# Stochastic Resonance in a Hysteretic Circuit

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Stochastic resonance is demonstrated in a regenerative bistable comparator. The setup is a variation on an experiment performed by S. Fauve and F. Heslot [1]. The data confirms that by inputting both a noise signal and a pure sine wave to the same input of a regenerative bistable comparator, rather than separate inputs, stochastic resonance is maintained. A peak signal to noise ratio is located when the added noise signal has a standard deviation of $\sigma = 0.18488$ V.

## I. INTRODUCTION

The goal of any system of communication is to get one thing to resonate with another. This is often achieved by sending a signal from one object to another in hopes that the second object will receive the signal, or resonate with it. But sometimes the signals we send are too weak to grab the attention of their intended audience. However, it may be possible that adding some chaotic element to a signal could help it communicate. Stochastic resonance is a phenomenon where the introduction of random noise allows a signal to be detected by an instrument or perceiver that could not detect the same signal in the absence of extra noise. The concept was originally suggested to explain the reoccurring cycle of the planet earth's ice ages [2]. The planet is thrown into glaciation at a regular interval of around $10^5$ years, and the only force that acts on the earth in a similarly large period is far too weak to cause a shift in state as large as an ice age. In this first proposed model, short term climate fluctuations, which we experience as weather, are taken as the random noise that allows a very weak signal to translate into a large scale climate shift.

To conceptualize how random noise could possibly *aid* the communication of a signal, imagine you are in a crowded social function, standing in a corner of the room, enjoying some snacks and chatting with the people adjacent to you. Your friend Nick is standing at the opposite corner of the room. Now, it happens to be that you are standing near a window and you see that Nick's car is being towed outside the building. You want to get the message to Nick so that he can exit the party and stop the towing company before he loses his vehicle, but you also do not want to leave your spot next to the snack tray. So you tell the people near you that Nick's car is being towed, in hopes that people will pass on the message until it reaches him. Now if this party is full of people who are exhausted from long days at their respective jobs and don't know each other at all, then there is not very much conversation going on and the chances of your message making it across the room, person-to-person until it reaches Nick, are quite slim. However, you can imagine another version of this gathering where everyone in attendance is full of energy and excited to see one another. In this state, the message will easily travel across the room and Nick will hear the bad news about his car. In this scenario, the weak signal is the message you wish to send, and the noise is the energy level of the crowded room. Without noise, the signal does not pass through, with noise it does. We can also easily imagine the point where the noise becomes too powerful and once again the signal would be inhibited. If everyone in the room was yelling top volume, there is once again very little opportunity for Nick to learn of his poor parking choice. This overwhelmed scenario also exists in more nuanced systems exhibiting stochastic resonance.

Since the initial proposal relating to environmental shifts by Benzi et al. [2], stochastic resonance has been identified in many other systems on a much smaller level. These systems are often bistable, meaning they can exist in one of two states at any point. Some electronic systems exhibit such bistable properties, to the point of blocking weak signals [3]. Stochastic resonance can be employed to push such signals through bistable circuits [4]. One of the first, and maybe the simplest demonstration of stochastic resonance in an electronic circuit was performed by Fauve et al. on a simple Schmitt trigger [1]. In their work, they directly demonstrated bistable stochastic resonance in an analog scenario, clearly mapping out their circuit as a metaphor for a mechanical double potential well. In the experiment presented here, the circuit board setup is very similar to the one employed by Fauve et al., however, the input technique is modified, to show a mode of stochastic resonance that falls somewhere between the two current models, threshold and bistable stochastic resonance. This experiment bridges the gap between these two models by inputing both the pure signal and the noise signal to the same input. The system is bistable, but breaking a threshold value is the method of triggering transitions between these two states.

This experiment consists of a bistable circuit with one input point and two separate signal generators feeding into the same location. One signal is a weak sine wave, the other is random Gaussian noise. Without the noise, the sine wave does not pass through the circuit. With the addition of noise, the output begins to display a signal. Too much noise causes the output to return to no signal at all. So, Fourier transform is employed to analyze the range of noise with an active output, and a signal to noise ratio is established across a variety of noise amplitudes to find the peak location of stochastic resonance.

