The Application of Idealized Models to Isolate Dominant Features of a Physical Double Pendulum

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The double pendulum is a pendulum with a second pendulum attached to the end of the first. The double pendulum is a chaotic system whose motion is difficult to model over time. While idealized models enable computational simulations which can predict the pendulum's motion, they do not reflect the behavior of a physical double pendulum. The inability to model the motion of a physical double pendulum accurately does not yield the model ineffective. Rather, it isolates a unique behavior of the double pendulum. This experiment used a purposely limited mathematical model and computer simulations to discover what proprieties of the double pendulum are so dominant that they appear both in computer simulations and in the physical double pendulum. By comparing computational results to experimental results of the time it takes for one of the pendulums in the double pendulum to flip, the double pendulum was found to have a distinct transfer of energy between the two pendulums which is based on the initial difference in angle between the two pendulums. Ultimately, a pendulum was more likely to flip if they were out of phase by an angle of $\pi/2$ with one the other. That is if the pendulums were in phase the energy transfer was potential to linear kinetic energy, and no flip occurred, and if they were out of phase the energy transfer was from potential to rotational kinetic energy and a flip did occur.

I. INTRODUCTION

The legacy of Daniel Bernoulli is for his work in fluids dynamics and the early development of probability theory. However, a lesser-known fact is Bernoulli was the first to analyze the double pendulum in 1733 [1]. The double pendulum is when two pendulums are attached such that the pivot point of the second pendulum is at the end of the first. Bernoulli modeled the motion of small-angle oscillations of the double and n-chained pendulum. While Bernoulli is credited as the first, the historical continuity for the double pendulum's motion after his initial discovery is contentious. Some imply Euler and Bernoulli began to work together due to their friendship [2]. Other's suggest that first Bernoulli, then Euler, and lastly Bernoulli's father developed their equations independently [3]. However, Bernoulli claimed his father stole his work and improved it to appear as his own [1]. Bernoulli's father nearly developed the differential equations of motion for the double pendulum yet failed. Finally, D'Alembert discovered the differential equations of motion for the double pendulum, yet it was the application of his results to derive the first wave equation which he is most known for, not the equations themselves [4].

The use of differential equations to model the dynamical system is a powerful application of calculus to model physical systems [1]. Yet certain differential equations are problematic; they appeared unsolvable and represent phenomena that had no order. The lack of order is a result of the double pendulum's nature: it is inherently chaotic. That is, there is no analytical solution for the pendulum's position as a function of time. Eventually, computers solved the equations of motion through numerical integration, and the double pendulum was analyzed with renewed interest. The lack of historical consistency perhaps represents how computational analysis enabled a deeper conception of chaotic systems and bolstered chaos theory into mainstream physics.

To analyze how computers changed the modern conception of dynamics, the double pendulum will be analyzed through the application of computations, Lagrangian formulation, and experimental results. While computers model complicated dynamics beyond human ability, some systems are idealized to neglect aspects of physical systems which are difficult to model accurately. Modern computational programs model real-world phenomena with incredible accuracy and precision, yet real-world chaotic systems prove difficult to model as a slight deviation in initial conditions results in divergent behavior over long periods.

This experiment used a purposely limited computational model and mathematical model of the double pendulum to discover the fundamental properties of the system. As mentioned prior, predicting the motion of the double pendulum is not possible without the most careful considerations to the physical system. Thus, a different aspect is tested; if one of the pendulums can *flip* or makes an angle greater than π to its rest position while in motion. That is, if the one of the pendulums can make a full revolution about its pivot point. If one of the pendulums can flip, then the time it takes is compared between different initial positions. For the mathematical model, qualitative observations were made and compared to the model to see if the observed behavior conflicts or corroborates the mathematical model.

II. THEORY

The double pendulum is a chaotic system whose solutions to the equations of motion cannot be solved analytically. However, the Lagrangian for the double pendu-

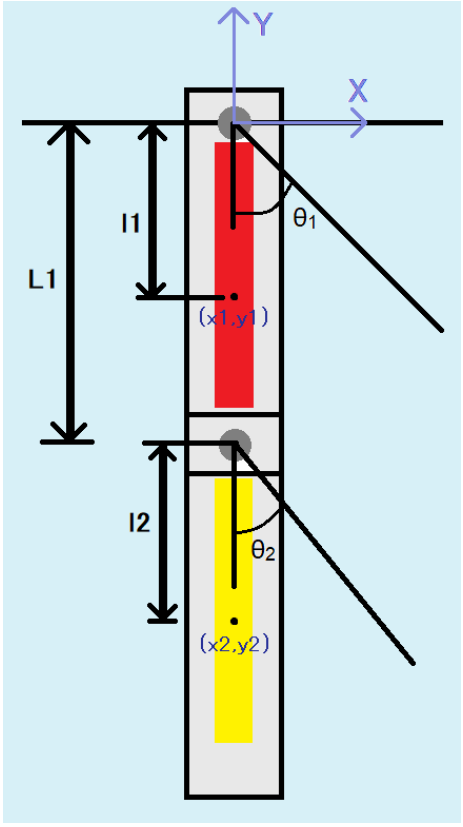

FIG. 1: A double pendulum with the center of mass, lengths to the center of mass, angles, and coordinate system which is used to derive the Lagrangian.

lum is possible to construct, which models the energy of the system. For our purposes, the Lagrangian is used to develop a qualitative description of the behavior of the double pendulum.

