Surface Ripples as Thermal Excitations

Carlos Owusu-Ansah
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In this paper, we confirm that a model devised for thermally excited waves can be applied to ripples moving along the surface of the water. Ripples with frequencies ranging from 180Hz to about 330 Hz were generated on the surface of the water and their wavelengths are measured by scattering red light over the surface of the water. The ratio of the exponent of the wavenumber to the exponent of the angular frequency was determined to be 1.46 ± 0.04 which agrees with the value 1.5 predicted by the model. Using an equation derived from the model, the surface tension of the water is determined to be 0.070 ± 0.003 N/m which agrees with the widely accepted value at 0.072 N/m. We conclude that the dynamics of the ripples formed are dominated by thermal effects.

I. INTRODUCTION

Imagine that it is drizzling outside. The water droplets create circular ripples on the surface of a puddle: thousands of ripples interfering constantly with each other, and reflecting along the boundaries of the puddle. It is all very intriguing and complicated, but suddenly, the rain stops, and the surface settles into a clear motionless membrane. There is nothing very intriguing here anymore, you might say, but you would be wrong. What if there were hundreds of thousands of microscopic ripples governed by the same dynamics as the visible ripples?

In the conduction model of heat transfer, heat propagates through a material via random collisions between its molecules. Due to the high mobility and density of the molecules in liquids, these collisions cause displacements that propagate through the liquid and lead to time-dependent oscillations at the liquid-air interface. At room temperature, these oscillations will be microscopic because they represent the lowest energy state of the liquid's surface. In this paper, we derive the wave equation for thermal excitations at the surface and show that it can be used to model surface ripples by using it to accurately determine the surface tension of water.

II. THEORY

Our derivation for the wave equation closely follows what is outlined in [1]. We begin by assuming that the liquid is incompressible. This means that the rate at which molecules enter a volume element equals the rate at which they leave the element. Mathematically, this condition is imposed by stipulating that the divergence of the velocity vector $\nabla \cdot \vec{v}$ at any point in the liquid is 0.

We also assume that there is no rotational flow in the liquid. In other words, if we placed a ball with negligible mass and uniform density at any point in the liquid, it would not rotate. This assumption is valid because we are referring to a liquid whose internal dynamics results solely from the thermal oscillations of its molecules. Perhaps, at an instant, there may be a small torque on the ball because of collisions at one side of the ball transfer more momentum to that side of the ball than collisions at the other side. However, since thermal motion is random, these effects generally balance out. Mathematically, the absence of rotational flow implies that the curl of the velocity vector $\nabla \times \vec{v}$ is 0.

Any vector with a vanishing curl can be represented as the gradient of a scalar quantity [1]. Therefore, we can express the velocity vector at any point as the gradient of some scalar potential function

$$\vec{v} = \nabla \phi. \tag{1}$$

By taking the divergence of this equation and using the result that the divergence of the velocity vector is 0, we obtain the differential equation

$$\nabla \cdot \vec{v} = \nabla \cdot (\nabla \phi) = \nabla^2 \phi = 0. \tag{2}$$

This is the Laplace equation. It has a well known solution that can be expressed as:

$$\phi = X(x)Y(y)Z(z) \tag{3}$$

where

$$Z(z) = (e^{kz} - Ae^{-kz}) \tag{4}$$

$$X(x) = (Be^{i(kx-\omega t)} + Ce^{i(kx+\omega t)}) \tag{5}$$

$$Y(y) = (Ee^{i(ky-\omega t)} + Fe^{i(ky+\omega t)}). \tag{6}$$

The asymmetry between the terms $Z(x)$, $X(x)$ and $Y(x)$ must be present for these expressions to solve the Laplace equation. The terms A, B, C, ω and k are quantities independent of the position variables. The time variable t was included in the solution because it is reasonable to hypothesize that the potential function ϕ is time dependent.

