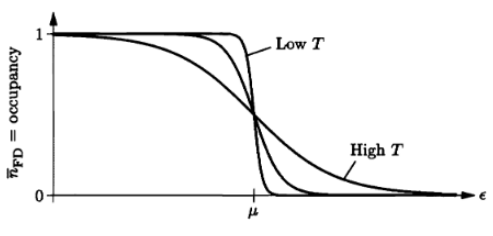

FIG. 4: Fermi-Dirac distribution at high, medium, and low values of T. Borrowed from D. V. Schroeder [5].

Expanding the squared expressions into their bases, the equation becomes

$$f(p) = \frac{1}{2}\left(1 + \frac{e^{a(p-p_c)} - e^{-a(p-p_c)}}{e^{a(p-p_c)} + e^{-a(p-p_c)}}.\right) \tag{7}$$

The numerator is now in the exact form of the hyperbolic sine function, and the denominator in the form of the hyperbolic cosine function, which simplifies to the hyperbolic tangent function. Simplifying the expression, we are left with the much simpler function

$$f(p) = \frac{1}{2}(1 + \tanh a(p-p_c)), \tag{8}$$

which has parameters giving the critical probability and the steepness of the curve at this point. It may not be obvious that this equation properly describes the relationship of percolation; however, previous research has determined that the distribution will closely follow this function [4].

III. PROCEDURE

A. Initial Design

To create a physical model for site percolation, we devised a system to represent water flowing through a porous solid with vacancy assigned randomly. To do so, slats with material randomly removed to a known vacancy value, *keys*, were placed together to create a grid, as shown in FIG. 5. Other options that would have worked to illustrate this could have been matches randomly placed, so that adjacent matches ignite each other; or a plane of conductive and non-conductive beads. However dramatic a fire may have been, the concept can be just as easily demonstrated by water with a much lower risk of damage.

To simulate a random grid of a known vacancy, thirty keys of acrylic were prepared. Each strip was approximately half of an inch in width and height, and six inches in length. A Mathematica program was used to produce schematics of the keys with a set number of quarter-inch cuts, distributed randomly about sixteenth-inch increments. The resulting shape is replicated in acrylic and calculated to find a value for each key's vacancy. An example diagram is presented in FIG. 6.

In design, the keys consisted of two halves, separated by a horizontal plane. The bottom half was completely solid, and only served as a back plate to keep the key together. This portion had no impact on the actual data collection, and was not figured into the total vacancy value. The top half of the key contained all of the material that actually constructs the percolation grids. Gaps allowed water to pass through, while solid areas blocked its flow. When keys were put together into grids, gaps could fall in line with one another to produce a route for the water's flow.

The initial set of ten keys was produced with a vacancy value around 0.59, approximately the value for two-dimensional critical probability found in previous studies [1, 3]. After analysis of this group, it was determined that further sets should be produced with lower vacancies, in an attempt to find a full range of data from a percolation probability of 0 to a percolation probability of 1.

B. Data collection

For each run, ten keys were selected and aligned together to produce a roughly square grid. The vacancy value for each key was averaged to find the total vacancy value for the setup. The grid was analyzed visually to determine if the system percolates or not, and a result was noted. Then, the same ten keys were randomly shuffled, flipped, rearranged, and reassembled into a grid of the same vacancy value, and the process was repeated to gather twenty possible configurations.

Determining if a system percolates is similar to solving a maze, in a sense that for a liquid to flow through, there need be only one route from one side to the other. Liquids solve mazes by themselves, filling in every possible route to the end, as in FIG. 7. Flooding could have been initiated at the top, to simulate liquid passing down through the grid; or from an area in the center, to simulate liquid spreading laterally as it is poured down onto a particulate substrate. The choice of flooding from the top of the apparatus and not from a point in the center was made for simplicity's sake, since previous experimentation found little difference between results of top-flooded and center-flooded experiments [3].

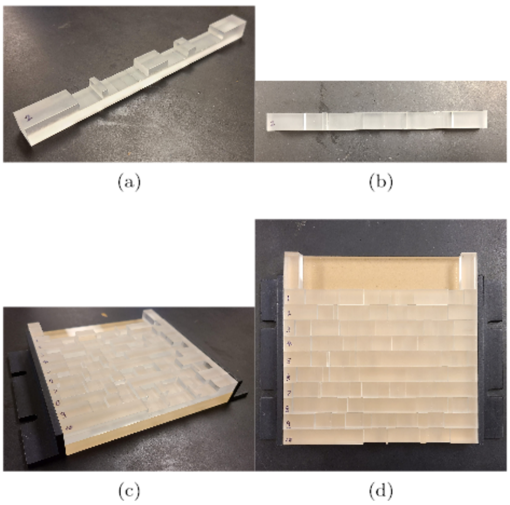

FIG. 5: Multiple angles of the keys and grids.

