Measuring Thermal Conductivity by Ångstrom's Method

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In this experiment, κ, the thermal conductivity of a rod of brass, was measured by periodically heating one end of the rod. Turning the heater attached to the rod on and off with a 10 mHz square wave signal, waves of heat were sent along the length of the rod, measured by two thermistors a distance Δx apart. Taking these temperatures as a function of time, a Fourier transform was applied to a 15 period long segment of data, allowing us to use the amplitudes γ_n and phases φ_n of the first three harmonics to make three distinct measurements of the thermal conductivity of the brass rod. For the first three contributing harmonics in the Fourier transform, $n = 1$, $n = 3$, and $n = 5$, an average thermal conductivity value of $\bar{\kappa} \pm \sigma_\kappa = 103 \pm 2$ W/(m K) was found. This result gives a small error of 5% with regards to the accepted value of $\kappa = 109$ W/(m K).

I. INTRODUCTION

In thermal physics, thermal conductivity is a material's ability to conduct heat; the greater a material's thermal conductivity, the faster heat will spread throughout. To give an example, copper has an extremely high thermal conductivity, second only to silver. This property leads to its use in heat sinks, where heat is able to quickly pass through the sink and away from a sensitive electronic or mechanical component, as well as the prevalence of copper pots, whose high thermal conductivity allows the surface of the pot to heat evenly and quickly.

Though heat manifests as an observable macroscopic phenomenon, the quantitative study of heat was not scientifically feasible until Daniel Fahrenheit's early 18$^{\text{th}}$ century invention of an accurate mercury thermometer [1]. From that point, the field of thermal physics has expanded dramatically, quite notably in the 1807 to 1811 work of Joseph Fourier. During this period, Fourier conducted experiments and created mathematical techniques which were the first to estimate thermal conductivity [1]. Half a century later, in 1861, Anders Jonas Ångstrom published a method to measure the thermal conductivity of a material by periodically heating one end of a rod. This method was later modified to employ Fourier analysis of a square wave heating function, where the heater is only active every other half cycle.[2]

Within this lab, Ångstrom's experiment was recreated with a rod of brass, an alloy of copper and zinc. Heating the end of the rod at regular intervals and measuring its temperature at two separate points along its length, two periodic waves of temperature were measured as a function of time. Employing Fourier analysis of these data, the thermal conductivity of the brass rod shall be calculated.

II. THEORY

In this rendition of Ångstrom's experiment, the end of a brass rod switches periodically between being heated and being left to sit, producing a heat wave that propagates along the length of the rod.

Basing the derivation off of that given in the lab manual [2], this heat wave shall be expressed through the function $T(x,t)$, which is the bar's temperature relative to the air. For the moment, temperature will be varied at the end of the rod according to the sinusoidal function

$$T[0,t] = T_0 \cos(\omega t)\,, \tag{1}$$

where T_0 is the amplitude of temperature oscillation, and ω is its angular frequency. However, the heat wave is unable to simply travel along the rod unimpeded. As it travels further, the heat will diffuse along the bar, creating a smoothing effect on the wave function's shape, and convect away through the air, lowering the amount of thermal energy remaining in the rod itself, via Newton's cooling law. These factors, along with the geometry of the rod, combine to give a differential equation of

$$\frac{\partial T}{\partial t} = D\nabla^2 T - \epsilon T, \tag{2}$$

where $D\nabla^2 T$ corresponds to the diffusion of heat through the bar, and $-\epsilon T$ to the rate at which heat is emitted into the air. In the first case, the thermal diffusivity $D = \kappa/s\rho$, where κ is the material's thermal conductivity, s is its specific heat, and ρ is its density. The other term, the emissivity $\epsilon = RC/s\rho\mathcal{A}$, where R is a constant emission coefficient, C the circumference of the rod, and $\mathcal{A}$ the cross-sectional area of the rod.

As a result of this diffusion and emission, the temperature is expected not only to fluctuate as a function of time t and distance from the heated end of the bar, x, but also decay exponentially along the length of the bar. Thus, a solution of the form

$$T(x,t) = Ae^{-ax}\cos(\omega t - bx), \tag{3}$$

where a and b are constants, will be assumed.

To begin to solve for κ, the thermal conductivity of the material Eqs. 2 and 3 shall be combined.