This experiment falls within the context of many other ongoing projects surrounding stochastic resonance. Similar concepts have already been applied as a means of developing night vision goggles that can detect very dim or over-saturated light signals [5]. This particular experiment comes closer to modeling an auditory regime, as the voltage signals used herein fall into the human hearing bandwidth, and could be easily connected to a speaker and projected as such. This work could be directly applied to circuits within hearing aids, potentially as a means of boosting pure tones while abandoning any surrounding noisy environment.

## II. THEORY

There are two models often used to describe stochastic resonance in a general sense – threshold and bistable [6]. The circuit employed in this experiment exhibits bistability, so the model I will work from is the bistable model. This model will now be described in order to provide ample conceptual framework for the system at hand. Bistable stochastic resonance can be thought of in terms of a classical particle in a quartic double potential well, as shown in the top left corner of Fig. 1. We can imagine a driving signal applied to this potential well, causing the two divots to alternate in depth. This driving signal takes the form

$$x_i(t) = A\cos(\Omega t - \phi), \tag{1}$$

where $A$ is the amplitude of the signal, $\Omega$ is the frequency, and $\phi$ is the phase. Now, if this sinusoidal signal is powerful enough, the particle will jump between the two stable states at a regular frequency $\Omega$. The resulting motion is demonstrated in Fig. 1. However, in a situation where the driving signal is too weak to cause the particle to shift states, we must introduce a Gaussian noise signal, $\xi(t)$ with autocorrelation

$$\langle \xi(t)\xi(0)\rangle = 2\sigma^2\tau\delta(t), \tag{2}$$

where $\sigma$ represents the spatial standard deviation of the noise, and $\tau$ is the folding time of the correlation function, characteristic of its temporal width [7]. Angle brackets in Eq. 2 denote a correlation function. For simplicity's sake, here we will define the intensity of the noise, as $D = \sigma^2\tau$. In this over-damped case, with the presence of Gaussian noise, the particle is once again able to move between the two stable states. For experimental purposes, we are more concerned with the rate at which the particle jumps between the two wells. In the case where the pure signal is strong enough to drive the system alone, the rate will simply be the driving frequency, $\Omega$. The weak driving signal, with the aid of stochastic resonance, will give a very similar switching rate, but the amplitude of the prominent frequency is complicated by the presence of noise. This amplitude is dependent on the noise intensity, $D$, and is given by

$$X_o(D) = \frac{A_o\langle x_i^2\rangle_o}{D}\frac{2r_k}{\sqrt{4r_k^2 + \Omega^2}}. \tag{3}$$

$X_o$ here represents the amplitude of the periodic component of the particle's motion, and $\langle x_i^2\rangle_o$ is the variance of the system in the absence of the pure sine wave input [7]. There is a new and important element to this equation, $r_k$, known as Kramers rate. The Kramers rate is the rate at which the system will switch between the two stable states when driven only by our Gaussian noise, $\xi(t)$ [8]. This rate is given by

$$r_k = \frac{1}{\sqrt{2\pi}}e^{-\Delta v/D}. \tag{4}$$

$\Delta V$ here represents the height of the barrier between the two wells. Examining Eq. 3 in conjunction with our $D$-dependent Kramers rate, Gammaitoni tells us we can see

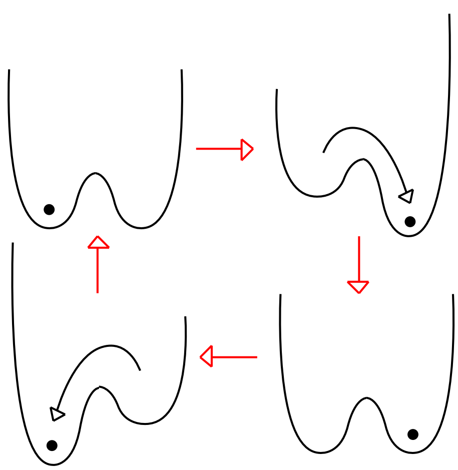

FIG. 1. Classical particle in a quartic double potential well. Here a signal is resonating with the particle, as the particle is moved to the lower position in potential shifts. Under a weak signal, the particle would remain in one side of the well throughout the cycle depicted here.

