Finding the Figure-Eight Solution to the Three-Body Problem for a Double Binary Star System Using Mathematica

Shivam Bhasin
Department of Physics, The College of Wooster, Wooster, Ohio 44691, USA
(Dated: May 2, 2021)

A simulation was created on Mathematica to visualize the figure-eight solution to the three-body problem discovered by Cristopher Moore in 1993. The Mathematica notebook was further developed to create a system of a double binary star system. A search for this solution was performed using the phase angle difference of one of the binary star systems with respect to the other as a parameter. All the other factors affecting the dynamics of this system were kept constant to idealize the model. Fifty thousand different values of the phase angle for one binary star were simulated to look for a trinary star system. The orbits of these resulting systems were evolved in time for analysis. These systems, being the candidates for the figure-eight solution, were visualized using plots. Out of twenty-one candidates for figure-eight orbits, none were the desired solution. This result makes practical sense considering the physical restrictions on the modeled system. Therefore, even in this heavily idealized model, the probability of finding a figure-eight orbit is less than 1/50000.

I. HISTORY OF THE THREE-BODY PROBLEM

The three-body problem is a subset of a generalized field of the n-body problem, which has been of special interest to the greatest minds. Isaac Newton solved the two-body problem when he developed his theory of gravity. While working on his theory, he accounted for the gravitational attraction between two bodies and predicted their motion [1]. He also considered the effects of a larger body, like the Sun, on a two-body system, like the Earth and the Moon. It was one of the earliest occurrences of the three-body problem. The interest in this problem started growing in the 18^{th} and the 19^{th} centuries, when the brightest minds of the time tried finding the solution to the three-body problem.

Sweden's King Oscar II in the 1800s established a prize for the person who solves the n-body problem. Even though no one was able to solve the three-body problem, let alone the n-body, Poincare was awarded the prize for his valuable insights, which proved to be sufficiently important for the work that followed. To this date, the general three-body problem remains unsolvable and is also claimed to be impossible. However, many advances have been made in this field, including the work of Montgomery in the 2000s, which increased the number of classical particular solutions for the three-body problem [2].

Since most of the three-body dynamical systems result in chaos, there are only a handful of stable solutions that exist. Most of the solutions are highly sensitive to small perturbations and are unpredictable. The first stable solution was discovered by Leonhard Euler in 1765 for a special case where all three bodies are collinear at any given time and lie on the same plane as depicted in FIG. 1 a). These conditions are necessary for this solution to exist. In this orbit, two bodies travel in an ellipse, with the center as the third body. One of the three bodies is assumed to have negligible mass as compared to the other two. The three-body problem with this assumption is now called a restricted three-body problem. Another solution by Joseph-Louis Lagrange was discovered in 1772, where the three bodies form an equilateral triangle about their center of mass like in FIG. 1 b), at any given instant of time.

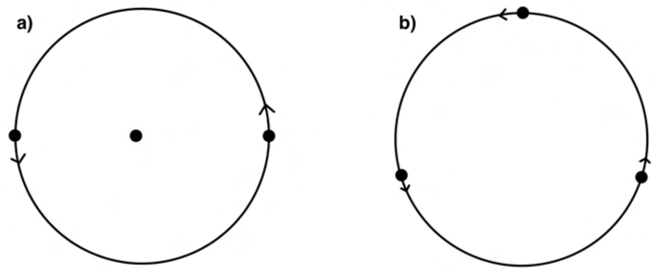

FIG. 1: Depiction of a) Euler's collinear solution, and b) Lagrange's triangle solution. The black circular curve represents the path traced by the two bodies in a) and by three bodies in b). The black dots represent the three bodies themselves and the arrows indicate the direction of travel.

Cristopher Moore, in 1993, discovered a special solution to the three-body problem. He found that three equal masses interacting gravitationally form a figure-8 orbit given the right initial conditions [3]. The figure-eight orbit was proven to be stable from the work of Carles Simó in 2000 [4]. During the same time, an independent research by Alain Chenciner and Richard Montgomery analyzed this solution using different numerical methods [2].