A. Boundary Conditions

To understand the physical meaning of our solution, we impose boundary conditions. The force over the surface of our liquid is expressed as

$$f = -\oiint p \; ds \tag{7}$$

where p is the pressure over an area element ds of the surface. Using the divergence theorem, we convert this surface integral into the volume integral

$$f = -\iiint \nabla p \; dV. \tag{8}$$

This equation indicates that the force on a volume element in the liquid is directly proportional to the gradient in pressure $-\nabla p$ at that point. Therefore, Newtons second law for a volume element of the liquid may be expressed as

$$-\nabla p = \rho \frac{d\vec{v}}{dt}. \tag{9}$$

We simplify this equation by considering that the velocity vector at each point in the liquid is a function of both position and time. Therefore, the time derivative of the velocity vector is

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{d\vec{x}}\frac{\partial \vec{x}}{\partial t} = \frac{\partial \vec{v}}{\partial t} + (\nabla \cdot \vec{v})\vec{v} \tag{10}$$

which, when substituted into Eq 9, gives the equation

$$-\frac{\nabla p}{\rho} = \frac{\partial \vec{v}}{\partial t} + \vec{v}(\nabla \cdot \vec{v}). \tag{11}$$

Since thermal oscillations at the surface have extremely small amplitudes, we ignore terms that are second order in $\vec{v}$. By letting $\vec{v}(\nabla \cdot \vec{v}) = 0$, and making the substitution $\vec{v} = \nabla \phi$ from Eq 1, we obtain

$$-\frac{\nabla p}{\rho} = \frac{\partial (\nabla \phi)}{\partial t} \tag{12}$$

which implies that

$$-\frac{p}{\rho} = \frac{\partial \phi}{\partial t}. \tag{13}$$

The implication holds because the density is constant. Our goal is to calculate the surface tension of the liquid using the wave equation so we now seek an expression for the pressure in terms of the surface tension.

Surface tension is the tendency of the surface of the liquid to act like a stretched elastic membrane. It arises because the cohesive forces between the molecules of the liquid are greater than the adhesive forces between the liquid and the environment. The surface of our liquid behaves like a membrane along the x-y plane with mass density σ and tension α undergoing small-amplitude oscillations. Therefore it obeys the wave equation

$$\alpha \nabla^2 u = \sigma \frac{\partial^2 u}{\partial t^2}. \tag{14}$$

The variable u is the amplitude of the oscillations in the z direction [1]. Comparing this wave equation to Newton's second law, we can see that net force per unit area is

$$f = \alpha \nabla^2 u. \tag{15}$$

This net force per unit area is the difference between the pressure from the air molecules above the membrane and the pressure from the liquid molecules below the membrane. Therefore, it is the z-component of the net pressure at the interface with the opposite sign. Substituting this pressure into Eq 13, we obtain

$$\alpha \nabla^2 u = \rho \frac{\partial \phi}{\partial t} \tag{16}$$

We approach our final expression for the surface tension. The time derivative of u is the z-component of the velocity of any point on the membrane. In Eq 1, we defined the velocity vector $\vec{v}$ to be the gradient of the potential ϕ. Therefore the time derivative of u is the z component of $\nabla \phi$. Taking the time derivative of Eq 16 and making this substituting gives us

$$\rho \frac{\partial^2 \phi}{\partial t^2} - \alpha \nabla^2 \frac{\partial \phi}{\partial z} = 0 \tag{17}$$

When the solution generated in Eq 6 is substituted into Eq 17, and the result is simplified we obtain

$$-\rho \omega^2 \phi + \alpha k^3 \phi = 0. \tag{18}$$

Now, we derive that the surface tension of the membrane is given by

$$\alpha = \frac{\omega^2 \rho}{k^3} \tag{19}$$

Observe that the only boundary condition imposed on the liquid is that a stretched membrane lies above it in the x-y plane. The absence of other boundaries means that the depth and breadth of the liquid in our model extends infinitely. This is not possible, but it is a reasonable assumption given that thermal oscillations are unlikely to propagate from one end of the liquid container to another end. We also assume that the effects of gravity on thermal oscillations are negligible.

B. Surface Ripples

We apply our model for thermal oscillations to visible ripples along the surface of the water to determine the surface tension of water. Ripples differ from thermal oscillations in some key ways. Since they are generated by forced oscillations at the surface, we cannot be certain that the resulting flow beneath the surface is irrotational. Ripples are also affected by gravity and have significantly larger velocity vectors than thermal oscillations. By correctly calculating the surface tension with Eq 19 using measurements on ripples, we ascertain that ripples of certain frequencies and wavelengths behave similarly to

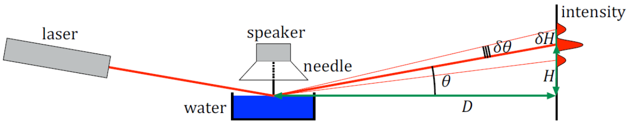

FIG. 1: A schematic of our experimental setup. The distance D is the distance between the point of reflection and the wall. The height H is the height of the location where the specularly reflected ray is incident on the wall relative to the height of the surface of the water, and δH is the distance from this point to the nearest point of constructive interference. The angle θ is the angle of reflection of the specular ray while $\delta\theta$ is the angle between the reflected specular ray and the constructively interfering rays above it. This figure was obtained from [2].