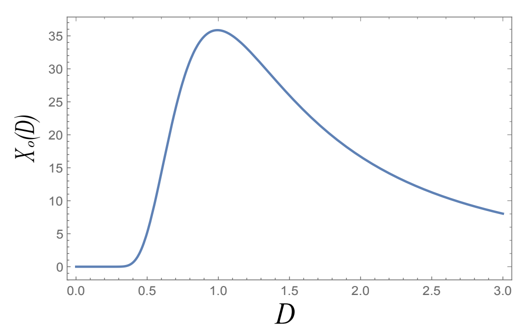


FIG. 2. Plot of Eq. 3 generated in Wolfram Mathematica. This figure confirms the behavior described by Gammaitoni et al. in their review paper on stochastic resonance [7], as we can see that there is a peak where the pure signal is strongest. We expect that the final result of this experiment will depict a similar shape to the one shown here.

that the amplitude of our desired output will be small when $D$ is very small, increase with $D$, then eventually decrease as $D$ once again gets large [7]. The behavior of Eq. 3 is shown in Fig. 2. This behavior is the key element of stochastic resonance. The goal of this experiment will be to locate the regime in $D$ where the output signal is strongest.

The bistable double potential well does a good job of eliciting the concept of stochastic resonance, but it is not fully descriptive of this experiment. This experiment is a demonstration of stochastic resonance specifically in an electronic setting, so to complete the conceptual background some information on circuitry is required. The circuit under scrutiny here is a bistable regenerative comparator, commonly referred to as a Schmitt trigger. The circuit is a simple one, an operational amplifier with a voltage divider connecting the output to the non-inverting input, as shown in Fig. 3. The primary component is an operational amplifier (Op-Amp) in the non-linear regime, with driving voltage values $V_d$ and $-V_d$. This means the Op-Amp functions to compare the inverting and non-inverting inputs. If the inverting input has a higher voltage, then the Op-Amp will output $-V_d$. If the non-inverting output has a higher voltage, the Op-Amp will output $V_d$. Given that the output is then fed back to the non-inverting input, the circuit has two different threshold values. We can find these values through simple analysis of our voltage divider,

$$V_t = \frac{V_o R_2}{R_1 + R_2}, \tag{5}$$

where $V_t$ is the threshold voltage, $V_o$ is the output voltage, and $R_1$ and $R_2$ are the two resistors that make up the voltage divider [9]. We can see that since the Op-Amp only has two possible output voltages, the Schmitt trigger will only have two possible threshold voltages.

What distinguishes this circuit from other comparators is the fact that it possesses a bistable regime. Given that the feedback is connected to the non-inverting input, a shift in output voltage opposes the sign of the input when the threshold is crossed. This means that there is a bistable regime when $-V_t <$ (input voltage $V_i$) $< V_t$, the area between the two dashed lines in Fig. 4. This particular bistability is called hysteresis, and is very important to the field of electronics as it presents a natural opportunity for memory in circuits. While the input lays within the bistable regime, the output of a Schmitt trigger is dictated by whichever non-bistable state the system most recently occupied, this is demonstrated in Fig. 5. In other words, the system remembers its most recent output. For our purposes here, the hysteresis effect simply serves to allow the possibility of a sub-threshold signal to enter a bistable system.

If we input a signal, $V_i(t)$ with the form of Eq. 1 to the circuit with the condition $A > V_t$, then the output voltage will be a square wave of frequency $\Omega$. This tracks onto our double potential well metaphor as the case where the signal is powerful enough to move the particle on its own. The weak signal case also tracks back to the Schmitt trigger if we input a signal with the form of Eq. 1 given the condition $A < V_t$. In this scenario, the entirety of our signal takes place within the hysteretic regime, meaning the output voltage will never switch. So, in order to push the signal through the circuit, we introduce a Gaussian noise signal, $\xi(t)$. The noise and the pure input add to create a new input voltage of

$$V_i(t) = A\cos(\Omega t - \phi) + \xi(t). \tag{6}$$

Following the metaphor of the double potential well, there is a predicted range of noise intensity $D$ where the output voltage will display a strong square wave of frequency $\Omega$. The strength of this output signal will vary proportionally to the amplitude of Eq. 3.