C. Implications of Setup

The acrylic keys in this experiment were of a fixed length, height, and width. The height of the acrylic, and indeed the height of the cuts, played no role other than to allow the passage of water. The width of the pieces does not matter either; as the cuts in the keys are all in this direction, the only consideration for percolation is if there is the presence of material to block the path. For example, a key with a width of one unit will effect the possible paths water could take the same as a key with a width of one hundred units. The length, however, does play a role: with shorter lengths, the half-inch cuts would take up a larger proportion of the total surface, and therefore would alter the vacancy value of the key. If the width of the cuts was altered proportionally to the change of the length, then this dimension could be scaled as well.

Now, consider the radius of the cuts themselves and the spacing between their possible locations. The cuts in this experiment are all a quarter inch in radius. Some cuts were within this length of another, creating an overlapping gap and a wider, continuous area of vacancy, allowing for the random particle sizes desired in this experiment. The location of every cut is randomly determined on sixteenth-inch intervals; while this granularity could have been continuous, it was much easier to calculate and machine cuts determined as they were. In addition, gaps closer than a sixteenth of an inch would have presented problems due to the surface tension of water. Closer than that, the water may have been more inclined to flow over the apparatus than to actually pass through the gap. However, with deeper cuts, a less viscous fluid, or higher pressure, a continuous interval could be physically tested with a similar setup.

The width of the cuts, on the other hand, play a massive role in the outcome of the experiment. Cut widths were the major determining factor in the grid size; larger cuts would result in a more smoothed-out relationship between vacancy and percolation probability, while thinner and more numerous cuts would approach the phase-change transition relationship of large and infinite grids.

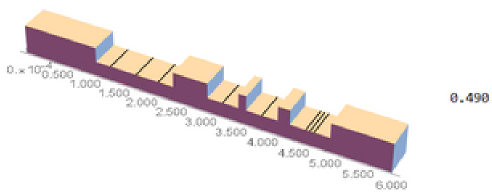

FIG. 6: Diagram of a key produced in Mathematica. The image represents the form that the final key will take. The black lines are the center lines where each half-inch cut will be made. To the right, the value represents the vacancy of the particular piece, on a scale from 0 to 1, 0 being an entirely vacant key, and 1 being an entirely solid key.

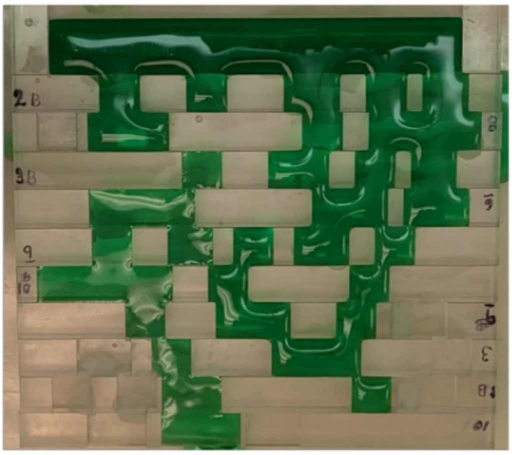

FIG. 7: An arrangement that exhibits percolation, the liquid finding an interesting route through the grid.

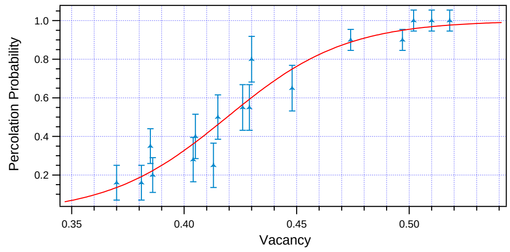

FIG. 8: Graph of percolation probability dependent upon vacancy, fit to Eq. 8. A slope at the critical probability is found to be $a = 19 \pm 2$, the critical probability is found to be $p_c = 0.42 \pm 0.03$, with $\chi^2 = 11.6$.

Finer cuts would allow for a closer estimate of the critical vacancy, but would be physically intensive in creating the keys and shuffling the setup. As the size of the cuts decreases, say by a half, the grids would require twice as many keys.

The reasoning behind making cuts with a chance to overlap over a strict grid is to more closely emulate what I would expect in natural phenomena. In soil and other organic particulate matter, particles are rarely of a fixed size and in fixed locations, as they often are in a computational approach. In this physical experiment, I sought to more closely replicate the variability of physical phenomenon such as liquids passing through organic particulate matter, such as soil. Although the variety of particle size is only in the horizontal direction, I imagine this could simulate particulate substrates where material has been layered flatly due to pressure or some similar circumstance.