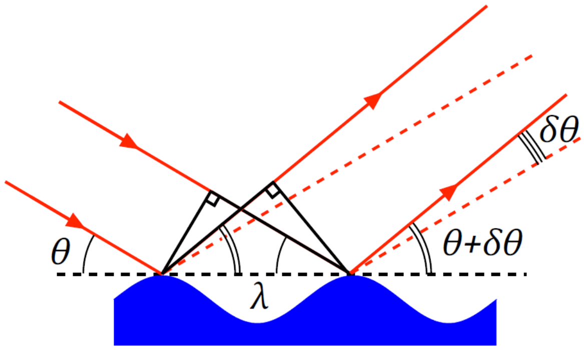

FIG. 2: The reflection of two rays that constructively interfere above the point of specular reflection. The specularly reflected rays are represented by the dashed lines. The distance λ is the wavelength of the ripples. This figure was obtained from [2].

thermal oscillations.

In our experiment, ripples are generated by periodically nudging the midpoint of the surface of the water in a reservoir with a needle that oscillates at some frequency. To use Eq 19 to calculate surface tension, we need to record the frequency of the needle's oscillations and measure the corresponding wavelength of the ripples. The wavelength is measured using a method described in [2]. A laser beam is incident on the rippled surface and reflected onto a distant wall. Depending on the frequency of the oscillating needle, the specular reflection point may be enveloped by two or more bright spots where light rays constructively interfere with each other. This result is illustrated in Fig 1. The distance between the points of constructive interference and the point of specular reflection is used to calculate the wavelength of the ripples.

To understand the calculation we consider the interaction between the light and the surface of the water. Since the surface is not uniform, it causes both diffuse and specular reflection to occur. However, we expect that the bright spots close to the specular reflection point were reflected off wave crests because the gradient at wave crests is closest to the horizontal.

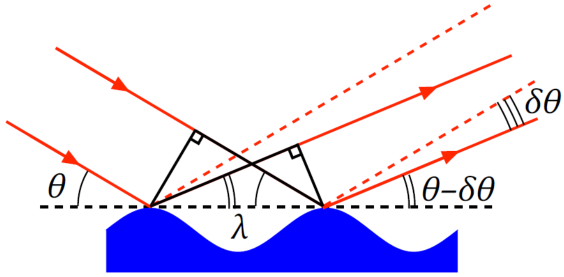

FIG. 3: The reflection of two rays that constructively interfere below the point of specular reflection. The specularly reflected rays are represented by the dashed lines. This figure was obtained from [2].

Fig 2 illustrates two light rays that were incident on neighboring crests and reflected at an angle $\delta\theta$ greater than the angle of incidence. Similarly, Fig 3 illustrates two different light rays that were incident on neighboring crests and reflected at an angle $\delta\theta$ less than the angle of incidence. The quantity λ is the wavelength of the water wave. We assume that the reflected rays in Fig 2 constructively interfere to form a bright spot above the point of specular reflection. Similarly, the reflected rays in Fig 3 constructively interfere to form a bright spot below the point of specular reflection even though the rays are parallel. This assumption is false. However, since the wall is far compared to the distance between the interfering rays, it is standard practice in optics to make the simplifying assumption that the rays are parallel.

Since the rays in Fig 2 and Fig 3 constructively interfere, we may assume that the path difference is equal to the wavelength of light λ_l. By analyzing at Fig 2, we can deduce that the path difference is between the two rays is

$$\lambda\cos(\theta) - \lambda\cos(\theta + \delta\theta) = \lambda_l. \quad (20)$$

Similarly, by analyzing Fig 3, we obtain a similar equation for the path length difference:

$$\lambda\cos(\theta - \delta\theta) - \lambda\cos(\theta) = \lambda_l. \quad (21)$$

