Calculating the Feigenbaum Constants In A Nonlinear RLD Circuit

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Period doubling bifurcation and chaotic behavior were observed in a diode's voltage connected in series with a resistor, an inductor, and a sinusoidal function generator. The circuit's driving voltage was varied while the frequency is held constant to observe the evolution of the circuit's route to chaos which is plotted in a bifurcation logistic map. The voltages at which period doubling occurred were recorded and used to calculate Feigenbaum's constants. The first constant which describes the rate at which period doubling occurs was calculated to be 4.6 ± 0.2 which is within the accepted value of $\sigma = 4.669$ with an error percentage of only 1 %. The second constant which describes the ratio of tines to subtines in the bifurcation map was calculated to be 2.7 ± 0.3 which is also within the accepted value of $\alpha = 2.503$ with an error percentage of 7%.

I. INTRODUCTION

Had the author of this paper chosen to drink hot chocolate over apple cider on a cold wintry day when he was five, this paper would have never been written. This statement can be labelled as nothing but an outrageous conjecture, yet it holds the essence of chaos theory. In this example, the existence of this paper depends on a seemingly insignificant decision made over a decade ago; this "life" system can be described as highly sensitive to small changes in its parameters.

The notion of chaos can be interpreted as the absence of order and the unpredictability of outcomes in a system highly sensitive to change. This definition of chaos challenges the fundamental axiom of causality that physics is built upon; after all, if the cause in a deterministic system is known so should the effect. This prompted some physicists to attribute chaotic behavior to insufficient knowledge of the variables contributing to chaos. Others argued that the high sensitivity of chaotic systems make them impossible to predict and hence can only be understood in terms of the underlying patterns that arise in a large class of chaotic systems. Regardless of stance on chaos, this paper explores the concept as a not fully understood phenomenon that piques the interest.

The concept of chaos first appeared in publication when the French theoretical physicist Henri Poincarè attempted to solve three-body problem in the 1880s. This problem seeks to determine the motion of three points in space according to Newton's laws of motion and the universal laws of gravitation given all needed initial conditions such as position and velocity. Poincarè showed that the problem has no general solution as the the motion of the three bodies is non-periodic and non-repeating except in special cases. This is an example of a deterministic system that exhibits chaotic behavior.

It was not until 1961, however, that the American mathematician and meteorologist Edward Lorenz established *chaos theory.* Lorenz had developed a computer simulation to model air movement in the atmosphere to predict weather changes. Lorenz noticed that in performing the same calculation rounding three rather than six digits gave drastically different weather outcomes. This gave rise to famous *butterfly effect*, where a butterfly flapping its wings in Brazil, in analogy to the minute change of rounding off an extra digit in the simulation, sets off a tornado in Texas a year later.

The complete randomness of chaos was shattered in 1975 with the discovery of the period doubling route to chaos by the American physicist Mitchell Feigenbaum. This theory describes the transition from regular dynamics to chaos with a series of period doubling responses, known as bifurcations, to a varying driving parameter. This theory also states that the ratio of distance between two consecutive bifurcations is a constant, known as the *Feigenbaum constant*, found in a multitude of chaotic systems. Suddenly, chaos contained some element of order characterized in the universality of this transition. The universality of this constant in chaotic phenomena implies that there is something fundamental about chaos irrespective of system. Understanding the deep structure and dynamics of one chaotic system would then shed light on chaos theory as a whole [1].

In this paper, we examine the period doubling and chaos behavior in a simple circuit containing a resistor, inductor, and diode. The period doubling behavior is observed in the diode's voltage response to increasing driving voltage. The driving voltages at which period doubling occurs was then used to calculate the first Feigenbaum constant. The bifurcation logistic map is also plotted and the width of its tines is measured to calculate the second Feigenbaum constant.