Now, the double potential well model provides us with a qualitative description of what sort of behavior characterizes stochastic resonance, but it does not fit our Schmitt trigger perfectly. There are a few key differences that stop us from simply translating the equations from the model to the circuit. First, in a hysteretic circuit, the barrier between the two bistable states is not smooth. The double potential well does not represent a transition as stark as the one present in this circuit. The

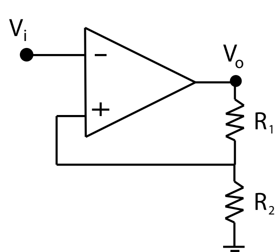


FIG. 3. Circuit diagram of a basic Schmitt trigger. $V_i$ represents the input voltage, and $V_o$ represents the output voltage.

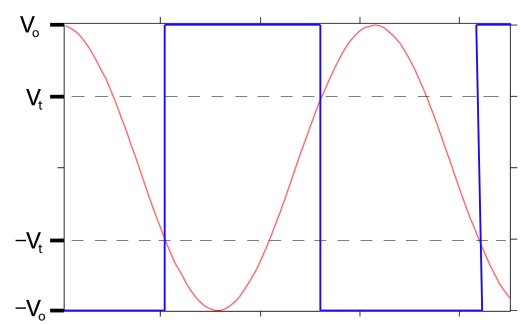


FIG. 4. Theoretical prediction of the input to output relationship in a Schmitt trigger. The input is shown here in red, the output in blue. As the input descends below the threshold voltage, the output inverts, between two stable states, thus the name inverting bistable comparator.

second difference is also what distinguishes this experiment from the original work performed by Fauve et al. . In their work, the pure sine wave signal is applied to the non-inverting input of the Schmitt trigger, and the Gaussian noise is still placed on the inverting input – this method is slightly closer to the double well model, as the pure sine wave raises and lowers the thresholds rather than adding directly to the noise value [1]. The method employed in this experiment takes the Schmitt trigger slightly farther away from the double potential well model, however, stochastic resonance will still take place. This leaves us with only one further issue to resolve. How can we quantitatively determine if stochastic resonance is occurring?

The solution is simply to take the Fourier transform of the output

$$\widetilde{V}_o(\omega) = \int_{-\infty}^{\infty} V_o(t)e^{-i\omega t}dt. \quad (7)$$

This operator takes us from the temporal spread of the output to the spread across frequency, $\omega$ [9]. If stochastic resonance is occurring in the system, $\widetilde{V}_o(\omega)$ will have a distinct spike at $\Omega$. Once we have located this spike, we can find the signal to noise ratio (SNR), simply defined as the height of the peak divided by the average of the surrounding frequency spectrum. By plotting SNR against corresponding values of $\sigma^2$, defined as the standard deviation of the voltage spread in the noise signal as before in the double well model, we can determine if the system is exhibiting the qualitative properties of stochastic resonance. $\sigma^2$ is used here rather than $D$, as $\tau$ will remain constant over all amplitudes of noise, meaning $\sigma^2$ is the only varying aspect of $D$.

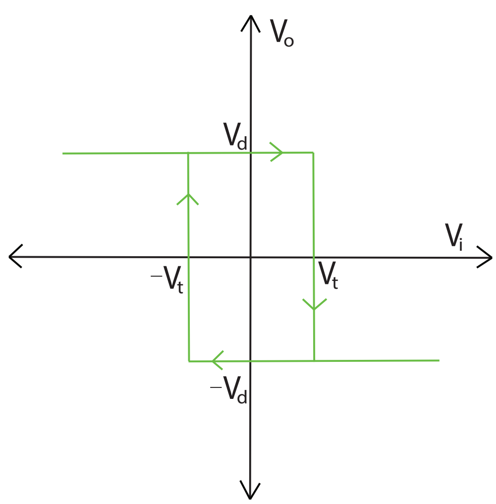


FIG. 5. Theoretical prediction of output voltage as a function of input voltage. Hysteresis is the shown as a signal descending from a positive input voltage to the bistable regime maintains a negative output, while a signal ascending from a negative input maintains a positive output.