We use the double angle formula to expand the terms with $\delta\theta$ in Eq 20 and Eq 21 and add the resulting equations. Simplifying, we obtain the expression

$$\lambda_l = \lambda\sin(\theta)\sin(\delta\theta). \quad (22)$$

From the perspective of Eq 19, a more useful form of this Eq 22 may be expressed as

$$k = k_l \sin(\theta)\sin(\delta\theta). \quad (23)$$

The wave number of the ripples $k = 2\pi/\lambda$ and the wave number of light $k_l = 2\pi/\lambda_l$. By inspecting Fig 1, we see that θ is calculated using the relation

$$\theta = \arctan\left(\frac{H}{D}\right). \tag{24}$$

The quantity $\delta\theta$ is calculated using the relation

$$\delta\theta = \arctan\left(\frac{H+\delta H}{D}\right) - \theta. \tag{25}$$

Using Eq 24 and Eq 25, we can compute the value for k at each frequency f using Eq 23. The relation

$$\omega = 2\pi f \tag{26}$$

allows us to use the values of k and ω to calculate the surface tension of water using Eq 19.

III. EXPERIMENTAL DETAILS

The schematic of our experimental setup is illustrated in Fig 1. In this section, we discuss the finer details of the experiment. The source of our beam was a helium-neon laser that produces light with a wavelength of 632.8 nm. An adjustable stand was used to elevate and orient the helium-neon laser so that the incident beam was reflected close to the midpoint of the dish. We kept the incident beam close to the midpoint to prevent ripples generated by the needle from interfering with ripples reflected at the boundaries of the surface of the water.

To obtain accurate measurements for the wavelength λ of the ripples, we ensured that the direction of the incident beam was perpendicular to the wavefronts of the circular waves. This condition guarantees that our calculated value for λ is the shortest distance between two crests. The point of specular reflection on the wall becomes elongated as the incident beam becomes parallel to the wavefronts. Therefore, we were able to ensure that the incoming beam was perpendicular to the wavefronts by adjusting the position and orientation of the laser and observing the intensity pattern on the wall.

The container for the water was about 6 cm deep. During our experiment, it was filled so full that the meniscus was distinctively convex. Since our model for thermal oscillations assumes that the upper boundary of our liquid is an interface between only two different mediums, it was important to minimize the adhesion between the container and the surface of the water. The container for the water was placed on an anti-vibration table to insulate the water from disturbances in the surroundings.

The oscillating needle was attached to a loudspeaker which was connected to a Pasco function generator. The function generator allowed us to control both the amplitude and frequency of the needle's oscillations. The generator provided a sinusoidal signal causing the speaker and the needle to oscillate at the specified amplitude and frequency. The amplitude was chosen so that the needle repeatedly hits the surface of the water at the midpoint of the container. The range of frequencies used in our measurements was chosen to allow the distance between the point of spectral reflection and the points of constructive interference directly above and below it to be measured accurately. We selected frequencies ranging from about 180 Hz to about 330 Hz because it was difficult to distinctly identify the interference pattern beyond this range.

To accurately determine the height H and δH in Fig 1, a grid was placed on the wall so that the intensity pattern is displayed on this grid. The distance between the grid-lines was approximately 1 cm. The measurement process consisted of adjusting the frequency using the function generator and measuring the distance between the point of specular reflection and points of constructive interference on the grid.

Despite the anti-vibration table, the point of specular reflection and the points of constructive interference oscillated slightly during the experiment. We found it convenient to take pictures of the intensity pattern on the grid after every adjustment in frequency. To keep track of the correlation between the frequencies and the pictures, we preceded each picture of a grid with a picture of its corresponding frequency displayed on the function generator. Distance measurements using pictures of the intensity pattern on the grid are more reliable because the bright spots are stationary. Our capacity to zoom-in on the pictures also improves precision.

IV. ANALYSIS AND DISCUSSION

Using a tape measure, the distance D in Fig 1 was measured to be 3.78 m. After measuring H and δH over a range of frequency values, we used Eq 24, Eq 25 and Eq 23 to determine wave numbers k corresponding to these frequency f values. The angular velocities were computed from the frequency values using Eq 26 to obtain a table containing the angular velocities ω of the ripples and their corresponding wave-numbers k.

If our model for thermal oscillations holds for the ripples created in our experiment, then the angular velocity ω and the wave number k of the ripples should obey Eq 19. By taking the natural logarithm of both sides, Eq 19 may be rewritten as

$$\ln(\omega) = \frac{3}{2}\ln(k) + \frac{1}{2}\ln\left(\frac{\alpha}{\rho}\right). \tag{27}$$

Evidently, the model predicts that the slope of a plot of $\ln(\omega)$ against $\ln(k)$ should be 1.5. Fig 4 is a plot of $\ln(\omega)$ against the $\ln(k)$ for our measurements on ripples moving along the surface of water. The slope of this plot is 1.46 ± 0.04 which is in fair agreement with the prediction of the model. Therefore, the ratio of the exponent of the wave number k to the exponent of the angular velocity ω of the ripples is accurately predicted by the model.