To account for the left side of Eq. 2, the partial derivative with respect to time of Eq. 3 is taken, finding

$$\frac{\partial T}{\partial t} = -\omega A e^{-ax}\sin(\omega t - bx). \tag{4}$$

Similarly, the Laplacian of the temperature function is taken. Creating the variable $\xi \equiv (\omega t - bx)$ to conserve space and improve readability,

$$\nabla^2 T = Ae^{-ax}[(a^2 - b^2)\cos(\xi) - 2ab\sin(\xi)]. \quad (5)$$

Thus, Eq. 2 can be expressed as

$$-\omega Ae^{-ax}\sin(\xi) = Ae^{-ax}D[(a^2 - b^2)\cos(\xi) - 2ab\sin(\xi)] - \epsilon Ae^{-ax}\cos(\xi). \quad (6)$$

Dividing out A and the exponential term, and bringing everything to one side,

$$0 = \omega\sin(\xi) + D[(a^2 - b^2)\cos(\xi) - 2ab\sin(\xi)] - \epsilon\cos(\xi). \quad (7)$$

Separating out the sine and cosine terms,

$$0 = \cos(\xi)[D(a^2 - b^2) - \epsilon] + \sin(\xi)[\omega - 2Dab], \quad (8)$$

and because sine and cosine of ξ are orthogonal functions, it must be true that $[D(a^2 - b^2) - \epsilon] = 0$ and $\omega - 2Dab = 0$. From this come the equations

$$D = \frac{\omega}{2ab} \quad (9)$$

and

$$\epsilon = (a^2 - b^2)D. \quad (10)$$

Now, the temperature of the bar is monitored as a function of time at two different points, located at distances x_{close} (x_c) and x_{far} (x_f) away from the heated end of the rod. With these two points, it is possible to derive a simple formula used to find the thermal conductivity of the material.

Using Eq. 3, the ratio of amplitudes A_f and A_c found at the arbitrary points is

$$\frac{A_c}{A_f} = exp{-a(x_c - x_f)}, \quad (11)$$

with a a phase difference of

$$\varphi_c - \varphi_f = b(x_c - x_f). \quad (12)$$

Inverting these two relations, gives the parameters

$$a = \frac{Log[A_c/A_f]}{x_f - x_c} > 0, \quad (13)$$

and

$$b = \frac{\varphi_c - \varphi_f}{x_f - x_c} > 0. \quad (14)$$

Thus, it is possible to take the equation for thermal diffusivity, $D = \kappa/(s\rho)$ and rearrange it, subsequently bringing in the results of the last few equations to show that

$$\kappa = s\rho D = s\rho\frac{\omega}{2ab} = \frac{s\rho\omega}{2}\frac{\Delta x^2}{(\varphi_c - \varphi_f)\log[A_c/A_f]}. \quad (15)$$

Though Eq. 15 gives a method to find κ from a single cosine heating wave, this experiment heats the end of the rod through a square wave, which can be thought of as an infinite superposition of sinusoids [2]. Applying Fourier analysis to a time series of temperatures at point x, it is found that

$$T_x(t) - < T_x > = \sum_{n=1}^{\infty}(\alpha_n\cos(\omega_n t) + \beta_n\sin(\omega_n t)), \quad (16)$$

where $< T_x >$ is the expectation value for the temperature of the rod at point x at a steady state and the harmonic frequency $\omega_n = n\omega_1 = n2\pi/\tau$, where τ is the linear period of the heating function. As components of the Fourier transform, the α_n term denotes the real portion of the Fourier transform, and β_n is the imaginary portion.

To make this complex Fourier transform compatible with the previous derivation, it is consolidated into a real sum of cosines,

$$T_x(t) - < T_x > = \sum_{n=1}^{\infty}\gamma_n\cos(\omega_n t + \varphi_n), \quad (17)$$

where

$$\varphi_n = \arctan(\beta_n/\alpha_n), \quad (18)$$

and

$$\gamma_n = (\alpha_n^2 + \beta_n^2)^{1/2}. \quad (19)$$

Thus, the value of κ is given by the n$^{\text{th}}$ harmonic of the heating function's Fourier transform by using the angular frequency ω_n, phases φ_n, and amplitudes γ_n in Eq. 15.

III. PROCEDURE

For this experiment, conducted with the equipment shown in Fig. 1, heat was applied to the end of a brass rod, approximately 1 cm in diameter and 1 m in length, wrapped in insulating foam and bubble wrap in order to minimize heat loss along its length. To create the periodic heating, a square wave function with a long period of $\tau \approx 10^3$ s was created by the Tektronix function generator. Next, the Kepco power supply offset and amplified the signal such that it would switch between on and off every 500 seconds. This amplified signal then powered a thermofoil heater attached to a reservoir in contact with the end of the rod, alternating between heating and inactivity every half-period of the square wave.