A. Assumptions and Coordinates

First, a few assumptions are made to simplify the derivation: frictional forces and potential products of inertia are assumed to be negligible, the total energy of the system remains constant, and the gravitational potential energy is zero at the origin of our coordinate system. Next, we define the coordinate system for our model. The double pendulum is constructed in the form of Fig. 1 where the first pendulum is colored red and the second pendulum is colored yellow and the grey circles represent the pivot points for each respective pendulum. The first pivot point is where the origin is set, as it does not move while the pendulums are in motion.

To model the motion of the pendulums, we identify the center of mass for each pendulum as two sets of coordinates, which are denoted in Fig. 1 as (x_1, y_1) and (x_2, y_2) for the first and second pendulum respectively. Lastly, in Fig. 1, we define parameters θ_1 and θ_2 to represent the angles of the pendulums from the $-y$ axis, l_1 and l_2 be the lengths to each pendulums center of mass from their respective pivot points, and L_1 to be the total length of the first pendulum. Now we apply the assumptions and coordinate system to model the motion of the double pendulum.

The location of the center of mass for each pendulum is

$$\begin{aligned} x_1 &= l_1 \sin\theta_1, \;\; y_1 = -l_1 \cos\theta_1 \\ x_2 &= L_1 \sin\theta_1 + l_2 \sin\theta_2 \\ y_2 &= -L_1 \cos\theta_1 - l_2 \cos\theta_2, \end{aligned} \tag{1}$$

where x_1, y_1, x_2, and y_2 is the position of the center of mass, l_1 and l_2 are the length to the center of mass for each pendulums respective pivot point, L_1 the length between the two pivot points for the top pendulum, and θ_1 and θ_2 is the angle of the respective pendulum to the normal. The time derivative of these functions are

$$\begin{aligned} \dot{x_1} &= l_1\dot{\theta_1}\cos\theta_1, \;\; \dot{y_1} = l_1\dot{\theta_1}\sin\theta_1 \\ \dot{x_2} &= L_1\dot{\theta_1}\cos\theta_1 + l_2\dot{\theta_2}\cos\theta_2 \\ \dot{y_2} &= L_1\dot{\theta_1}\sin\theta_1 + l_2\dot{\theta_2}\sin\theta_2, \end{aligned} \tag{2}$$

where the new term $\dot{\theta}$ is the angular velocity of the the respective pendulums. Armed with the location and change in location over time for each pendulum's center of mass, the Lagrangian for this system is constructed.

B. The Lagrangian

To construct the Lagrangian, we need kinetic and potential energy of the pendulums as the Lagrangian is defined to be

$$\mathcal{L} = T - V,$$

where $\mathcal{L}$ is the Lagrangian, T is the kinetic energy, and V is the potential energy. However, before we find the kinetic and potential energy, let us redefine some of the parameters of the double pendulum to simplify the model.

First, L_1 and l_2 are defined in terms of l_1, which is

$$\begin{aligned} a = \frac{L_1}{l_1} \implies al_1 = L_1 \\ b = \frac{l_2}{l_1} \implies bl_1 = l_2 \end{aligned} \tag{3}$$

where a and b are numerical constants depending on the design of the double pendulum. Second, m_1 is defined in terms of m_2, which is

$$c = \frac{m_1}{m_2} \implies cm_2 = m_1 \tag{4}$$

where c is the ratio of the mass of the pendulums. With these simplifications, we model the kinetic and potential energy.

1. Kinetic Energy

The total kinetic energy of our system is comprised of linear and rotational kinetic energy. Recall the total kinetic energy is defined as

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I_{cm}\dot{\theta}^2,$$

where m is the mass, $\dot{x}$ and $\dot{y}$ are the components of the linear velocity, I_{cm} is the moment of inertia about the center of mass, and $\dot{\theta}$ is the angular velocity. For the double pendulum, the kinetic energy is

$$\begin{aligned} T = &\frac{1}{2}m_1\left(\dot{x_1}^2 + \dot{y_1}^2\right) + \frac{1}{2}I_{1\ cm}\dot{\theta}^2 \\ &+\frac{1}{2}m_2\left(\dot{x_2}^2 + \dot{y_2}^2\right) + \frac{1}{2}I_{2\ cm}\dot{\theta}^2. \end{aligned} \tag{5}$$