## III. PROCEDURE

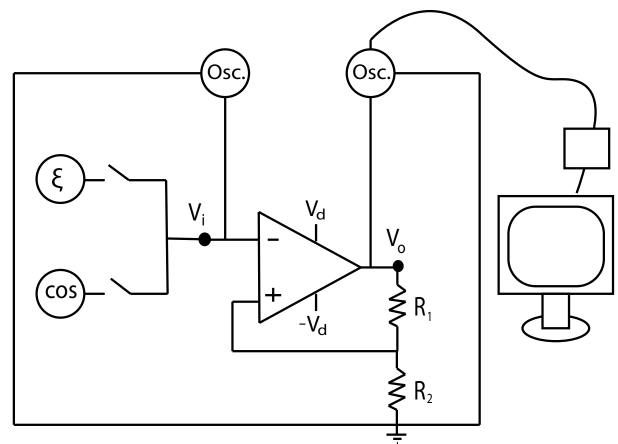


FIG. 6. Complete circuit diagram for the setup of this experiment. The circle containing $\xi$ represents the noise source, the circle containing "cos" represents the pure signal source. The two switches connecting these sources to the input show that they are able to be connected and disconnected for observation. The two circles labeled "Osc." represent the two locations where the oscilloscope is connected. these readouts are overlayed on one screen in Fig. 7. The line running from the right oscilloscope to the computer represents the connection made to the PASCO 850 interface, depicted here as a box.

The apparatus in use consists of two function generators connected to the input of a Schmitt trigger, as shown in Fig 6. One generator is producing a sub threshold signal of frequency $\Omega = 400$ Hz, and amplitude $A = 0.79$ V, while the other is producing Gaussian noise of variable amplitude. There is also an oscilloscope with two readouts, one connected from the input voltage $V_i$ to ground and the other from the output voltage $V_o$ to ground. The signal from $V_o$ is also sent to a computer running PASCO Capstone, via a PASCO 850 Universal Interface.

The focal point of this experiment is simply a Schmitt trigger as shown in Fig. 3. This circuit is constructed from a 741 Op-Amp, and two resistors with resistance $R_1 = 100$ k$\Omega$ and $R_2 = 10$ k$\Omega$. Using these values, Eq. 5 gives a simple formula for the threshold voltage, $V_t = 0.09V_o$. The two possible output voltages are determined by the driving voltage of the Op-Amp, in this case that is $\pm 15$ V. This driving voltage suggests threshold values of $\pm 1.35$ V. Examination of the circuit by vary the input voltage reveals a slightly lower threshold of $\pm 1.31$ V. This discrepancy is most likely due to internal resistance within the 741 Op-Amp. Knowledge of the threshold values allows us to conclude that a signal with a peak to peak voltage difference less than 2.62 V will not pass through the circuit.

An Aligent 33220A function generator supplies a sine wave to the input of the Schmitt trigger with a peak to