IV. RESULTS & ANALYSIS

Keys were selected randomly from a group of thirty to obtain a range of average vacancies, but also specifically selected in a few cases to achieve vacancies at the upper and lower extremes of the possible combinations. Once a group of ten keys was selected, the group was randomly shuffled and flipped to create twenty random combinations of each grid's average vacancy.

Configurations were visually inspected to determine if percolation is possible. Once each of the twenty configurations were determined to percolate or not, the ratio of percolating grids to non-percolating grids is calculated, and a graph of percolation probability dependent upon average vacancy was constructed, as shown in FIG. 8.

A particular combination of keys led to results that were far outside of the expected relationship in four data points, due the location of the cuts allowing most systems to percolate, despite the orientation of the pieces. For example, regardless of the order or orientation of the pieces, a route almost always existed near the edges of the grid. These four data points were removed to obtain a more accurate representation of the relationship. In a more expansive test with more variety in keys, I would not expect these outliers to exist in in any significant proportion.

After collection of the percolation probabilities, it became apparent that in this system of overlapping vacant squares, the critical probability was much lower than what was expected from the more traditional grid setup used in computational experiments. While the upper end of vacancies, systems with a percolation probability of one, were well represented, the available keys did not have vacancy values low enough to produce any data runs with a percolation probability of zero.

The resulting percolation probabilities were averaged in groups of four of five similar values to find a standard deviation, which is used as the error values for each group of points. The logic behind this is that some grids of a known vacancy exhibited a much larger variance in percolation probabilities than others. For example, on the upper end of the vacancies, where percolation probability was higher, the data was more consistent. The vacancies on the lower end was the second most consistent, then variance increases towards the middle of the distribution. Calculating uncertainty in this way allows for a better weighted fit, where more precise data is valued more heavily then the more spread out points.

The fit function, Eq. 8, is the theoretical expectation, but it is found that $\chi^2 = 11.6$ when aligned to the data. I attribute this to the pseudo-random nature of the configurations; combinations of the same keys tended to favor either percolating or not, even when they are randomly shuffled. I would not expect this in a truly random grid configuration.

The critical probability is determined to be $p_c = 0.42 \pm 0.03$, which was around the value I expected to find it at during data collection. Yet, this is 29% lower than the predicted value, at 0.59. This discrepancy may have been the result of the pseudo-randomness; but with such a significant difference, it is much more likely that random particle size does play a significant role in critical probability.

V. CONCLUSION

The experimental setup of acrylic keys randomly assigned with cuts to allow for the observation of percolation around a critical threshold. As the keys are selected to create grids and shuffled randomly to obtain a percolation probability of a known vacancy value, the expected Fermi-Dirac relationship was somewhat apparent in the data. Initial research of critical probability p_c in two dimensional systems suggested a value of about 0.59, but through experimentation using this setup designed to more closely replicate organic systems, a p_c value of 0.419 ± 0.003 was found. This is 29% from the expected value, but I might suggest that allowing vacancies to overlap and assigning them on an interval smaller than the radius of the cut width will always result in a lower critical probability then a typical grid format. Logically, this conclusion makes sense, because even the smallest gaps between particles allows for percolation to occur. To confirm this, however, I would first like to run several tests on larger-scale grids, which would allow for a more precise estimation of p_c.

In the very least, I believe that the results of this experiment suggests a difference between computational results and those that may be found in organic systems. For example, I believe that the critical probability for the spread of a bush fire may be lower then what one may expect. Extending this suggestion, I believe that a similar test in three dimensions where vacancies are randomly assigned and allowed to overlap may result in closer estimates of real-world percolation problems, such as the passage of water through soil.

It is possible that allowing for overlapping vacancies changes the 'aspect ratio' of the grid; for example, allowing the fourth-inch cuts to be made in sixteenth-inch increments may actually have the effect of producing a grid that is in effect four times shorter, since the cuts are effectively four consecutive vacant spaces in a grid that is effectively divided into sixteenth-inch squares. This, too, would require further investigation to confirm or disprove.

VI. ACKNOWLEDGMENT

I would like to acknowledge Dr. John Lindner for his assistance in devising this equipment; Mr. Timothy Siegenthaler for his incredible work in creating the experimental setup; and Dr. Susan Lehman, instructor of the lab, for allowing access to the facility required to execute this experiment and granting guidance through the experiment.

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