Substituting the results from Eq. 2 and 3 into Eq. 5, we find

$$\begin{aligned} T = &\frac{1}{2}m_1 l_1^2\dot{\theta_1}^2 + \frac{1}{2}I_{1\ cm}\dot{\theta_1}^2 \\ &+\frac{1}{2}m_2\left(a^2 l_1^2\dot{\theta_1}^2 + l_2^2\dot{\theta_2}^2 + 2al_1l_2\dot{\theta_1}\dot{\theta_2}\cos\left(\theta_1 - \theta_2\right)\right) \\ &+\frac{1}{2}I_{2\ cm}\dot{\theta_2}^2 \end{aligned} \tag{6}$$

and if we apply the parallel axis theorem

$$I = I_{cm} + ml^2,$$

where l is the distance between the center of mass and the point of rotation, Eq. 6 reduces to

$$\begin{aligned} T = &\frac{1}{2}I_1\dot{\theta_1}^2 + \frac{1}{2}I_2\dot{\theta_2}^2 \\ &+\frac{1}{2}m_2\left(a^2 l_1^2\dot{\theta_1}^2 + 2al_1l_2\dot{\theta_1}\dot{\theta_2}\cos\left(\theta_1 - \theta_2\right)\right) \end{aligned} \tag{7}$$

which is the kinetic energy of our system. Notice the rotational kinetic energy is the first two terms and the third term contains two terms: the linear motion of the pivot point of the second pendulum and a more complicated form of linear motion. With the kinetic energy modeled, we now model the potential energy.

2. Potential Energy

The potential energy of our pendulums is potential gravitational energy, which is modeled by

$$V_{grav} = m_1 g y_1 + m_2 g y_2$$

where m is the mass of the object, g is the acceleration due to gravity, and y is the height of the respective pendulums. From Eq. 1, 3, and 4, the potential energy of our system is

$$V_{grav} = -m_2 g l_1\left(c\cos\theta_1 + a\cos\theta_1 + b\cos\theta_2\right),$$

and thus the potential energy of the double pendulum is

$$V = -m_2 g l_1\left(\left(c + a\right)\cos\theta_1 + b\cos\theta_2\right). \tag{8}$$

With both the kinetic and potential energy modeled, we can finally construct the Lagrangian, which is

$$\begin{aligned} \mathcal{L} = &\frac{1}{2}I_1\dot{\theta_1}^2 + \frac{1}{2}I_2\dot{\theta_2}^2 \\ &+\frac{1}{2}m_2\left(a^2 l_1^2\dot{\theta_1}^2 + 2al_1l_2\dot{\theta_1}\dot{\theta_2}\cos\left(\theta_1 - \theta_2\right)\right) \\ &+m_2 g l_1\left(\left(c + a\right)\cos\theta_1 + b\cos\theta_2\right). \end{aligned} \tag{9}$$

This result is confirmed by Hans Jürgen [5] to be the correct derivation of the Lagrangian of the double pendulum, where the constant values are changed depending on the physical design. The equations of motion are possible to find, but the solution to the equations provides little insight for our purposes. Next, we shall derive the potential energy required for a flip to occur.

C. Potential to Flip

For the double pendulum to flip, there must be enough initial energy in the system for one of the pendulums to make a complete revolution. With the pendulum released from rest, the kinetic term in Eq. 9 goes to zero and thus the gravitational potential energy must be energetically enough for a flip to occur. The lowest amount of energy for one of the pendulums to make a complete revolution is either $\theta_1 = \pi$ or $\theta_2 = \pi$. Let us assume that $b < c + a$ which implies that the second pendulum requires less energy to flip. Therefore, from Eq. 8, the minimum potential energy required is

$$\begin{aligned} V &= -m_2 g l_1\left(\left(c + a\right)\cos\left(0\right) + b\cos\left(\pi\right)\right) \\ &\implies V_{min} = -m_2 g l_1\left(c + a - b\right), \end{aligned}$$

where V_{min} is the minimum energy required for a flip. Thus, the requirement of the potential energy for a flip is

$$\begin{aligned} V &> V_{min} \\ -\left(\left(c + a\right)\cos\left(\theta_1\right) + b\cos\left(\theta_2\right)\right. &> -\left(c + a - b\right). \end{aligned}$$

If we remove the negative signs, then we find that a flip is *impossible* if

$$\left(\left(c + a\right)\cos\left(\theta_1\right) + b\cos\left(\theta_2\right) > \left(c + a - b\right)\right. . \tag{10}$$

With the values of a, b, and c, then the angles which it is energetically possible for a flip to occur is found. Although, even though it may have enough energy for a flip to occur, that does not mean a flip will always occur. In the next section, we apply the results of our model to discover what set of angles have the same potential energy.