This project focuses on finding the figure-eight solution to the three-body problem in a system of two binary star systems. I aim to successfully simulate the solution using Mathematica and then develop the code to look for the figure-eight solution. The objective of this experiment is to check if any figure-eight orbit can be achieved by changing the phase angle of one of the binary star systems with respect to the abscissa in the system's initial position plane. This phase angle geometrically refers to the tilt provided to one binary star system as an initial condition while keeping the tilt for the other binary system zero. The phase angle will vary from 0 to π in an interval of 1/50000. This will provide us with 50,000 different systems to check the interaction between the two binary star systems. The solutions (or the orbits of masses) in which three masses are close to each other and the fourth leaves the system will be saved as the candidates for the figure-eight solution. The orbits of these candidates will be analyzed in a frame of reference of the center of mass of the remaining three bodies to look if any system forms a figure-eight orbit after a long period of time.

II. FIGURE-EIGHT ORBIT

Among all the solutions which exist for the three-body problem, the figure-eight solution is the most fascinating. Due to its existence, a trinary star system with a figure-eight orbit can occur in the Universe. The solution can be derived using knowledge of Lagrangian Mechanics, which is beyond the scope of this paper. The three-body problem is solved using Newton's equations of motion for three objects, and the initial conditions computed by Simó were used [1, 4]. The masses of all three bodies have to be equal and the net angular momentum of the system has to be zero for the initial conditions of the system. In Mathematica, the equations were used in the form

$$m_1\ddot{r_1} = -G\,\frac{m_1 m_2}{r_{12}^2}\,\hat{r}_{12} + G\,\frac{m_1 m_3}{r_{31}^2}\,\hat{r}_{31}, \tag{1}$$

$$m_2\ddot{r_2} = +G\,\frac{m_2 m_1}{r_{21}^2}\,\hat{r}_{21} - G\,\frac{m_2 m_3}{r_{23}^2}\,\hat{r}_{23}, \tag{2}$$

$$m_3\ddot{r_3} = -G\,\frac{m_3 m_1}{r_{31}^2}\,\hat{r}_{31} + G\,\frac{m_3 m_2}{r_{32}^2}\,\hat{r}_{32}, \tag{3}$$

where $\ddot{r_i}$ is the second time derivative of the position or acceleration of the i^{th} body and G is the gravitational constant. These equations account for the gravitational interaction among the three bodies to calculate for their orbit as time evolves. The values of the gravitational constant along with the equal masses were set as 1 for computational simplicity. Since the accepted value of $G = 6.67 \times 10^{-11}\,\mathrm{Nm^2kg^{-2}}$ is just a scaling factor in the code, it was ignored. The equations of motion, being three second-order differential equations, were solved using a Mathematica function called NDSolve. A set of initial conditions originally computed by Simó were provided to the code alongside the equations for the particular solution [4]. These conditions included the initial values of their position vectors along with their velocities. These initial conditions were set considering that the total angular momentum of the system has to be zero. The time period of the figure-eight orbit was also provided as an initial condition [4].

A manipulator with a time slider was constructed to visualize the figure-eight orbit's evolution. We observe in FIG. 2 that the black curve, which maps the path of the three bodies, is in a figure-eight. The three bodies are marked by three different colors on the plots and each plot corresponds to a different value of time labeled on their top. After a time period of $t = 6.32591$, the system returns to its original positions, which is the computed period of this orbit.