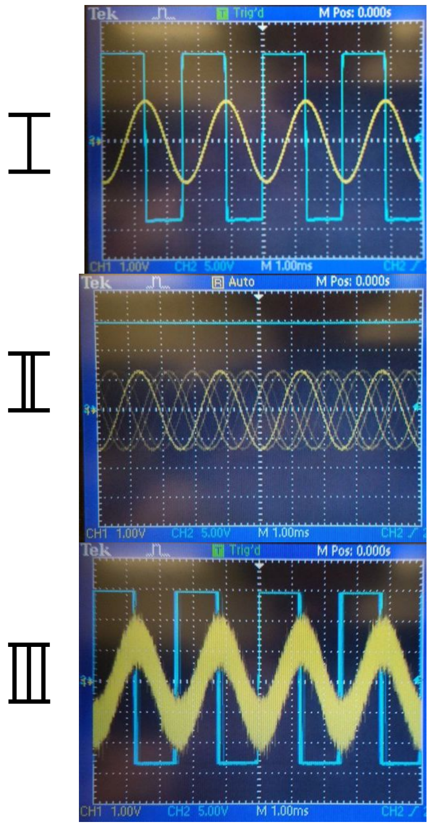


FIG. 7. Photos of the oscilloscope reading for three possible scenarios within the setup. The input voltage is shown in yellow, the output is shown in blue. (I) depicts a signal strong enough to trip the threshold values and cause the Schmitt trigget to output a pure square wave of frequency $\Omega$. This readout confirms that our Schmitt trigger is operating according to the theory presented in Fig. 4. (II) depicts a clean sub-threshold input signal, with no additional noise, meaning the output of the circuit is a constant voltage. (III) shows the same input signal as II with the addition of a Gaussian noise signal, causing the circuit to once again output a square wave, this time with a dominant frequency of $\Omega$. The input here takes the form of Eq. 6. We can observe stochastic resonance qualitatively in this image.

peak voltage of 1.58 V, so that the thresholds are never tripped, and the output remains constant. A Hewlett Packard 33120A function generator then supplies a Gaussian noise signal to the same point, causing the two signals to add. The signal becomes very noisy, and the combined input trips the threshold values, causing the output to switch between the two possible values. Progression between these three operating modes is shown in Fig. 7. This output waveform is recorded using PASCO Capstone, and exported as a text file of Voltage vs time. The output voltages were recorded at a resolution of 100 kHz, much larger than the driving frequency, $\Omega = 400$ Hz. This text file is then imported into Igor Pro for analysis. The relevant range of noise amplitude was established before taking data, and a text file was gathered for peak to peak noise voltages between 2.5 V and 6.5 V at ascending increments of 0.5 V. The noise waveforms alone were also recorded through the same interface in order to identify values of $\sigma$.

## IV. DATA AND RESULTS

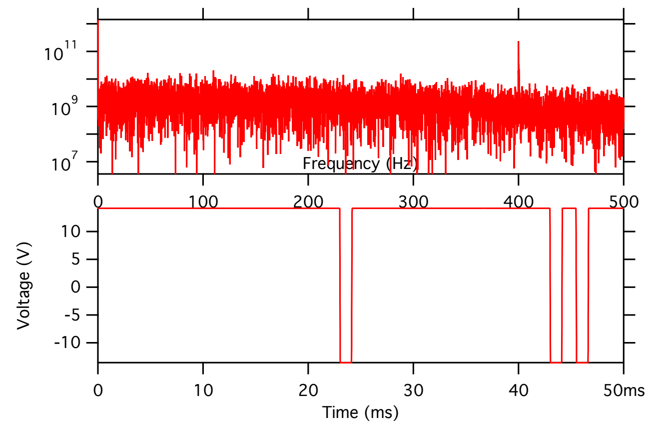


FIG. 8. Graphic representations of data from a run with a $\sigma = 0.11$ V. **Bottom**: direct graph of 0.05 seconds of data from the output of the circuit. **Top**: Fourier transform of 5 seconds of the same data set. Note that the bottom graph shows the output 'firing' every now and then with a width associated the with the frequency $\omega = 400$ Hz, and there is a spike the Fourier transform at $\omega = 400$ Hz.

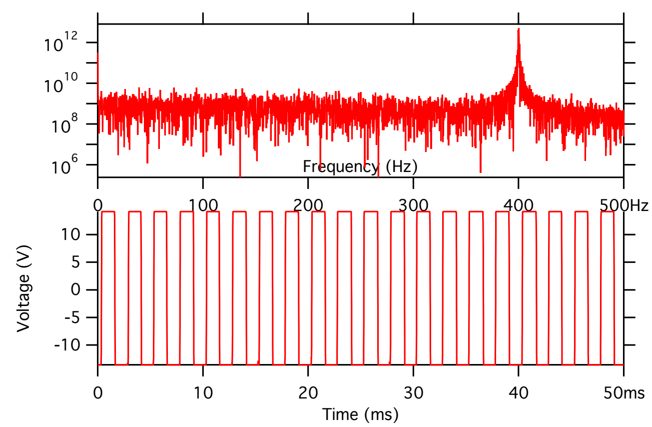


FIG. 9. Graphic representations of data from a run with a $\sigma = 0.18$ V. This is the data run identified as optimal for stochastic resonance in this circuit. **Bottom**: direct graph of 0.05 seconds of data from the output of the circuit. **Top**: Fourier transform of 5 seconds of the same data set. Note that the square wave on the bottom is firing regularly, appearing almost as a pure square wave of frequency $\omega = 400$ Hz. The spike in the Fourier transform at $\omega = 400$ Hz has grown very defined.