With periodic pulses of heat moving down the rod, the temperature was measured by two thermistors placed within, located at distances x_c and x_f away from the end of the rod and $\Delta x = (x_c - x_f) = 15.1 \pm 0.1$ cm apart from each other. As the temperature of the brass rod changed, so did the resistance of each thermistor. In series with the thermistors were a reference resistor of 15 kΩ, used

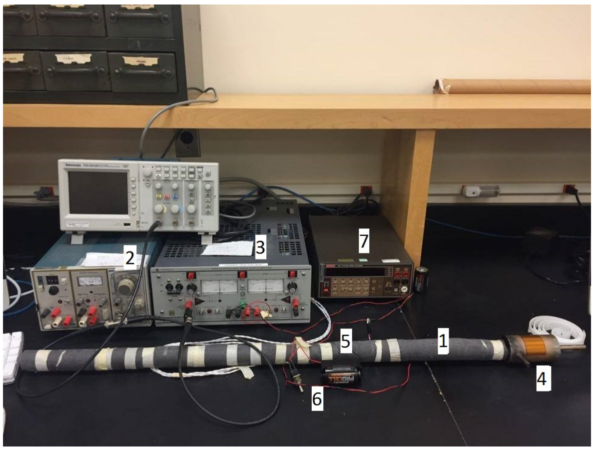

FIG. 1: Ångstrom experiment equipment. 1: Foam-wrapped brass rod. 2: Tektronix FG 501 function generator. 3: Kepco BOP 100 Power Supply. 4: Heater reservoir. 5: Two YSI 44004 Precision thermistors (embedded in rod). 6: 1.5 V dry cell battery and switch. 7: Keithley 199 scanner.

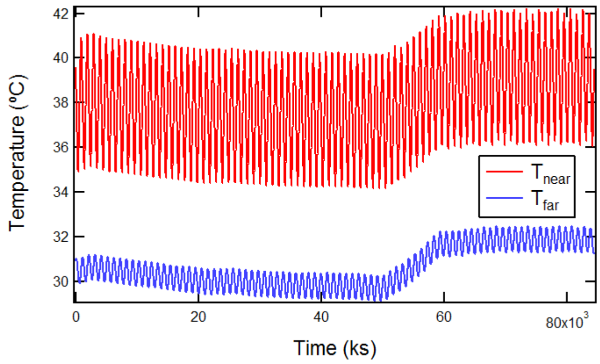

FIG. 2: A plot of the second experimental run, over the course of about 24 hours. The blue shows the data for T_f, cooler and less strongly oscillating than the red, T_c. As a result of changing ambient temperatures, the average temperature is either falling, as on the left, rising, as in the middle, or unstable, but overall constant, as in the rightmost region.

to determine the current through the circuit, and a 1.5 V dry cell battery, used as a constant source of voltage. By activating the Keithley 199 scanner, connected to each resistor via separate channels, and flipping the switch of the circuit containing the resistors to on, thus allowing current to flow, the voltage across each thermistor was measured.

Once the experiment had begun, collection of these data was automated by a LabVIEW program running on a computer connected to the the Keithley scanner. As the waves of heat traveled across the rod, the LabVIEW program received the scanner's output of the voltage across each thermistor and converted the data into readings of temperature within the rod as a function of time. Once the experiment was completed, the data were saved as a text file, to be analyzed in Igor.

Two runs of data were collected, each approximately 24 hours long in order to allow the heated rod to reach and maintain a steady state. The second run of data was begun after preliminary analysis of the first showed inconsistency within the steady state.

IV. RESULTS AND ANALYSIS

Once data from the second run of the experiment were imported into Igor, the first step was to redimension the temperature waves, determining the time interval between each point. This was accomplished quickly by plotting the wave of time data against its index and finding the slope of the resulting linear fit, finding an interval of $\Delta t = 5.328$ seconds.

With this interval determined, the waves of temperature T_c measured close to the heater at point x_c and temperature T_f, measured further away at point x_f, were plotted as a function of time, as shown in Fig. 2. In the figure, T_c is shown in red and T_f in blue. Closer to the source of heat, the T_c waves have both a greater amplitude and a greater equilibrium temperature than the more distant T_f heat waves. Diffusion of heat within the rod not only caused the sharp waves visible at x_c to become more gradual and rounded by the time they reach x_f, but emission of heat into the air has decreased their amplitude, as well.

