Estimating G Using a Gravitational Torsion Balance

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In this experiment, we measured the gravitational constant G using a torsion balance. We used two methods to estimate a change in distance between two masses due to their gravitational attraction. In the first method, we compared the gravitational force between the two masses in two different configurations after collecting data for one hour. In the second method, we measured the force between the masses in one configuration until the torsion balance reached equilibrium. For the first method, the gravitational constant was measured to be $G = (6.6 \pm 0.1) \times 10^{-11}$ N^{2}m^{2}kg^{-2}, and for the second $G = (7.3 \pm 0.1) \times 10^{-11}$ N^{2}m^{2}kg^{-2}. Compared to the known value of $G = 6.67 \times 10^{-11}$ N^{2}m^{2}kg^{-2}, we saw that only the first method had the known value to be within its uncertainty. Using the determined G values from each method, we calculated the mass of the Earth to be $(5.9 \pm 0.1) \times 10^{24}$ kg and $(5.4 \pm 0.1) \times 10^{24}$ kg. Compared to the known mass of the Earth $m_E = 5.97 \times 10^{24}$ kg, we found only the first method included the accepted value within its uncertainty.

I. INTRODUCTION

Although Isaac Newton predicted the gravitational constant G in 1687 in his universal law of gravitation, he never measured it. Today, when we think of the first measurement of G, we attribute it to Henry Cavendish's 1797 experiment. In the late 1700s, The Royal Society was determined to find density of the Earth. Cavendish devised an experiment using a torsion balance, and measured the density to astounding precision [1].

Today, using a modernized version of Cavendish's apparatus, we can rearrange variables and instead, directly measure the famous gravitational constant. For this experiment, we used our measurements for G and our knowledge of forces to calculate the mass of the Earth.

II. THEORY

The gravitational constant G is fundamental in determining the gravitational force between two objects. The gravitational force between two masses m_1 and m_2 is

$$F = \frac{G m_1 m_2}{b^2}, \tag{1}$$

where b is the distance of separation between their centers of mass. A top view diagram of the apparatus is shown in Fig. 1, where the small masses m_1 are suspended from the torsion balance, separated from the large m_2 by the distance b. The suspended masses are attached to each end of a lever arm of length d, and can rotate about the suspension point. Due to the rotation of the masses, we need to consider the torques acting on the axis of rotation from Fig. 1. The general form for a torque τ is

$$\vec{\tau} = \vec{r} \times \vec{F}. \tag{2}$$

In this case, since the gravitational forces generating the torque are perpendicular to the lever arm $\vec{r}$, then we can define the "gravitational" torque τ_g as

$$\tau_g = 2Fd, \tag{3}$$

where d is the length of the lever arm [2]. We note the factor of 2 in Eq. (3) since we have the two gravitational forces causing the arm to rotate in the same direction. Due to the resistance of the torsion band, a torque is created in the opposite direction of Eq. (3) where

$$\tau_b = -\kappa\theta, \tag{4}$$

with a torsion constant κ and θ is the rotation angle for the system. The gravitational force perturbs the small mass system, causing it to oscillate. For the system to reach equilibrium, the torques from Eq. (3) and (4) should be equal such that $\tau_g = -\tau_b$ where

$$2Fd = \kappa\theta. \tag{5}$$

If we substitute Eq. (1) into Eq. (5) , and solve for G, we see that

$$G = \frac{\kappa\theta b^2}{2 d m_1 m_2}. \tag{6}$$

The change in distance between our masses m_1 and m_2 can be determined from our experimental setup. We will use the angle of deflection θ from Fig. 1 such that $\tan(2\theta) = \Delta S/2L$ and ΔS is the change in distance. Since we are dealing with such small angles, we can make a small angle approximation so $\tan(\theta) \approx \theta$ thus

$$\theta = \frac{\Delta S}{4L}. \tag{7}$$

The length L in Eq. (7) is the distance between the mirror attached to axis rotation and the detector. As a result, the change in distance between m_1 and m_2 ΔS, can be characterized by the rotation of the axis picked up from the detector and the length between them.