II. THEORY

A. Deterministic systems

A deterministic system follows the idea of causality, where any one state is dependent of the previous state. One can imagine snapshots in time of a set of dominoes falling, where the state of each snapshot is directly caused by the previous. A non-deterministic system, on the other hand, is characterized by the independence of its states. Rolling a three this turn provides no insight on the next roll. That being said, rolling a die is a determin-

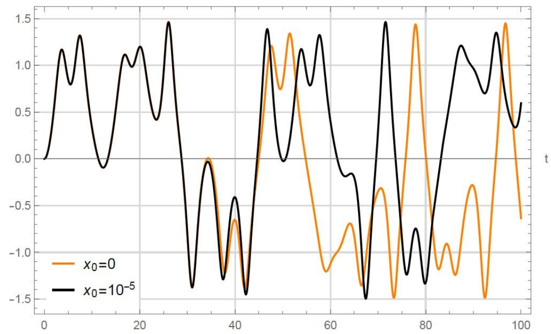

FIG. 1: Sensitivity to initial position change in a Duffing oscillator simulation plotted on Mathematica. Original code from [2] (modified).

istic system; After all, one can hypothetically calculate the final state of a dice by calculating the rotation and trajectory using kinematics assuming all parameters are known precisely.

In our experience, we perform actions with a certain expectation of the result; we expect a moving car to be closer to us in the future so we move away. It is easy then to assume that causality implies predictability, and for the most part we would be correct. Some deterministic systems, however, exhibit chaotic behavior, take the three body problem for example. The question becomes, how precisely can one know the initial conditions in a system?

B. Nonlinear Systems

A nonlinear system is a system where the change in output is not proportional to the change in input. The volume of a sphere, given by $V = \frac{4}{3}\pi r^3$, is an example of a nonlinear equation, where doubling the radius results in a eight-fold change in volume. Nonlinear systems are more sensitive to change in input than linear systems are. In most cases, an infinitesimal change in input leads to a diverging change in output in time dependent systems. In fact, most chaotic systems are nonlinear system.

C. Chaos Theory

To understand the level sensitivity of chaotic systems we briefly examine the Duffing oscillator system named after the German engineer Georg Duffing. The Duffing oscillator is a damped and driven oscillator described by a nonlinear second order differential equation which is known to exhibit chaotic behavior. This system is modeled by the Duffing equation given by

$$\frac{\partial^2 x}{\partial t^2} + \delta\frac{\partial x}{\partial t} + \alpha x + \beta x^3 = \gamma\cos(wt), \qquad (1)$$

where x is position, t is time. We pay no attention to the constants α,β,γ, and δ as we only care about the chaotic behavior. The position in time of a Duffing oscillator is plotted twice in Fig. 1 with an infinitesimal change on the magnitude of 10^{-5} to initial position. The two traces start off the same then quickly diverge to give different outputs. Note that a change in input of any magnitude eventually leads to diverging outputs in a chaotic system given enough time.

D. Diode

In this experiment, a simple resistor, inductor, and diode (RLD) circuit is used to model chaos, with the diode being the nonlinear part of the setup. It is then essential to explore the structural components of a diode to understand the reason it behaves chaotically. A diode is an crystal containing two types of extrinsic semiconductor material, namely p-type and n-type, forming a p-n junction that allows the current to pass in only one direction. We now proceed to explain this definition of a diode.

1. Semiconductors

Semiconductors are materials with electrical conductivity in the middle ground between conductors and insulators. In terms of band structure, intrinsic semiconductors have a filled valence band and a small gap between valence and conduction band. When enough energy is supplied to promote electrons to the conduction band, these electrons can now move freely if an external field is applied.

The promoted electrons leave *holes* behind, also allowing electrons in the valence band to move if a field is applied. It is more useful in this situation to keep track of the hole movement instead of the collective electron movement. Hence, promoting an electron to the conduction band has created an *electron-hole* pair that contribute to the conductivity of the material. It is often more useful in application to have semiconductors where one type of charge carrier predominates. We can achieve this by adding impurities to our intrinsic semiconductor in a process called doping.

2. p-type

A semiconductor with the majority of charge carriers being holes is called a p-type, "p" for positive charge carrier. The majority of semiconductors are elements from group IV such as silicon, hence p-type doping can be achieved by adding an element of group III. Adding boron to a silicon semiconductor results in a creation of a hole as boron covalently binds to four silicon even though boron has three valence electrons as shown in Fig. 2.a.