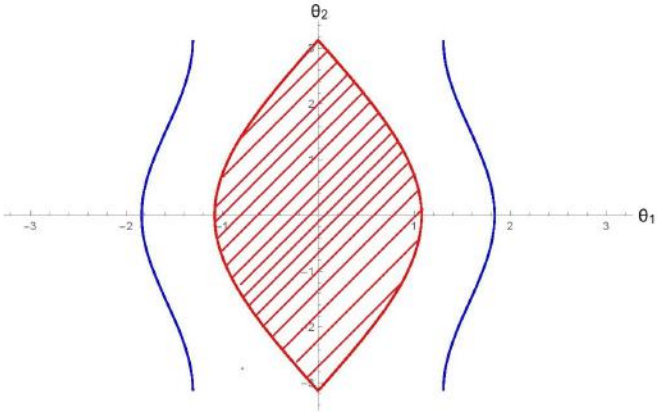

FIG. 2: The computational result where the x-axis represents the initial release angle, θ_1, for the first pendulum and the y-axis represents the initial release angle, θ_2, for the second pendulum. The blue line is the possible initial angles combinations for when the potential energy equals zero and the red line is the possible initial angle combinations for minimum energy required for a flip to occur. Inside the shaded region of the red line is where a flip is impossible to occur regardless of any combinations of different combinations of initial release angles.

III. THE ANGLES FOR CONSTANT POTENTIAL ENERGY

To determine what corresponding angles of θ_1 and θ_2 have the same potential energy, the results from Eq. 8 and Eq. 10 were applied. First, Eq. 8 was simplified such that $m_2 = g = l_1 = 1$; this was done as the value of V is not important but rather that it remains constant. To denote the difference between the actual potential energy and the numerical potential energy, V_c is defined to be the computed potential energy. Next, the constants a, b, and c were measured to be $a = 1.556$, $b = 0.778$, and $c = 1.400$, which are needed to solve Eq. 8 numerically. Finally, the initial potential energy was chosen to be the value for Eq. 8 when $\theta_1 = \theta_2 = \pi/2$, which was 0. This process resulted in the equation

$$V_c = 0 = 2.956\cos\theta_1 + 0.778\cos\theta_2. \tag{11}$$

To find angles that satisfy this equation besides $\theta_1 = \theta_2 = \pi/2$, the solutions for Eq. 11 was computed in Mathematica, which yielded a θ_2 as a function of θ_1, and thus yielded the corresponding angles which $V_c = 0$ held. The same process was applied to Eq. 10 to find which corresponding angles a flip is energetically impossible. The solutions to Eqs. 11 and 10 are represented in Fig. 2 with θ_1 on the x-axis and θ_2 on the y-axis, with the blue line representing $V_c = 0$ and the red area representing which angles a flip is impossible to occur.

With the desired angles found, the physical double pendulum was constructed to incorporate the computed angles. Before any trials were conducted, the physical double pendulum was refined to be consistent with the assumptions made for our model.

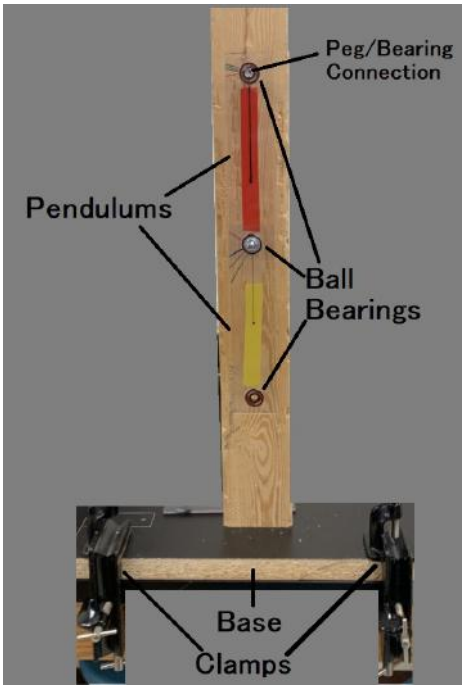

FIG. 3: The physical double pendulum's final design.

IV. EXPERIMENTAL DESIGN

The theoretical model developed prior assumed aspects of the system which are not true for the physical double pendulum, such as no frictional forces. Therefore, the double pendulum was constructed specifically to reduce any properties which we did not model. Fig. 3 was the final design for the physical double pendulum. In the figure, notice that the base of the pendulum is clamped down which was to ensure that it did not move as the pendulums were in motion. The pendulums were comprised of identical acrylic rectangles and low friction ball bearings, with the first pendulum attached to a peg secured in the board. In Fig. 3, the ball bearing attached to the peg was secured by parafilm between the inner portion of the ball bearing and the peg itself to ensure that the bearing did not slide along the peg.