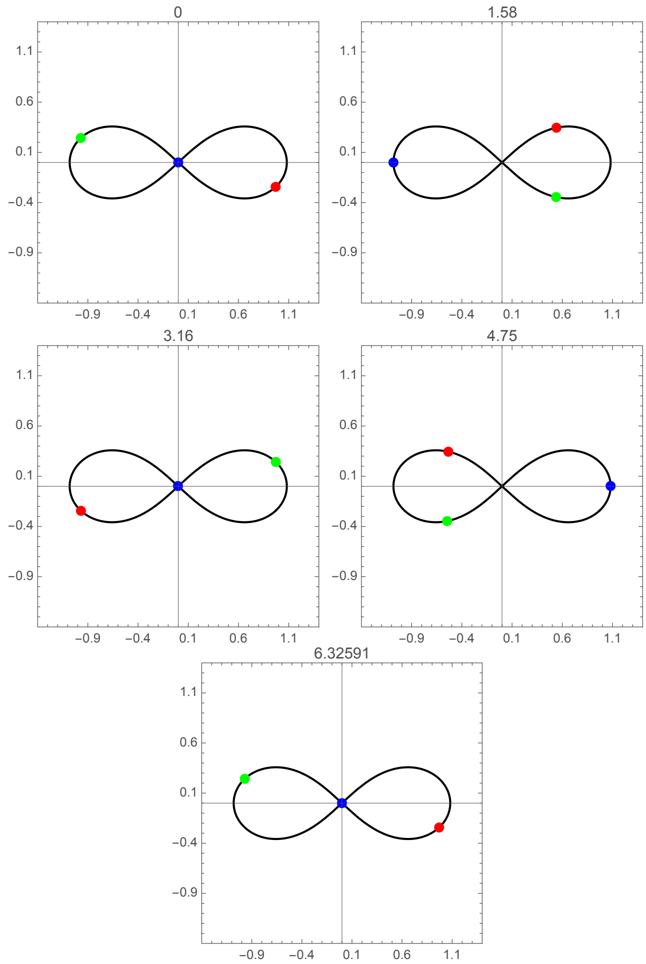

FIG. 2: Time evolution of the figure-eight solution to the three body problem simulated in Mathematica. The three bodies are depicted by the three colored dots tracing out the black curve as their orbit. The label of different plots indicate the time of that orientation. The time period of the orbit is $T = 6.32591$ given on the label of the last plot.

III. INTERACTION OF TWO BINARY STAR SYSTEMS

The goal of this experiment is to look for figure-eight orbits in a double binary star system. It is done by changing the phase angle difference of one of the binary systems with respect to the other. Other factors like the initial distance between the two double binary systems and also the distance between the stars themselves are kept constant. The simulated model coded in Mathematica can be seen in FIG. 3, where the two binaries are coming together along the abscissa in their position plane while

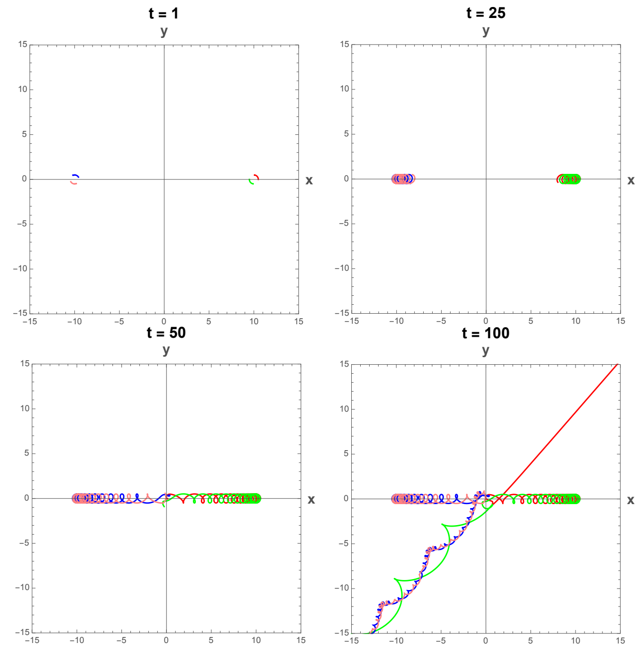

FIG. 3: Time evolution plot of one of the simulated double binary system with a phase angle value of $534\pi/3125$. The four different curves, represent the path traced by four different bodies when interacting gravitationally. The four plots represent different times into the evolution of the system, which is given as the label. At $t = 1$, we observe that the blue-pink binary system has a non-zero phase angle like the red-green system. Due to the initial velocities provided, the binaries come closer and finally interact amongst themselves at $t = 50$. The final resulting path for the bodies can be seen in the last plot at $t = 100$. Here, the resulting system is a trinary (blue-pink-green) with the fourth body (red) flying out in a different direction.

gravitationally interacting together. The four colored curves correspond to the path traced by the four bodies in a double binary system as time progresses. At the time $t = 1$, the phase difference of the two binaries can be seen where the pink-blue system is tilted left with respect to the abscissa. After some time, the four stars come together to interact closely at the origin, which results in a different orbit for all the bodies than they previously had. Due to high sensitivity to the initial conditions, the orbits look drastically different if small changes are made to those conditions. Since only the phase angle is varied in this system, I look for 50000 different values of the angles ranging from zero to π. The objective of the experiment is to look for a resulting trinary star system with the fourth body flying out of the system. Looking for this case reduces the sample size of the orbits of different phase angles and act as candidates to the figure-eight solution.