Though the second round of data collection began with the rod well into the heating process, the data still showed variability. After reflection upon the shape of the graph, and rough extrapolation of the time of day corresponding to the times on the plot, it is likely that the variance in equilibrium temperature to be a result of changes in the ambient temperature of the building, caused by its heating system. While bubble wrap and foam insulate the sides of the rod, the seam where the thermistors are inserted into the rod was partially open to

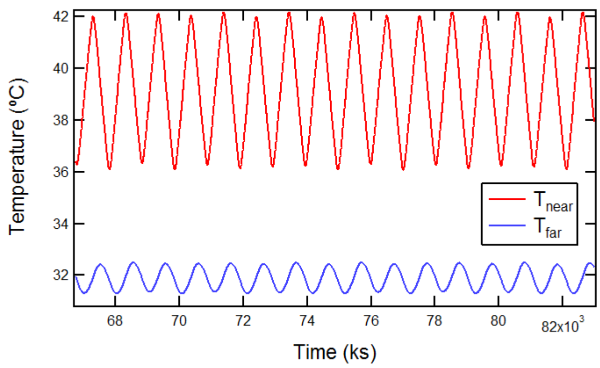

FIG. 3: A closer look at the region of data over which the Fourier Analysis was applied. The temperature close to the heated end is in red and the far temperature is in blue. There is an uncertainty of ±0.1 C in T_c and an uncertainty of ±0.05 C in T_f.

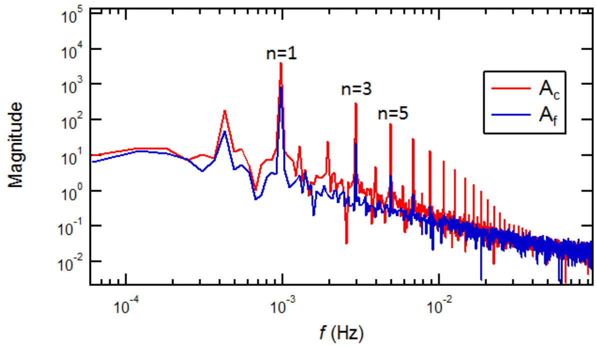

FIG. 4: The amplitudes of both Fourier transforms, graphed together. Manifesting peaks at the same frequencies, the blue γ_f peaks, further from the heat source, diminish much more quickly than the closer red γ_c, due to the dispersion of heat within the rod and emission of heat into the air. The peaks for the first, third, and fifth harmonics are shown, at 0.00098 Hz, 0.0029 Hz, and 0.0049 Hz, respectively.

the air, likely allowing external changes in temperature to noticeably alter the temperature of the rod. Looking at Fig. 2, there is a consistent slope of decrease over what would be the afternoon, corresponding to the building cooling while its heat is off. This trend is followed by a sharp rise in temperature, corresponding to a rise in temperature as the heaters turn back on. Finally, there is a region where $< T >$ remains relatively constant, but the amplitude of the heating wave fluctuates, probably a result of the heating system turning off and on to keep the ambient temperature of the building around an optimal level. This effect could likely be reduced by re-taping the thermistor portion of the rod to ensure better insulation against external effects.

For Fourier analysis of the data in Igor Pro, the temperatures were analyzed from the region near the end of data collection, chosen because it was the only portion of the data with a consistent value of $< T >$. In an attempt to minimize the effect of the inconsistent amplitudes of temperature in this region, a wide range of 15 periods, or about 3 hours, was analyzed, shown in Fig. 3. Boundaries were selected to ensure that the analyzed region began and ended on the same phase of the waves, so as to minimize error in Igor Pro's Fourier transform tool. This region was chosen as it was the only one with a relatively constant ambient temperature, and the large period of time was selected to minimize any error caused by the fluctuating temperature within. Measuring the difference between peak amplitudes for each wave, the uncertainty in T_c is ± 0.1 C, and the uncertainty in in T_f is ± 0.05 C.

With an input of 16 336 points, Igor Pro was able to provide a Fourier transform consisting of 1533 complex points for each temperature wave. Using Eqs. 18 and 19, the phase φ_n and amplitude γ_n of each Fourier transform were determined, as seen in Table I. For this analysis, the 1^{st}, 3^{rd}, and 5^{th} harmonics were selected as the three greatest contributors to the simple square wave of heat input. They were found by their peaks at f_n in Fig. 4.