The pendulums were connected by a bolt and nuts with washers spacing the pendulums which resulted in Fig. 4 to prevent contact between the pendulums. However, the number of washers used was only enough to barely avoid contact as increased space between the pendulums resulted in torque being applied to the peg. While parafilm reduced the motion on the peg large amounts of torque resulted in motion along the axis of rotation. Moreover, the washers added lowered the center of mass for the first pendulum. Done to a large extent, the behavior of the double pendulum changes drastically.

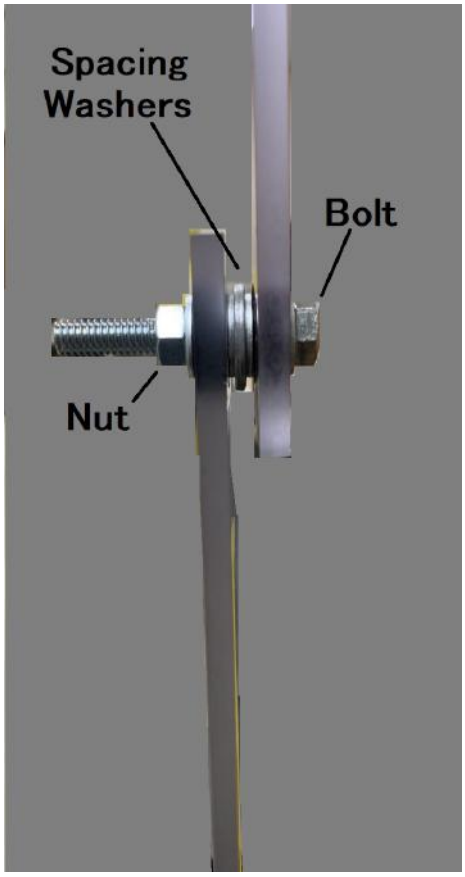

FIG. 4: How the two pendulums were connected and spaced through the use of a nut, bolt, and 3 spacing washers.

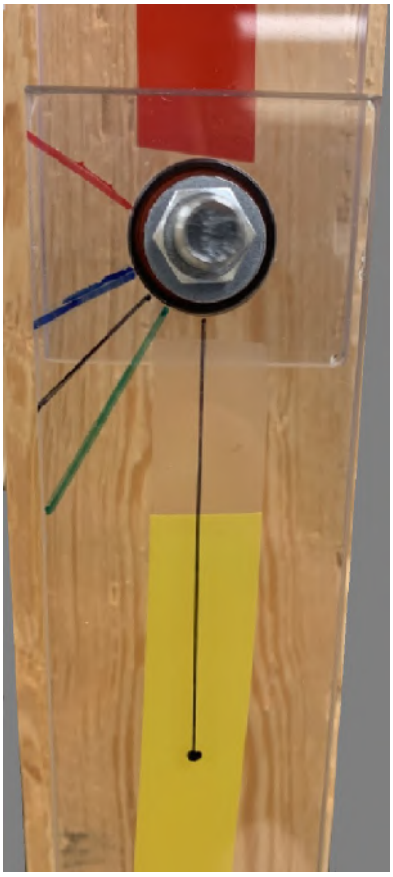

FIG. 5: The different angles which were computationally found and how they were denoted and color-coded on the pendulums.

Fig. 5 demonstrates how the angles found prior were denoted on the pendulums, with the angles drawn on the second pendulum colored to correspond with angles drawn on the first pendulum. The larger black lines in Fig. 5 is the position of the center of mass for the pendulum and was used as the reference to draw each of the colored angles. The matching colors represent the angles that each pendulum required for $V_c = 0$. Lastly, the pendulums were recorded by a camera to measure the drop angle and the time it took for a flip to occur. The camera was placed on a stand about 1.2 meters away from the pendulum and positioned to have both the pivot in view. With the pendulum carefully constructed, we now can develop the experimental method to analyze the system.

V. EXPERIMENTAL PROCEDURE

The experiment intended to select pairs of θ_1 and θ_2 such that all pairs had equivalent initial energy. Therefore, the pendulums were dropped from rest, and the angles were measured such that Eq. 8 is constant. Each corresponding angle was dropped multiple times, as ensuring the angles were exact proved to be difficult alone. However, with enough drops, the desired angles had a greater likelihood of being recorded. Ultimately, the angles dropped were somewhat consistent, with data containing angles that deviated slightly. This resulted in the initial potential energy not being constant, but it did provide an upper bound for the initial energy in the system.

The video recordings of the pendulums were analyzed qualitatively and the angles were measured again to ensure the precision and accuracy of each trail. A flip was determined to be once either pendulum reached the value of $\theta = \pi$ and proceeded to flip over. Alternatively, a flip could be determined for when a pendulum makes a complete revolution, returning to the initial angle. However, the pendulums completed a revolution with higher velocities than reaching $\theta = \pi$ and thereby was difficult to determine the time it took to achieve the flip. Once the desired angles were tested, the pendulum was dropped from drastically different levels of height to see how the level of energy stored in the pendulums altered the behavior of each pendulum.