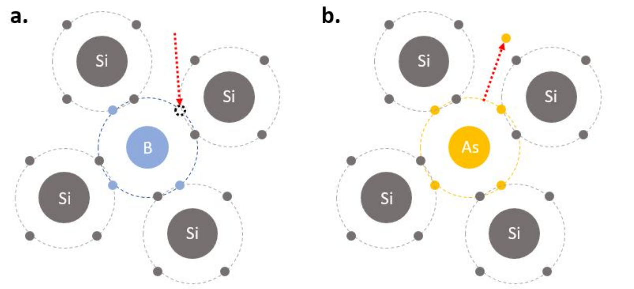

FIG. 2: Doping a silicon intrinsic semiconductor with (a) boron making it a p-type, or (b) arsenic making it an n-type.

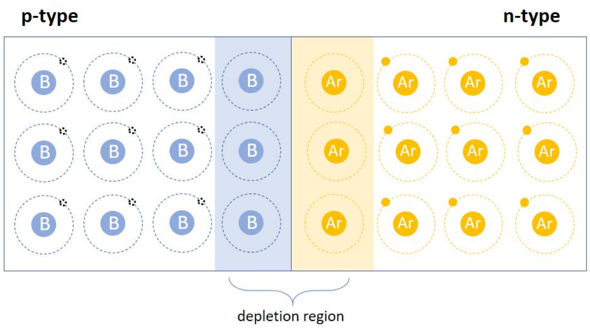

FIG. 3: Formation of the depletion region on the interface of a pn-junction.

3. n-type

A semiconductor with the majority of charge carriers being electrons is called an n-type, "n" for negative charge carrier. Doping silicon with an element of group V such as arsenic creates an n-type. Arsenic has five valence electrons and it binds with four silicon leaving one electron to loosely bound to arsenic as shown in Fig. 2.b.

4. p-n junction

Now that we have both p-type and n-type semiconductors, we can combine them to form a p-n junction. Once the interface between the two semiconductors is formed, electrons quickly migrate from n-type to the p-type semiconductor. The excess electrons from arsenic annihilate the boron holes as shown in Fig. 3 forming a depletion region near the interface. The potential difference between the two semiconductors appears in the form of a drop across the junction from p-type to n-type.

We will now discuss the behavior of the junction when an external field is applied. Connecting a pn-junction to a generator with the positive terminal attached to the p-type is called forward bias as it allows the flow of current through. In this case, electrons are flowing through the n-type which decreases the potential difference across the interface of the pn-junction. The electrons can then

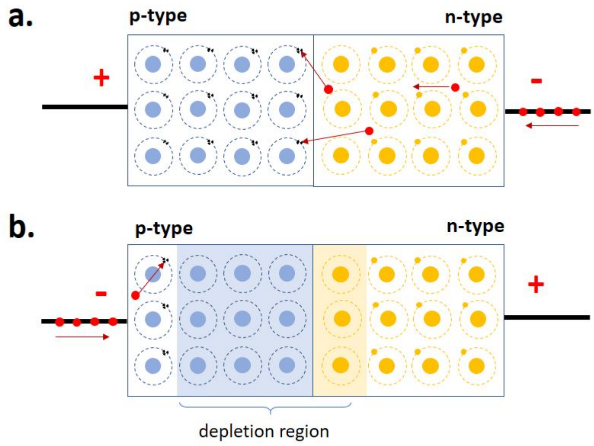

FIG. 4: Schematic of a pn-junction in operation .In (a) forward bias of pn-junction characterized by the depletion region getting smaller, and (b) reverse bias of a pn-junction characterized by the depletion region getting wider.

freely move through the p-type "jumping" between holes as shown in Fig. 4.a. The charge carriers in the p-type are holes which are the positive counterpart to the negative electron charge carriers in the n-type; an electric field is then formed directed from the p-type to n-type. The flow of electrons is unhindered as their direction is opposed to that of the electric field.

Alternatively, connecting the positive terminal of the generator to the n-type is called a reverse bias as the current is not allowed through the junction. The electrons flow through the p-type against the electric field which hinders their movement. These electrons end up annihilating the holes in the p-types which causes the depletion region to increase in size. The potential difference increases across the junction further blocking the flow of electrons. In a reverse bias then, current does not pass through a pn-junction [3].

5. Chaos in a Diode

We now know that the diode, which is the nonlinear part of the circuit, is just a crystal containing a pn-junction. The diode is connected with an alternating sinusoidal signal across of which means that the diode is alternating between its forward and reverse bias state. In forward bias, the depletion region decreases in size and more electrons can migrate through the diode, then switching to reverse bias where the depletion region increases in size and fewer electrons can pass through the diode. When a high frequency signal is supplied, the diode no longer has enough time to reach equilibrium between forward and reverse bias. The transition from one state to the other would then depend on the previous state. Hence, varying the driving voltage would cause chaos in the diode's voltage response [4].