Again, the rotation due to the gravitational forces between the small and large masses will cause the lever arm to oscillate until it reaches equilibrium. The period of oscillations T can be defined in terms of κ and the moment of inertia I of the suspended mass system so

$$T^2 = \frac{4\pi^2 I}{\kappa}. \tag{8}$$

Since the system experiences under damping due to the torsion ribbon, the position x of the masses can be modeled as a function of time t according to

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta), \quad (9)$$

where A is the amplitude, δ is the phase, and ω_1 is the angular frequency of oscillation [3]. As a result of under damping, the decay parameter β is much less than the natural angular frequency of the system ω_0. As a result, $\omega_1 \approx \omega_0$ so we can use ω_1 from observations to determine the period where $T = 2\pi/\omega_1$. The moment of inertia from Eq. (8) for the suspended masses is

$$I = 2m_2(d^2 + (2/5)r^2), \quad (10)$$

where m_2 is the small mass [2]. We now take Eq. (8) and solve for κ so that

$$\kappa = \frac{4\pi^2 I}{T^2}. \quad (11)$$

Next, substituting Eq. (10) into Eq. (11) gives us

$$\kappa = 8\pi^2 m_2 \frac{d^2 + (2/5)r^2}{T^2}. \quad (12)$$

Our final step is substituting Eqs. (7) and (12) into (6) to arrive at

$$G = \pi^2 \Delta S b^2 \frac{(d^2 + (2/5)r^2)}{T^2 m_1 L d}. \quad (13)$$

However, our final equation value G will need to be adjusted. We only considered the large masses m_2 interacting with the small masses m_1 right in front of them and not with those on the other end of the lever arm. If we consider the combination of those gravitational forces, we find that our calculated G needs to be adjusted such that

$$G_0 = \frac{G}{1 - b_0}, \quad (14)$$

where G_0 is the corrected value and

$$b_0 = \frac{b^3}{(b^2 + 4d^2)^{3/2}}. \quad (15)$$

The full derivation for the correcting force with included force-body diagrams can be found in the Lab Manual [2]. We can also use our measured values for G to estimate the mass of the Earth. If we consider the force due to gravity on an object at sea level is equal to the force of gravitational attraction, we find that the mass of the earth is

$$m_E = \frac{gR_E^2}{G}, \quad (16)$$

where R_E is the radius of the Earth and g is the acceleration due to gravity at sea level.

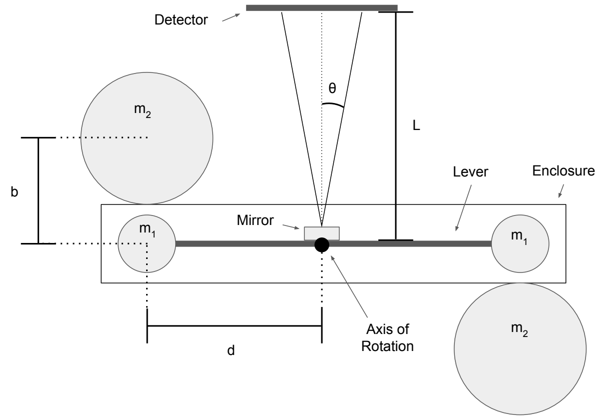

FIG. 1: Top view of the apparatus.

III. PROCEDURE

Our experimental setup was similar to Cavendish's 1798 torsion balance but with some modern modifications. A basic diagram of the apparatus is shown in Fig. 1. Two tungsten balls with mass $m_2 = (1.5 \pm 0.01)$ kg could be rotated around the axis connecting them, shown in Fig. 2, to allow for measurements in both $\pm\theta$ directions. Inside of a protective case, were two smaller tungsten balls of mass $(3.82 \pm 0.02) \times 10^{-2}$ kg, denoted m_1, connected by a bar. The center of bar was connected to an incredibly sensitive pendulum which could rotate according to a torsion balance located above the protective case.

The torsion balance had a beryllium copper ribbon which created the resistance to the rotation of the lever arm so the suspended masses could reach equilibrium. It should be noted that, once oriented, the large masses were locked in place so the change in distance between m_2 and m_1 was entirely due to the movement of the smaller masses. From Fig. 1, the distance d from one m_1 to the axis of rotation was 50 mm. This distance d denoted our lever arm. Located at the axis of rotation for the lever arm was a mirror used to deflect a laser beam in order to record the rotation. In line with the mirror was a detector in the line of sight the mirror so that any deflection of the laser could be detected. The measured distance between the mirror and the detector was $L = 12.8 \pm 0.1$ cm. The uncertainty found in L was large compared to the other known constants in the system, which resulted in a larger uncertainty in the final calculations.

The laser detection was accomplished using a Hamamatsu S3270 Position Sensitive Detector (PSD) connected to Hamamatsu C3683-01 circuit which recorded the change in laser position as a voltage. Both were connected to a computer via a NI-DAQ 6009 so the measured voltages could be converted back to distances using a proportionality constant $k = 37/20$ mm/V for the results and calculations [2].