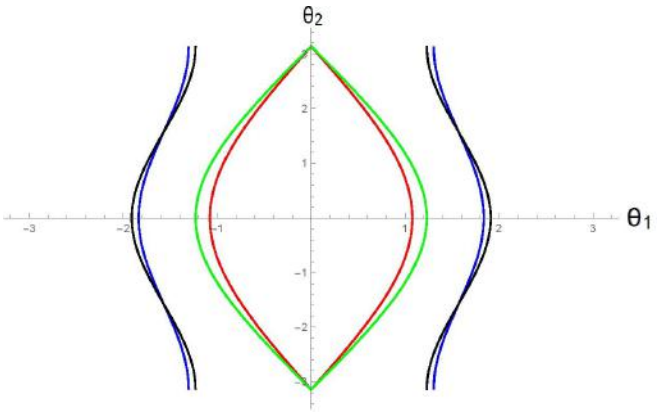

FIG. 6: The same figure as Fig. 2, except the numerical solution for Eq. 12 superimposed onto it. Again, the x-axis represents the initial release angle, θ_1, for the first pendulum and the y-axis represents the initial release angle, θ_2, for the second pendulum.

VI. COMPUTATIONAL RESULTS

The double pendulum which was experimentally tested was not modeled computationally. However, the time flip graph for the massive rod double pendulum was produced, and their equations for V_c were nearly identical. For example, with the process to construct Fig 2 applied for the massive rod pendulum, which resulted in the equation

$$V_c = 0 = 3\cos\theta_1 + \cos\theta_2, \tag{12}$$

which is nearly identical to Eq. 11.

Eq. 12 was added to Fig. 2, which resulted in Fig. 6. The black and green lines almost exactly correspond to the blue and red lines respectively. While the time flip graph may differ in the exact initial conditions a flip occurs, the properties of the time flip graph were analyzed to compare the experimental properties of the physical double pendulum.

The time flip graph is visualized in Fig. 7, where θ_1 is on the x-axis and θ_2 is on the y-axis. The graph is colored to denote how long it takes for a flip to occur, with green the shortest, red a moderate amount, and purple the longest. Each of these colors are on a scale of light to dark, where dark indicates a shorter amount of time and light indicates a longer amount of time. For example, the angles for light green indicate a flip takes longer to occur than the angles for dark green but takes less time than the angles for dark red. The white color represents where a flip does not occur, where we can see a similar shape to the red area in Fig. 2.

From Fig. 7, one conclusion is that the areas with the angles that are out of phase by $\pi/2$ seem to flip more quickly even at a lower level of potential energy whereas the areas which are in phase appear to take a longer time for a flip to occur. Eq. 9 corroborate this conclusion as the linear kinetic energy term is dependent upon the

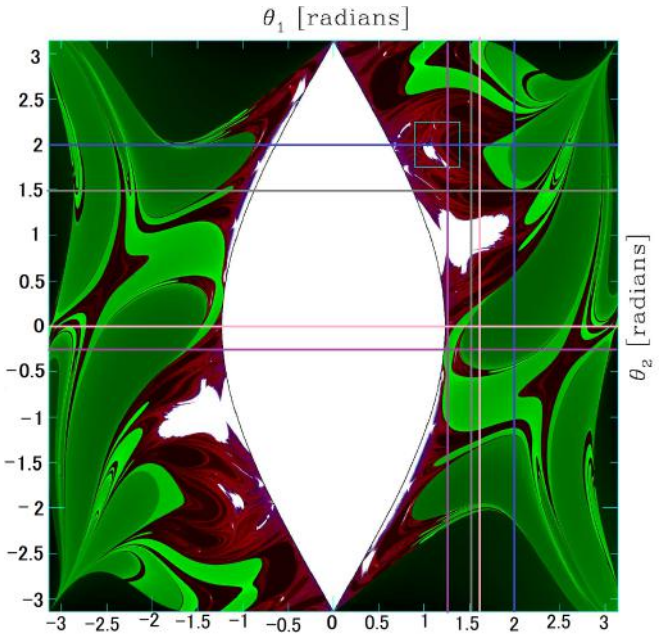

FIG. 7: The computational result for a double pendulum which is constructed with rods and identical locations for the center of mass. The corresponding pairs of angles are the initial conditions and the color is coded to represent the amount of time it takes for a flip when dropped from the particular angles. Modified from [6].

phase of the initial conditions. That is, to minimize the linear energy, if the pendulums are out of phase, then the stored potential energy will convert to rotational motion.