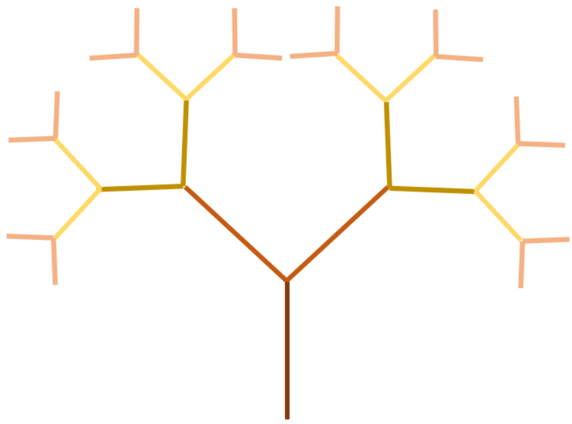

FIG. 5: Tree diagram showing the doubling behavior in bifurcation. Each branching vertex represents the point at which bifurcation occurred.

E. Bifurcation Theory

Bifurcation theory is the study of qualitative changes in a system with respect to a driving parameter. In a magical world where a system described by $y = x$ is altered to $y = x^2$ when a parameter is varied, we say that the system bifurcates. Initially, the system has one solution for every position, this is then changed to two solutions which would correspond to period doubling behavior. This concept is shown in Fig. 5. In our system, the depletion region in the diode is oscillating in a sense as the current is varied. This causes the diode to respond in different ways to different driving voltages.

The behavior of a nonlinear chaotic system building up to chaos can be modeled through a bifurcation logistic map. In the $y = x$ and $y = x^2$ example, the logistic map would have one value that doubles to two values after a certain parameter value. This model predicts that further doubling would occur at each of the two values generating four, then eight, sixteen, thirty two values... The bifurcations become so dense and are eventually indistinguishable from chaos

1. *Feigenbaum Constants*

Feigenbaum discovered universal constants that apply to all chaotic systems that depend on only one driving parameter. The first constant σ, called the *first Feigenbaum constant*, is given by

$$\sigma = \lim_{n\to\infty} \frac{\lambda_{n-1} - \lambda_{n-2}}{\lambda_n - \lambda_{n-1}} \approx 4.669, \tag{2}$$

where λ_n is the value of the driving parameter where the $n-$th bifurcation occurs. This constant states that period doubling occurs at regular intervals where the limit

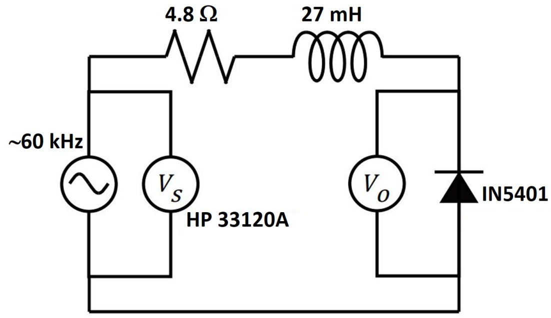

FIG. 6: The RLD circuit connection used in this experiment to model the chaotic behavior in the diode's voltage (Modified from ref. [6]).

of the ratio of the distance between two consecutive period doubling approaches $\sigma = 4.669$. The parameter λ in this experiment is the driving voltage; we can then rewrite Eq. (2) as

$$\sigma = \lim_{n\to\infty} \frac{V_{n-1} - V_{n-2}}{V_n - V_{n-1}}, \tag{3}$$

where V is the driving voltage.

The second constant α, called the *second Feigenbaum constant*, is

$$\alpha \approx 2.503, \tag{4}$$

and is a measure of the ratio of width of a tine and one of its two subtines in the bifurcation logistic map. The tine is the pitchfork shape of period doubling. Note only one of the two subtines can be used for this ratio and that this constant does not apply to the second period doubling [5].