As discussed before for the three-body system, the masses of all four bodies in this system are also equal to one to satisfy the initial conditions required by the figure-eight orbit. For the two binary star systems, the equations

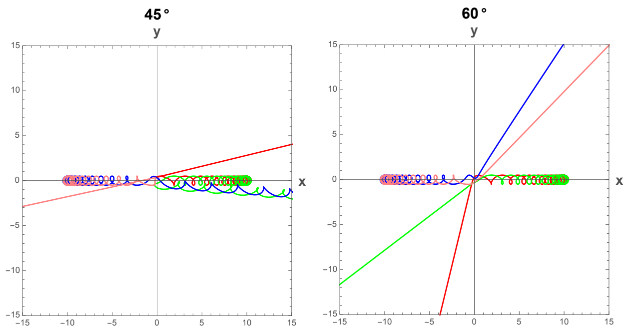

FIG. 4: Two plots indicating two double binary system with different initial phase angle as given on their label. The different colored curves represent the path traced by different bodies as the time evolved. The motion is calculated after taking in account the gravitational affects on a body due to the other three. For an angle of 45°, the system results into a generation of a new binary with the other two bodies leaving the system in different directions. For 60°, all the bodies fly off in different directions.

of motion for four bodies can be constructed the same way as Eqs. (1), (2), (3). However, for this system, the four second-order differential equations were expressed as eight first-order differential equations. Two equations for every body like

$$m_1 \dot{v_1} = -G \, \frac{m_1 m_2}{r_{12}^2} \, \hat{r}_{12} - G \, \frac{m_1 m_3}{r_{13}^2} \, \hat{r}_{13} + G \, \frac{m_1 m_4}{r_{41}^2} \, \hat{r}_{41}, \tag{4}$$

$$\dot{r_1} = v_1, \tag{5}$$

define the forces on one from the other three bodies and its velocity as the first derivative of its position. Similar equations for the remaining three bodies were also constructed and added to the code. This change was mainly done to use a different method of the NDSolve function on Mathematica called the Symplectic Partitioned Runge-Kutta (SPRK) method. The SPRK method uses Hamiltonian dynamical systems to numerically integrate for the evolution of that system. This method is better for the systems that need to be evolved for longer time periods, like the two binary star system. The simulation is also coded in such a way that the collisions among the four bodies are ignored in the output data. The exact working of this integration method is also beyond the scope of this paper, like the derivation of the figure-eight solution.

The resulting plots like in FIG. 4 shows the path traced out by all the four bodies in the system signified by four different colors. The two plots in the figure are the outcomes of two different phase angles for the blue-pink binary system given on the label. For one system, the tilt of the first blue-pink binary system with respect to the abscissa is 40° and for the other is 60°. As time progresses, the two binaries come together to gravitationally interact with each other. We can observe that in one of the cases, a new binary system of blue-green was formed. For the second case, all four bodies fly off in different directions. The code was further developed to run 50,000 different phase angles. This was done by ranging the phase angle from 0 to π and testing after every 1/50000 interval. Afterward, a series of checks were made to the resulting 50,000 simulations to look for cases where a trinary system is formed, and the fourth object ejects. These checks were performed using a condition that the distance between three out of four objects was less than 5 units. For reference, a distance of 0.5 units was set as the distance between two stars in a binary system. This reduced the output data set significantly, leaving some potential candidates for the figure-eight orbit solution. The angle values for these cases were noted and visually represented through plots. Also, to analyze the subset of the candidates orbits, the frame of reference was shifted from the origin of the initial positions of the four bodies to the center of mass of the resulting three-body system.

IV. RESULTS AND CONCLUSION

Out of 50,000 angle values, there were only twenty-one cases where a stable trinary star system was formed after simulating for a considerable time. Out of these twenty-one cases, none of them had a figure-eight orbit for the resulting trinary. All of the cases had a binary star system in a complicated orbit, with the third body tracing an elliptical or circular path outside the complex binary star system. Six of these cases can be seen in FIG. 5, where three different colors indicate different paths traced by the resulting three bodies in a trinary. As can be observed, all the cases have a binary system in a complicated orbit, with the third body orbiting on the outskirts of the binary. These orbits are simulated for a small period of time, long after the two binary systems started coming together. Choosing the right time scale was important to check for the stability of the orbit. Due to this, I can confidently claim that these orbits are the final orbits and that they do not change as time further evolves. After my successful simulation, I can conclude that the probability of finding a figure-eight orbit in a double binary star system with varying phase angle for one binary system is less than 1/50000.