We measured the change in the voltage of the laser over time, and fit the damped-oscillatory behavior to estimate

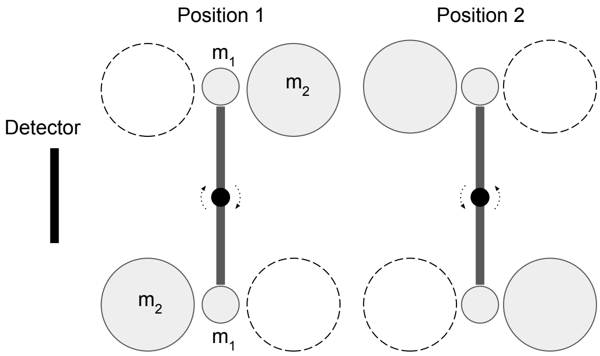

FIG. 2: Possible configurations for the large masses m_2.

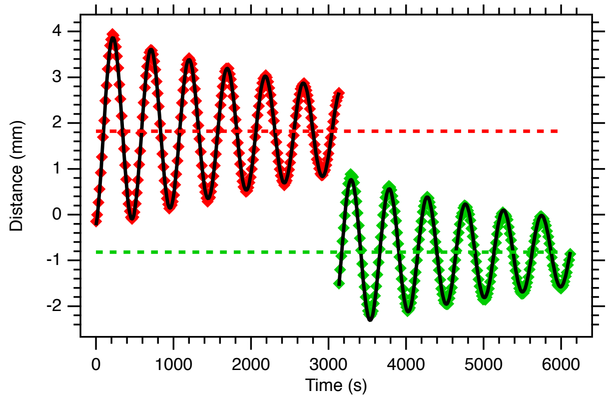

FIG. 3: Position of the laser measured as a function of time for method one. The red diamonds designate the position of the laser for m_2 set to position two, and the green diamonds for position one from Fig. 2. The green data set was taken at a later time and offset in the plot in order to better compare with the red data set. The black curves represent the fits used for each of the data sets using an under-damped, harmonic oscillator with a period of $T = 491 \pm 1$ s. The dashed red and green lines represent the determined equilibrium positions of the deflected laser with $h_2 = 1.82 \pm 0.002$ mm and $h_1 = -0.822 \pm 0.004$ mm respectively.

the period. The voltages were then converted to distances to estimate ΔS between the small and large masses due to the gravitational force.

We used two methods of determining this change in distance. For the first method, we measured the voltage every 15 seconds for one hour with the large masses in each position from Fig. 2. We then compared the determined equilibrium position for the small masses in each position, h_1 and h_2 respectively, and used $|h_1 - h_2|$ to find ΔS for Eq. (13) to solve for G.

For the second method, the voltage was measured every second for roughly 6×10^4 s with the large masses removed to establish an initial value in case the neutral equilibrium point for system was not located at $\theta = 0$. The voltage was measured again for 6×10^4 s, but with large masses in position one from Fig. 2, until the small mass assembly reached equilibrium. The ΔS value was calculating using the difference between an equilibrium position from Fig. 2 and the neutral equilibrium position.

The period of oscillations, as required by Eq. (13), was measured for both methods by fitting the respective curves using the general solution for an under-damped, harmonic oscillator from Eq. (9).

IV. DATA ANALYSIS

The two methods used to measure ΔS were: first, comparing the differences between the equilibrium positions from Fig. 2, and second, comparing the difference between one of the equilibrium positions and a central equilibrium position determined by removing the large masses.

The results for the first method are shown in Fig. 3. The deflection of the laser as a result of motion small masses, m_1, were fit with an under-damped, harmonic oscillator in both positions from Fig. 2. The equilibrium positions were measured to be $h_1 = -0.82 \pm 0.004$ mm for position one, and $h_2 = 1.82 \pm 0.002$ mm for position two. We took the difference of the two positions to find $\Delta S =$

TABLE I: Results from both methods of calculating the gravitational constant G and the Earth's mass m_E.

Method	G Calculated ($\mathrm{N^2m^2kg^{-2}}$)	m_E Calculated (kg)
I	$(6.6 \pm 0.1) \times 10^{-11}$	$(5.9 \pm 0.1) \times 10^{24}$
II	$(7.3 \pm 0.1) \times 10^{-11}$	$(5.4 \pm 0.1) \times 10^{24}$
Accepted G and m_E:	6.67×10^{-11}	5.97×10^{24}

2.64 ± 0.01 mm.

The results for both methods of calculating G can be found in Table I.

As expected, we found the period of oscillations to be the same for each position within uncertainty, where $T = 491 \pm 1$ s. With the rest of the variables in Eq. (13) being constants from Fig. 1, our estimation for G, using $\Delta S = 2.64 \pm 0.01$ mm and $T = 491 \pm 1$ s, was $G = (6.6 \pm 0.1) \times 10^{-11}$ $\mathrm{N^2m^2kg^{-2}}$ after making the correction calculation from Eq. (14). Compared to the known value of $G = 6.67 \times 10^{-11}$ $\mathrm{N^2m^2kg^{-2}}$, we saw that the known value fell within our uncertainty for method one as shown in Table I. The large uncertainty in calculated G was due to the large uncertainty in the measurement of the distance between the mirror and the detector L. The results for the second method are shown in Fig. 4. Using the same fit parameters as method one, we determined the equilibrium position for the m_1 assembly and when m_2 were removed. We found that the equilibrium position for position to be $h = 2.006 \pm 0.0002$ mm, and for the neutral position $h_0 = 0.5494 \pm 0.0001$ mm.