III. EXPERIMENT

A. Setup

The circuit used in this experiment is a simple resistor, inductor, and diode circuit with the diode being the nonlinear part exhibiting chaotic behavior. The resistor has resistance of 4.8 Ω, the inductor has inductance of 27 mH, and the diode used is an IN5401 connected as shown in Fig. 6. The driving voltage for this circuit is supplied by a HP 33120A function generator. This voltage along with the diode's voltage are both monitored on a Tektronix TDS 2012B. The driving voltage is varied for a given frequency and the period doubling in the diode's response voltage is observed.

B. Data acquisition

Calculating the first Feigenbaum constant requires finding the driving voltages at which period doubling occurs. This data was collected by slowing increasing the driving voltage and recording the voltage when period doubling is seen on the oscilloscope. The period doubling voltages where collected for several frequencies. Oscilloscope images were saved using the save/recall function and transferred via a 1 GB flash drive.

Calculating the second Feigenbaum constant required plotting the bifurcation logistic map. This could be done by saving the data on the oscilloscope for several values in the period doubling regime; the diode voltage would then be plotted against the driving voltage. This method was time consuming and did not provide enough resolution for the constant to be calculated.

Alternatively, the oscilloscope was filmed while the driving voltage was increased and the video was used to track the voltage peaks during the period doubling regime. The screen of the oscilloscope was aligned parallel the camera to avoid parallax effect. Tracking was done using a free software called Tracker developed by Douglas Brown [7].

IV. RESULTS & ANALYSIS

The period doubling behavior of the diode is observed for several driving frequencies. All these runs exhibited similar period doubling behavior and chaos at different driving voltages. We focus our behavior description on the frequency of 15 kHz as it showed the most resolution and clarity.

The most significant events in diode are shown in Fig. 7, where events occurred in order from top to bottom. In the case of multiple periods, the persist function on the oscilloscope was used to capture all periods in the same image. The waveform in (a) is the initial response of the diode at low driving voltages. This response continues to increase in amplitude gradually until it bifurcates to the waveform shown in (b) characterized by the splitting of the initial peak into two separate ones. A series of bifurcations then occur giving periods 4, 8, and 16 as shown in (c), (d), and (e). Bifurcations appear more often as the period number increases; the voltage had to be increased by 437 mA to get from period 2 to 4 and by 18 mA to get from period 8 to 16.

Period doubling occurs so frequently close to period 16 that the transition to period 32 (not shown) took required 4 mV increase in driving voltage. Bifurcations then quickly accumulates a period of an immensely large number that can no longer be distinguished from chaos as shown in (f). The chaotic behavior then persists for a couple millivolts then switches back to order in (h) after increasing the driving voltage sufficiently. The waveform in (h) appears to be period 5 with only five peaks, this is surprising as 5 can never be obtained by doubling 2. Increasing the driving voltage even further gives period 3 shown in (i). This is an important results as period 3 was proven to exhibit infinitely many period state [8]. In other words, there exists period-x, where x is any number you can dream up. This makes it more plausible that the waveform seen in (h) represents period 5.