To be more precise, Fig 7 has lines drawn to specific values of θ_1 and θ_2. First, consider the grey lines, when both θ_1 and θ_2 are in phase with a value of 1.5 radians. The region which they align is a dark red region, meaning that the flip occurs in a moderate amount of time. Now consider the pink lines, which $\theta_1 = \pi/2$ and $\theta_2 = 0$. The region in which they align is dark green and thus the pendulum flips in a shorter amount of time. More specifically, the potential energy stored in the in-phase case is $V_c = 0$ while out of phase the energy is $V_c = -1$. The initial conditions with more energy do not imply that a flip will occur in a shorter amount of time.

Now bring attention to the blue lines in Fig. 7. That is when the pendulums are released in phase at a value of 2 radians. A flip occurs within a relatively short amount of time. However, this does not contradict Eq. 9. If large amounts of energy are stored in the system, a flip should occur regardless of the phase. Now, consider the purple lines, which represent the pendulums being out of phase by about $\pi/2$. The purple lines are barely outside the region where it is energetically impossible for a flip to occur, yet a flip still occurs. Therefore, for lower amounts of energy, the phase is more consequential to whether or not a flip occurs. With the computational results determined, we move on to the experimental results.

VII. EXPERIMENTAL RESULTS

The experimental results are split into two groups: qualitative observations and quantitative observations. The qualitative observations were compared to the theoretical model and the quantitative observations were compared to the computational results mentioned prior.

A. Qualitative Observations: Height and Speed

Some of the observations made were difficult to reproduce exactly due to the differing motion of the pendulum due to slight deviations of initial conditions. However, qualitative properties of the double pendulum were consistent and recorded. First, the greater the initial angle that each pendulum is released, the greater the height can be for a flip to occur. It was noticed that for large angles, one of the pendulums made a revolution at a greater height than those which were released at smaller angles. For example, the flip which occurred at maximum height was when the initial conditions of $\theta_1 = \theta_2 = \pi$.

The second observation was that the speed of rotation for the second pendulum increased if the first pendulum's center of mass was at its lowest y position and had no rotational motion. Conversely, if the center of mass for the first pendulum was at a larger value of y and it was in rotational motion, then the second pendulum's rotational speed was at the lowest. For example, when the first pendulum was at an angle of zero and not rotating, then the rotation of the second pendulum was maximized for the trial. Furthermore, if the second pendulum managed to flip at a greater height, the speed of its rotation decreased compared to when it flipped at a lower height. Both observations were consistent but were difficult to quantify. However, they indicate particular properties of the double pendulum.

B. Quantitative Observations: To Flip or Not to Flip

The experimental results for the physical double pendulum time to flip is represented in Table I, with the initial angles for each pendulum, the amount of frame it took for a flip to occur if it did, the difference between the angles, and the calculated initial energy V_c.

From Table I, it is clear that experimental results did not seem to provide conclusive results into the behavior of the pendulum when the potential energy remains constant. The difference in V_c between the different trials makes it impossible to tell how energy alters the pendulum's flipping time. However, what did remain consistent is that when $\Delta\theta$ was closer to the value of $\pi/2$, the time to flip was decreased while when $\Delta\theta$ was closer to 0, the pendulum either did not flip or took an increased amount of time to flip. Moreover, while V_c was not consistent, it always remained less than 0. Therefore, we restricted the initial conditions to be that of low initial potential energy. With the qualitative and quantitative experimental results, we now compare to see if the theoretical model and computational model have isolated a fundamental property of the double pendulum.

TABLE I: Experimental results of the physical double pendulum.

θ_1	θ_2	Frames Till Flip	V_c	$\Delta\theta$
89.9°	90.1°	No Flip	≈ 0	0.2°
90.0°	91.2°	No Flip	≈ 0	1.2°
90.6°	91.6°	No Flip	≈ 0	91.6°
102.4°	66.9°	31	-0.3244	35.4°
103.3°	64.2°	20	-0.3278	39.0°
102.3°	59.6°	20	-0.3244	42.7°
110.0°	46.8°	16	-0.4794	63.2°
107.2°	43.5°	16	-0.3097	63.7°
108.8°	45.0°	17	-0.4024	63.8°
112.4°	34.7°	16	-0.4868	77.7°
112.4°	34.0°	16	-0.4814	78.4°
114.5°	36.1°	17	-0.5878	78.4°
115.9°	28.9°	10	-0.6100	87.0°

VIII. ANALYSIS: ARE IDEALIZED MODELS USEFUL?

A. Theoretical Model

The qualitative observations indicated the height when a flip occurs increased when the initial angles were increased and that the rotation of the second pendulum increased as the height and rotational velocity of the first pendulum decreased. Through Eq. 9 and specifically the potential energy term, larger angles indicate a greater amount of potential energy being stored in the system. Therefore, the increased amount of energy stored in the system implies there was more energy to be distributed to the terms of the Lagrangian. Eq. 10 demonstrated that there is a minimum amount of energy for a flip to occur, which must be in the form of rotational kinetic energy of the second pendulum. Thus, if the requisite rotational energy is met, then the rest of the energy can be stored in the other terms of Eq. 9, which could be in the potential energy term.