TABLE I: Fourier Transform and Calculation Results

	1st Harmonic	**3rd Harmonic**	**5th Harmonic**
f_n	9.8×10^{-4} Hz	29.4×10^{-4} Hz	49.0×10^{-4} Hz
A_c	4012.3	312.16	79.17
A_f	869.38	22.17	2.81
A_c/A_f	4.719	14.08	28.21
φ_c	-0.0083 rad	0.064 rad	-0.171 rad
φ_f	1.4175 rad	2.554 rad	3.244 rad
$\varphi_f - \varphi_c$	1.426 rad	2.554 rad	3.244 rad
ω_n	0.0062 rad/s	0.0185 rad/s	0.0308 rad/s
s	380 J/(g K)	380 J/(g K)	380 J/(g K)
ρ	8550 kg/m^3	8550 kg/m^3	8550 kg/m^3
κ	103.01 W/(m K)	101.22 W/(m K)	105.20 W/(m K)

With the Fourier data acquired, κ may be calculated after determining the remaining constants in Eq. 15. To start, the values of the gap between thermistors, $\Delta x = 0.151 \pm 0.001$ m, and the angular frequency $\omega_n \equiv 2\pi f_n$ are already known. Because brass is an alloy, a mix of metals, its density is not always a specific value, as the density of brass covers a range of 8400 to 8700 kg/m^3 [3]. Thus, a median value of $\rho = 8550 \pm 150$ kg/m^3 is estimated for the purposes of calculations. For the specific heat of brass, an accepted value of s = 380 J/(g K) [4] was chosen. Inserting these known values, as well as the experimental results into Eq. 15, κ was calculated, as shown in Table I.

This lab's results for the thermal conductivity of brass, calculated three times through the first three harmonics of the square wave heating function, may be found in Table I. With a spread of only 4.98 between the largest and smallest values, the results are quite precise. From the three, a sample mean of $\bar{\kappa} = 103.1$ W/(m K) was attained, less than a tenth of a percent difference from this experiment's result from the 1^{st} harmonic, and a standard deviation of $\sigma_\kappa = 2.0$ W/(m K). Thus, In comparison to the accepted value for the thermal conductivity of brass, κ = 109 W/(m K) [5], the results for κ give an average error of 5%. With this low level of error, the thermal conductivity of brass has been accurately measured.

Though there are several already identified sources of uncertainty and error in this lab, their contribution to the overall uncertainty of the results is fairly minor. The accuracy would improve by finding the exact density of the brass rod, but only slightly, as the ± 150 kg/m^3 range of possible densities is less than a 2% fractional uncertainty with respect to the central estimate of $\rho = 8550$ kg/m^3. At its greatest, the fractional uncertainty of the frequency, limited by the resolution of the Fourier series, was only 3% at the first harmonic. What seems to be the most significant known source of uncertainty is the fluctuation in the amplitude of the heating waves, which still only gives a fractional uncertainty of about 4%, which is significant, but not overwhelming. Even so, it could be an interesting exercise to attempt to neutralize the effect of changing ambient temperature on experimental data by tracking those changes with a nearby open-air thermometer. Alternatively, the insulation around the rod could be reapplied, covering areas currently exposed in an attempt to reduce heat transfer between the rod and environment.

V. CONCLUSIONS

This experiment set out to investigate the propagation of heat through a rod of brass and measure κ, its thermal conductivity, by periodically heating one end and analyzing the temperatures recorded at two points along its length. First, a method was derived to find the value of κ by assuming a perfect sinusoidal heating function and measuring the temperature of the bar at two separate locations, x_c and x_f. Because the heating function was a square wave, a Fourier transform of the previous equation was employed to find three values for κ, corresponding to the contributions of the first three harmonics.

Heating one end of the brass bar with a square wave frequency of $f \approx 10^{-3}$ Hz, two waves of temperature within the bar were recorded over the course of a day. The location closer to the heated end showed higher average temperature and amplitude of oscillation. The first three harmonics of these waves' Fourier transforms were used to find the thermal conductivity of brass. These three results were combined to produce an average result of $\bar{\kappa} \pm \sigma_\kappa = 103.1 \pm 2.0$ W/(m K), a 5% difference from the accepted value of $\kappa = 109$ W/(m K).

Though this lab's results for the thermal conductivity of brass were fairly accurate, a variety of small but solvable errors affected the result, such as not knowing the actual density of the rod, or slopes and fluctuations in average temperatures likely caused by changes in the ambient temperature of the lab. As these issues are clearly defined, it would not be difficult to take steps to neutralize their effects upon the experiment and achieve yet better results.

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