Since we only measured one position ΔS, according to

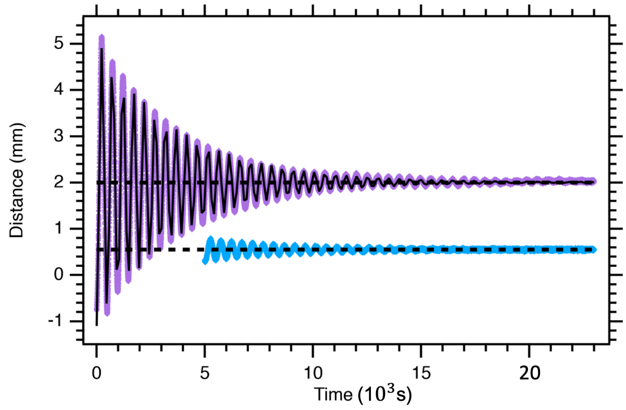

FIG. 4: The laser position is shown as a function of time for the masses in position two (purple diamonds) and masses absent (blue diamonds). The blue data is offset to better compare with the purple data though both data sets span the same time period. The black curve represents the fit used for the purple data set using an under-damped, harmonic oscillator with a period of $T = 491 \pm 1$ s. The dashed black lines represent the equilibrium positions of the deflected laser. The dashed line over the purple curve is located at $h = 2.006 \pm 0.0002$ mm, for the blue at $h_0 = 0.5494 \pm 0.0001$ mm.

Eq. (7), $\Delta S = 2|h - h_0|$ where the factor of two makes up for the lack of a second measurement. Our ΔS value for method one was $\Delta S = 2.914 \pm 0.001$ mm. The period was measured to be the same as in method one where $T = 491 \pm 1$ s. From substituting T and ΔS into Eq. (13) and making the correction from Eq. (14), the gravitational constant was calculated to be $G = (7.3 \pm 0.1) \times 10^{-11}$ N^{2}m^{2}kg^{-2}. We found that method two did not contain known G within its uncertainty as shown in Table I.

We used our measured G values from both methods to estimate the mass of the earth from Eq. (16). For the calculations, we assumed that the radius of the Earth was $R_E = 6347.5$ km, and used the acceleration due to gravity at sea level where $g = 9.807$ ms^{-2}. The accepted value for the mass of the Earth is 5.97×10^{24} kg. For method one, we calculated $m_E = (5.9 \pm 0.1) \times 10^{24}$ kg, and method two $m_E = (5.4 \pm 0.1) \times 10^{24}$ kg. Similarly with our comparison between the calculated and known G values, the large uncertainty in L, resulted in a large uncertainty for the calculated masses.

V. CONCLUSION

We measured Newton's gravitational constant G for two methods using the torsion balance. Our apparatus considered the deflection of a laser due to the rotation of a lever arm from the gravitational forces between masses. Using two measurement methods, we considered the rotation angle θ of lever arm from Fig. 1, and used it to determine the change in distance between the large and small masses.

The first method measured the change in distance ΔS by comparing the equilibrium locations of the lever arm between the two configurations of the large masses, as shown in Fig. 2. The second method compared one of the equilibrium locations with the equilibrium location of the small masses when the large masses were absent. In method one, G was measured to be $G = (6.6 \pm 0.1) \times 10^{-11}$ N^{2}m^{2}kg^{-2}, and $G = (7.3 \pm 0.1) \times 10^{-11}$ N^{2}m^{2}kg^{-2} for method two. Comparing both results to the accepted value $G = 6.67 \times 10^{-11}$ N^{2}m^{2}kg^{-2}, only method one had the known value within its uncertainty.

Using the calculated G values and known constants for the radius of the Earth, and acceleration due to gravity, we calculated the Earth's mass. The results for the two methods were $(5.9 \pm 0.1) \times 10^{24}$ kg, and $(5.4 \pm 0.1) \times 10^{24}$ kg respectively. Again, only method one included the mass of the Earth, 5.97×10^{24} kg, within its uncertainty.

Although most of the constants and calculated values had a low uncertainty, the length measurement L between the mirror and the detector was the highest. As a result, both our calculated G and mass values for the Earth had a 10% uncertainty. If L were to be measured exactly, we could see the accuracy of both calculations increase by one order of magnitude.

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