Moreover, when the second pendulum flipped, the first pendulum would either come to a stop or would drastically slow down. The linear terms of kinetic energy and the rotational kinetic energy of the first pendulum decrease as they are dependent on $\dot{\theta}_1$. The remaining terms are the potential energy for both pendulums and the rotational kinetic energy for the second pendulum. Thus, with more overall energy in the system, the potential energy terms could be greater and still have enough energy remaining for a flip to occur. If the initial potential energy is decreased, then the total energy is constrained to either be potential energy or rotational kinetic energy,

not both. Thus, it is expected to see the pendulum only flip at a lower height when the initial angle of release is small for both pendulums.

Even if the first pendulum effectively came to a standstill, a portion of the energy in the system is necessarily potential, and thereby the rotational kinetic energy of the second pendulum was less, meaning that $\dot{\theta}_2$ is decreased and the pendulum flipped at a slower rate. However, if the heights of the pendulums are at their lowest, then the energy is not potential and must go to the only remaining term: the rotational kinetic energy of the second pendulum. Therefore, if the first pendulum is not in motion and at its lowest y position, the energy in the system is contained only in the rotation of the second pendulum, and $\dot{\theta}_2$ is maximized. Both of the qualitative observations are well represented through Eq. 9, meaning the transfer of energy between the pendulums is a fundamental aspect as to how and why flips occur.

B. Computer Simulations Versus the Real World

The computer simulation was unable to accurately depict when flips occur in the physical double pendulum, as the trials for $\theta_1 = \theta_2 = \pi/2$ resulted in no flip while the simulation suggested a flip would occur. However, as mention in the computational results, if $\Delta\theta$ is increased, then lower amounts of V_c still result in flips occurring. More specifically, the computational results suggested that at lower initial potential energies, $\Delta\theta$ was the determining factor as to whether or not a flip occurs rather than V_c. Experimentally, these results were confirmed, as Table 2 indicates that the time for a flip to occur decreased as the phase angle increased. For small amounts of V_c, the fastest flip occurred when the pendulums were most out of phase, even if there is less overall energy stored in the system. The trials with the most initial energy were the trials that did not flip, which indicates that a fundamental aspect to the flip behavior of the double pendulum is the phase angle between the two pendulums.

From Eq. 9, the term $2al_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)$ models why this phenomenon occurred. If the angles are in phase, this value is maximized, which is a part of the linear kinetic energy term. If the angles are out of phase by $\pi/2$, then this term disappears, which implies the linear kinetic energy is minimized. Therefore, the initial potential energy in the system is converted to rotational kinetic energy if the starting angles are out of phase, meaning a flip is expected to occur more quickly. Conversely, if the angles are in phase, the potential energy will more greatly be converted to linear kinetic energy, and thus a flip will take longer to occur. Since frictional forces were acting on the double pendulum, this increase in time resulted in a flip not occurring, but this does not negate the computational simulation. Instead, it revealed a fundamental property of the double pendulum: how phase angles alter the conversation of potential energy to kinetic.

IX. CONCLUSION

The double pendulum is a chaotic system that does not lend itself to predictive motion. However, through the application of a highly idealized model and simulation, the double pendulum did seem to have a particular behavior that greatly impacts when a flip occurs: the phase angle between the two pendulums. However, more aspects of the double pendulum are potentially lost due to the limitations of the experiment. First, the computer simulation can be altered to retain more aspects of the physical pendulum constructed so the time flip graph is more representative of the physical system. Second, to improve the physical pendulum, ball bearings with lower frictional constants, a connection which aligned the pendulums exactly rather than one in front of the other as seen in Fig. 4, and pendulums with an identical center of masses can be used to make the conversion to computer simulation more achievable. As the design approaches the theoretical model, the differences between computational results and experimental results are readily comparable and may reveal more aspects of the double pendulum. Lastly, more data should be collected for the quantitative results and the qualitative results could be measured and provide a more exact description of how the double pendulum behaves. Nonetheless, while the experiment was imperfect, it still produced conclusive results.

One of the purposes of idealized models is to reduce a system such that its most fundamental properties which dictate its behavior are modeled effectively. Then, the model is expanded upon until it represents the natural world. Moreover, idealized models ensure the dominant proprieties of a system are most apparent and are a tool for the scientific process. They provide which components of a system should be varied first whenever experimentation is done. For the double pendulum, the use of simple models revealed that the initial energy of the system is less important than how the energy is transferred in the system. That is, the idealizations capture impactful components of the double pendulum; the phase angle between the two pendulums dictates the transfer of energy within the system.

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