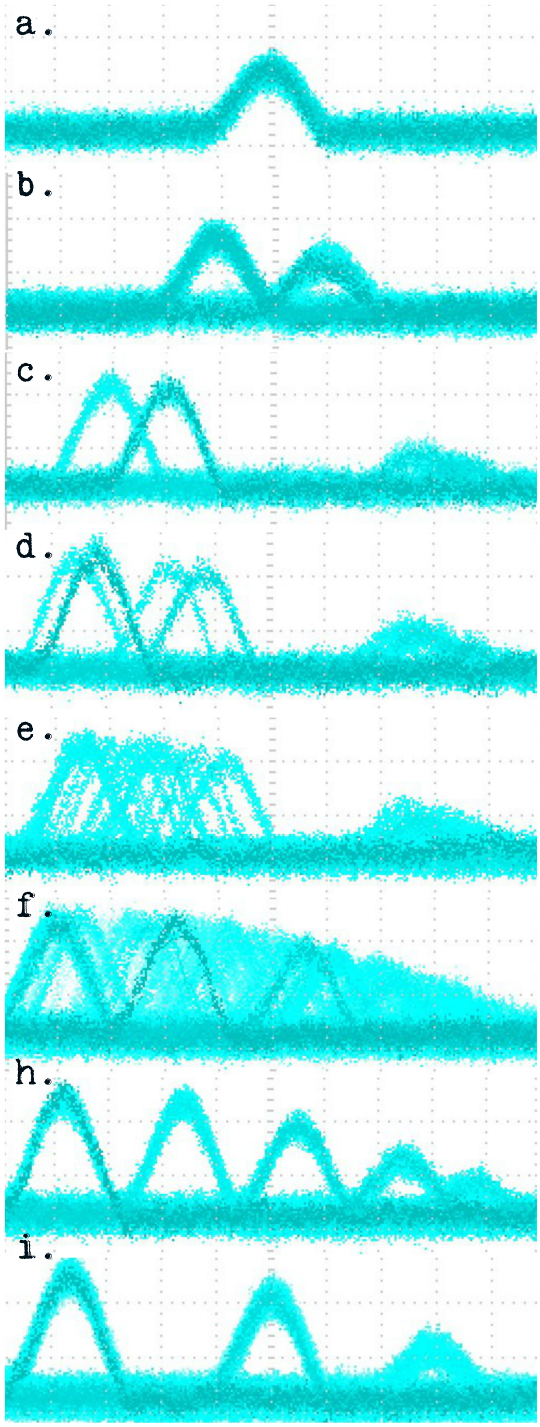

FIG. 7: Diode voltage response showing (a) period 1, (b) period 2, (c) period 4, (d) period 8, (e) period 16, (f) chaos, (h) period 5, and (i) period 3.

For several frequencies, the voltage at which period doubling occurred can be found in Table I. The individual Feigenbaum constants were then calculated using Eq. (3) and are shown in Table II.

TABLE I: Driving voltages at which period doubling occurred for various values of frequency. All voltages are in units of volts.

Frequency (kHz)	Period-2 (V_1)	Period-4 (V_2)	Period-8 (V_3)	Period-16 (V_4)	Period-32 (V_5)
13	2.702	3.286	3.393	3.417	-
14	2.353	2.850	2.943	2.962	-
15	2.081	2.518	2.604	2.622	2.626
16	1.874	2.266	2.345	2.362	-
17	1.711	2.064	2.138	2.155	-
18	1.577	1.898	1.968	1.983	-
19	1.465	1.764	1.826	1.84	-
20	1.371	1.648	1.711	1.724	-

TABLE II: Feigenbaum's first constant that calculated using Eq. (3) and the data in Table I.

Frequency (kHz)	σ_3	σ_4	σ_5
13	5.458	4.458	-
14	5.344	4.895	-
15	5.081	4.778	4.500
16	4.962	4.647	-
17	4.770	4.353	-
18	4.586	4.667	-
19	4.823	4.429	-
20	4.397	4.846	-
Mean	4.9 ± 0.4	4.6 ± 0.2	4.5 ± 0.0

The first constant represents the limit of σ_n as n tends to infinity, given by $\sigma = 4.669$; in our case periods later than 16 can rarely be determined as shown in Table I. We can then approximate $n = 4$ which corresponds to period 16 to be infinity in this experiment. We can then calculate the first Fiegenbaum constant to be 4.6 ± 0.2 which is well with in the accepted value of $\sigma = 4.669$ with only 1 % error.

The oscilloscope peak tracking data was also used to plot the bifurcation logistic map shown in Fig. 8. This map shows the evolution of period doubling behavior which was also previously shown in Fig. 7 (a)-(d). Zooming on the top right corner of the bifurcation map shows its fractal nature in the repeating forking behavior as seen in Fig. 9. We call the bifurcation shape of the plots as tines as they resemble tines of a pitchfork. The last plot, along with the bottom right corner of Fig. 8, can be used to determine the widths of the tines. We use the upper subtine in Fig. 9, to calculate the ratio in Eq. 4. The second Feigenbaum constant was determined to be 2.7 ± 0.3 which is also well within the accepted value of $\alpha = 2.503$ with an error of 7 %.

An extended map of Fig. 8 showing period doubling along with chaotic behavior of the diode's voltage is shown in Fig. 10. This plot represents the detailed evolution of the diode's voltage that is shown in Fig. 7, where the blank rectangles correspond to chaotic behavior shown in (f). Period five occurred after chaos twice in this system. We also observed the occurrence of period three that bifurcates into period six in between two chaotic phases. Finally we observed period four followed the period halving behavior resulting back into period one.