This simulation is highly impractical in the Universe due to all the assumptions that need to be true for the figure-eight orbit to form. To make the model more realistic, other factors like the distance between the two binary systems and the distance between the stars in the binary systems can be further parameterized to look for the figure-eight solution. Along with that, the initial positions and velocities can also be manipulated to replicate a practical situation of two binary stars coming together. Since the system is extremely sensitive to the initial conditions, making more aspects of this model a parameter will have a better chance of resulting in the desired figure-eight solution.

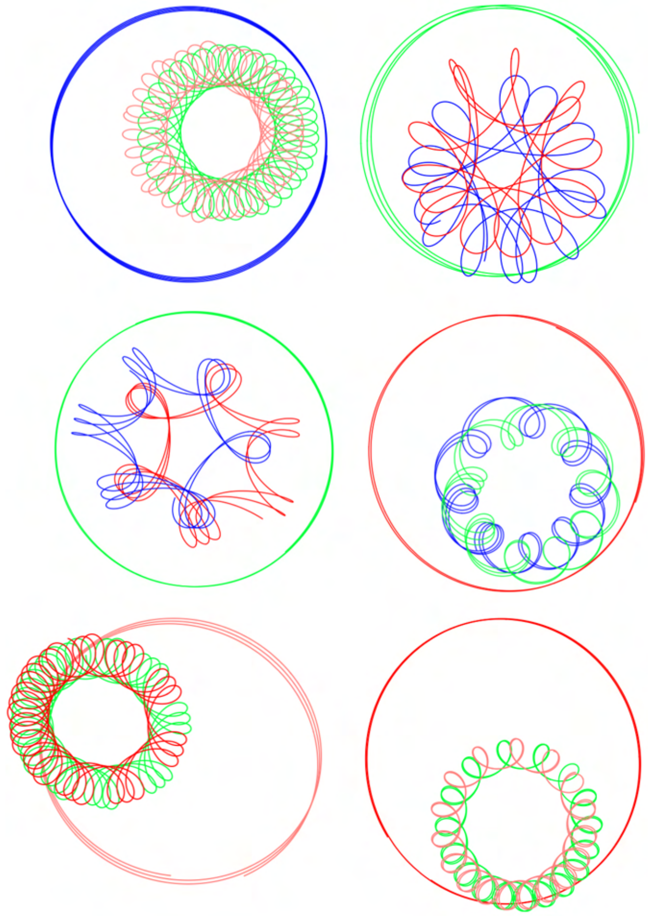

FIG. 5: Six of the twenty-one orbits for a trinary star system found after evolving a double binary star system in time. These cases were potential candidates for the figure-eight solution and were plotted to check their orbit. The different colors curves indicate the path traveled by different objects as time progressed. The plots are in a reference frame of the center of mass of these three bodies. All of the potential candidates were found to be a binary star system surrounded by a third star revolving around the binary system.

Throughout this experiment, several Mathematica notebooks were created, each of them focussing on one integral part of this experiment. First, a simulation of the figure-eight orbit was constructed using Newton's equations of motion along with the initial conditions calculated by Simó. This solution was simulated to visualize the center of this experiment and also to know how far the three bodies are from each other in this orbit. This information was used later to look for the candidates to the figure-right solution in the double binary system. After that, the double binary system was simulated such that the two binaries come together along one direction to gravitationally interact with each other. This system was checked for 50,000 values of the phase angle for one of the binaries, ranging from 0 to π with an interval of 1/50000. The resulting output simulations for these phase angle values were reduced to focus on the simulations with a final trinary star system and the fourth body flying out of the system. These candidates for the figure-eight solution were visually checked in a frame of reference of the center of mass of the resulting trinary star system. Out of 50,000 simulations, only twenty-one were the candidates with a trinary, out of which none of them was the figure-eight orbit.

[1] I. Newton, *Philosophiae Naturalis Principia Mathematica*, vol. 2 (typis A. et JM Duncan, 1833).

[2] A. Chenciner and R. Montgomery, Annals of Mathematics **152**, 881 (2000), ISSN 0003486X.

[3] C. Moore, Phys. Rev. Lett. **70**, 3675 (1993).

[4] C. Simó, in *European Congress of Mathematics*, edited by C. Casacuberta, R. M. Miró-Roig, J. Verdera, and S. Xambó-Descamps (Birkhäuser Basel, 2001), Progress in Mathematics, pp. 101–115, ISBN 978-3-0348-8268-2.