After the data were imported into Igor Pro, the Fourier transform was taken as in Eq. 7. Three out of the nine

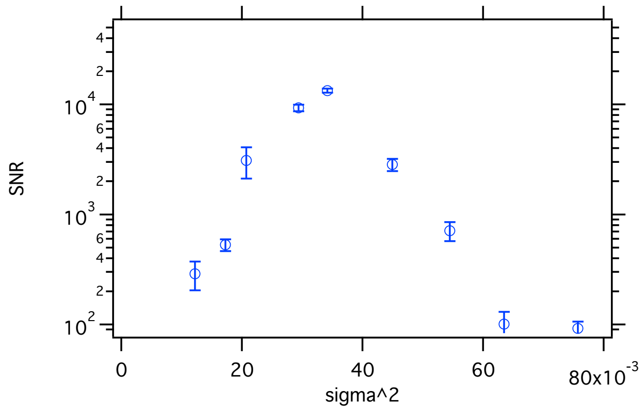


FIG. 10. Graph of SNR vs. $\sigma^2$. Here we can observe the behavior characteristic of stochastic resonance; increase in SNR to a peak, then decrease when the noise signal becomes too powerful.

Fourier transforms taken are displayed in Fig. 8 and Fig. 9. This allows us to locate the peak frequency, which as expected occurs at $\Omega = 400$ Hz, our driving frequency. The series of these graphs also allows for observation that as $\sigma$ increases, the height of the peak at 400 Hz increases as well, until it reaches a maximum, at which point it once again falls to the level of the background noise. To observe the process of increase and decrease characteristic of stochastic resonance, we measure the height of the peak and divide it by the average of the surrounding noise. This gives us the signal to noise ratio (SNR). SNR is then plotted against $\sigma^2$ to observe the change in resonance against the intensity of the noise signal, as displayed in Fig. 10. $\sigma$ was found using Igor Pro's wave stats function on a text file for the raw noise. Fig. 10 does not include a fit of the form in Eq. 3, given that the theory for a double well potential does not track perfectly to this electronic scenario–The main difference being that the decay tail after the peak dips much lower on the experimental plot than the tail in Fig. 2 does.

## V. CONCLUSIONS

The graph of SNR vs. $\sigma^2$ shown in Fig. 10 allows us to observe that stochastic resonance is in fact occurring. Just as in our theoretical double well model, there is a regime in which the addition of noise allows a signal to pass through the bistable system. This regime also behaves according to the predictions of Eq. 3, increasing at first, then decreasing [7]. This structure means we can locate a peak value, with a corresponding $\sigma$ that shows us the optimal noise input for stochastic resonance in this system given a driving frequency of $\Omega = 400$ Hz. This peak value is located at $\sigma = 0.18488$ V. These observations confirm that stochastic resonance is occurring even in an augmented version of Fauve and Heslot's first experiment. This augmentation doesn't necessarily adhere to the gold standard of bistable stochastic resonance set out in the double well model, but nonetheless stochastic resonance is observed

A system like this could be employed in further work developing hearing aids that listen for pure frequencies that may be sub threshold, or precise data collection devices that involve detecting very minimal signals. Another interesting extension of this work is as a model of a neuron. Similar to the behavior shown in the bottom half of Fig. 8, a neuron has a baseline energy level and fires in short bursts to a higher energy level when stimulated. If a large quantity of these hysteretic circuits was gathered, and placed in some sort of medium allowing feedback between the nodes, they could potentially be used as an electronic simulation of brain activity.

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