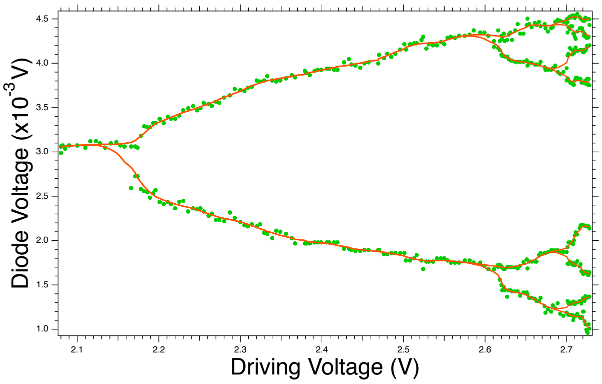

FIG. 8: Bifurcation logistic map showing the period doubling behavior of diode.

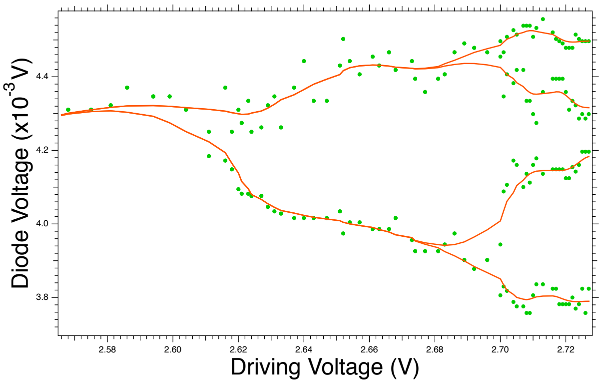

FIG. 9: Fractal behavior shown in the bifurcation logistic map. This plot is obtained by zooming the top right section of Fig. 8.

In general, the period of the diode's voltage increases when one peak splits into two. The period of the diode's voltage, however, decreases either as a result of peaks merging into one, or peaks decreasing all the way to zero. This is observed in the last section of the Fig. 10 where three peaks merge into one and the bottom period "dies out", giving back period one.

The extended bifurcation map also provide some valuable insight on the behavior of periods right before and after a chaotic phase. The map in Fig. 10 would look surprisingly continuous if chaotic phases are removed. In other words, one could hypothetically connect periods via imaginary lines through chaos. This is yet another fascinating property showing underlying order in chaotic behavior.

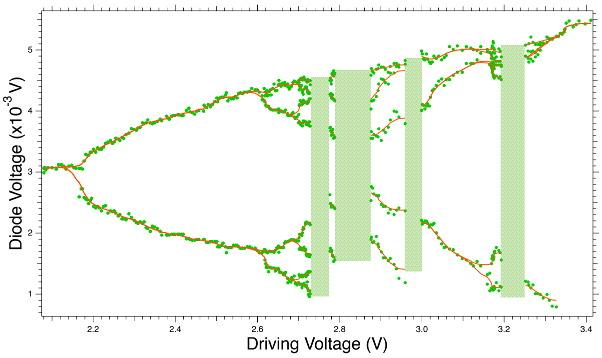

FIG. 10: Extended map of Fig. 8 showing the chaotic behavior of diode voltage and the periods in between chaos. The blank rectangles represent chaos.

The rate at which the period doubling occurred was increasing with the increase of the period number. This behavior is described by Fiegenbaum's first constant, where the ratio of the distance between three consecutive period doubling approaches a constant value. This constant was calculated to be 4.6 ± 0.2 which is well with in the accepted value of $\sigma = 4.669$ with only 1 % error.

The bifurcation logistic map was also plotted for a driving frequency of 15 kHz showing a more detailed period doubling and chaotic behavior. The forking behavior of the logistic map is shown in Fig. 8, where each period ends up branching into two more periods. The ratio of the width of a tine and one of its two subtines is described by Feigenbaum's second constant. This constant was calculated to be 2.7 ± 0.3 which is also well within the accepted value of $\alpha = 2.503$ with an error of 7%

V. CONCLUSION

The period doubling behavior of a diode in a simple RLD circuit was examined in this experiment. Periods up to period 32 were observed to occur in the diode's response voltage leading up to chaos. Period 3 and period 5 were also observed after the occurrence of chaos in what seemed like period halving route back into order.

VI. ACKNOWLEDGMENTS

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