To further ascertain the validity of the model, we

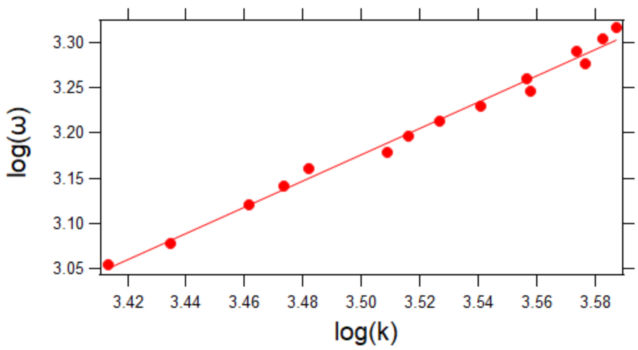

FIG. 4: The natural log of the angular velocity $\log(\omega)$ vs the natural log of the wave number $\log(k)$ for the ripples on the surface of the water. The slope of the line is 1.46 ± 0.04

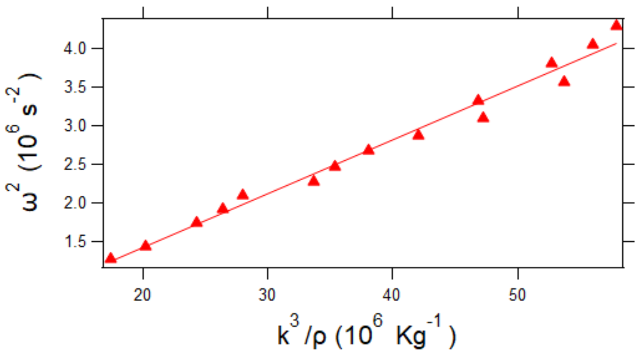

FIG. 5: The square of the angular velocity ω^2 vs the cube of the wave number k^3 divided by the density of water ρ. The slope of this line is 0.070 ± 0.003 N/m. This slope is our prediction for the surface tension of the water.

calculate the surface tension of water using Eq 19. According to Eq 19, a plot of ω^2 against k^3/ρ should produce a line with slope equal to the surface tension α. Fig 5 shows a plot of ω^2 against k^3/ρ using the values obtained in our experiment. It has the slope 0.070 ± 0.003 N/m. This value agrees with the accepted value for the surface tension of water which lies at 0.072 N/m. In a second trial with 6 fewer data points, we obtained the value 0.078 ± 0.002 N/m for the surface tension which differs from the accepted value by 6%.

We conclude the model devised for thermal oscillations is effective at predicting the behavior of surface ripples. The dynamics of the ripples created in our experiment are similar to those generated by thermal excitations. In other words, the ripples created in our experiment cause a flow that is largely irrotational and described by velocity vectors with amplitudes so small that second-order terms are negligible. Additionally, we may conclude that the surface ripples created in this experiment are not significantly affected by gravity since our model did not consider the effect of gravity.

V. CONCLUSION

In the first part of this paper, we develop a mathematical model for a liquid whose molecules interact solely via thermal collisions. This model is compared with measurements made on ripples on the surface of the water to determine its generality. The ripples are created by periodically nudging the surface of the water with a needle. The wavelength of the ripples is measured by scattering a beam of red light over the wave fonts. We compare the measured relationship between the wavelength and the frequency to the predictions of the mathematical model and find that it fits the model nicely. The measured ratio between the exponent of the wavenumber and the exponent of the angular frequency is 1.46 ± 0.04 which agrees the value 1.5 predicted by the mathematical model. We indirectly ascertain the applicability of the wave equation derived in the model by applying it to the measurements on the ripples to calculate the surface tension of water. We determine that the surface tension of water is 0.070 ± 0.003 N/m which agrees with the widely accepted value at 0.072 N/m. The ripples created in our experiment had frequencies ranging from 180 Hz to about 330 Hz. Therefore, we conclude that the dynamics of ripples formed at these frequencies over the surface of deep water reservoirs are largely dominated by thermal effects.

VI. ACKNOWLEDGMENTS

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[2] Department of Physics - College of Wooster, *Junior independent study manual* (2020).