Quantum Decoherence - The Emergence of the Classical World from The Quantum World

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(Dated: May 12, 2016)

This paper introduces the basic understanding of the quantum system, concepts of states and state operator. Applying the properties of state operator to the spin-1/2 particle, we were able to further examine the expectation values of the observable in the Stern-Gerlach experiment and the Bernstein experiment. The result shows a direct connection between the disappearance of the off-diagonal elements of the density matrix and the loss of the quantum information. This conclusion was further tested through the quantum particle exhibiting Brownian motion in the finite square potential well.

I. INTRODUCTION

Ever since the quantum theory was developed, the debate about the relation of quantum physics to the familiar physical world continued [1]. Though the border line between the quantum and the classical worlds still remains unclear to us, the previous study conducted by J. Gamble suggested that the transition between these two worlds is related to the loss of the quantum mechanical information from a quantum system to a classical environment [2].

Due to the correspondence between the state operator and the state of a quantum system, the physical change of a system from a state to another during such a transition will be shown through the change of its corresponding state operator. In our study, we will first derive the general form of state operator for a spin-1/2 particle. We will then use this result in the Bernstein thought experiment.

The Bernstein experiment is closely related to the Stern-Gerlach experiment, which measures the spin of a particle through its analyzer. In addition to the Stern-Gerlach apparatus, we will have a roulette wheel as a classical component to cause the leakage of quantum information. Comparing the expectation values and the state operators in the case of having a roulette wheel and the case of not having a roulette wheel, we will find that the off-diagonal elements of the density matrix indicate the emergence of the classical system.

Based on the conclusion we obtain from the Bernstein experiment, we will then apply to a different system that can rather show a gradual evolvement of a quantum system to a classical mixture. Hence, we will examine the state of a quantum particle in a finite square well that is immersed in the statistical gas.

Both the Bernstein experiment and the particle in a finite well experiment consist of a simple system component that is treated quantum mechanically, and a complex environmental component that is treated classically, both components interfering with each other. From such an interference, a classical world will emerge from a quantum mechanical one due to its loss of information, and indicated through the off-diagonal elements of its density matrix.

II. UNDERSTANDING THE QUANTUM WORLD

A. What is Hilbert space

To understand Hilbert space, we can compare it with a general vector space. A general vector space of dimension n consists of vectors $\vec{v} = (v_1, v_2, .., v_n)$ and the linear combinations of vectors $a\vec{v} + b\vec{u}$, where a and b are either real or complex numbers. The Dirac notation of a vector $\vec{a} = (a_1, a_2, ..., a_n)$ is denoted by ket $|\alpha\rangle$, and its conjugate transpose $\vec{a}^H = (a_1^*, a_2^*, .., a_n^*)$ is denoted by bra $\langle\alpha|$. The inner product of $\langle\alpha|$ and $|\alpha\rangle$ is defined as $\langle\alpha|\alpha\rangle = a_1^*a_1 + a_2^*a_2 + ... + a_n^*a_n$, where a_1^* is the complex conjugate of a_1. Elements of the vector in the equation above represent numbers are numbers while α is the name of the vector, which does not have any mathematical values [3]. A Hilbert space of dimension n is a general vector space that also contains the inner product. In this paper, we will be only dealing with Hilbert space with finite dimensions.

B. From Classical to Quantum

As R.B. Griffiths addressed in Consistent Quantum theory, a possible state of a classical particle in one dimension at time t can be represented by a point on the phase-space diagram. Such a point (x, p), where x is the location and p is the momentum of the particle, provides a complete description of the state of the particle. Two classical states have exactly the same physical meaning if and only if they can be represented by two overlaid points on the phase space diagram. Similarly, a complete description of the state of a quantum particle at time t is given by its wave function $\psi(x)$. Though for a particular value of $x = x_0$, there will be a corresponding value of $\psi(x_0)$, it does provide any physical meaning for the particle at location $x = x_0$. Instead, its physical meaning is provided by the integration of its absolute

value from $x = a$ to $x = b$ $\int_a^b |\psi(x)|^2 dx$, which indicates the probability of finding the particle between locations a and b [4]. The variables, such as x (position), p (momentum) and s (spin), that can, in principle, be measured are considered as observables [6]. The observables are represented by Hermitian operators that belong to Hilbert space. A quantum state can then be understood as a set of the probability distribution for all the observables [6]. Wave functions represent a possible state of a quantum particle, except for the state $\psi(x)$ where every value of x is zero. Since wave functions belong to Hilbert space, a linear combination of two wave functions $\psi(x)$ and φx in Hilbert space can be written as

$$\omega(x) = \alpha\psi(x) + \beta\varphi(x), \qquad (1)$$

where α and β are two complex numbers [3] that belong to Hilbert space, and $\omega(x)$ is a distinct wave function in Hilbert space. In general, wave function $\psi(x)$ expressed in Dirac notion is denoted as $|\psi\rangle$ and its conjugate $\psi(x)^*$ is denoted as $\langle\psi|$. Hence, we an express Eq. 1 in Dirac notation as

$$\langle x|\omega\rangle = \alpha \langle x|\psi\rangle + \beta \langle x|\varphi\rangle. \qquad (2)$$

C.   The State Operator

For each quantum state, there corresponds a unique state operator [5]. Compared with state vector, a state operator has its advantage of being able to represent a system in both pure state and mixed state. A quantum system is in a pure state when we have a complete knowledge of the state this system is in [5]. For example, for a spin-1/2 particle, we know exactly it is in a state $|\psi\rangle$ as a superposition of basis state $|\uparrow\rangle$ and $|\downarrow\rangle$. In this case, the state operator of this system is given by

$$\hat{\rho} = |\psi\rangle \langle\psi|. \qquad (3)$$

A mixed state can be seen as an ensemble of the pure states $\sum_{j=1}^{J} |\psi_j\rangle$ with p_j as the probability for the quantum system being in one of the state $|\psi_j\rangle$ [6]. A mixed state of a quantum system cannot be considered as the same kind of quantum state as a superposition. In fact, a mixed state only tells us that a system can be in one of the pure states with certain probability rather than being in an exact state as a superposition of these pure states. In another word, the mixed state is a statistical state that indicates our lack of knowledge of the system rather than an actual physical state os the system. Hence, a mixed state can also be seen as a state where we have an only particle or none of the knowledge of the system [5]. The definition of a state operator in a mixed state is given by

$$\hat{\rho} = \sum_{j=1}^{J} p_j |\psi_j\rangle \langle\psi_j|. \qquad (4)$$

Based on this information of the state operator, we will then derived the properties and the expression of the state operator of the spin-1/2 particle.

III.   THE BERNSTEIN EXPERIMENT

A.   The State Operator of a Spin-1/2 Particle

We will first consider the case for a silver atom with one unpaired electron that has an intrinsic property of spin-1/2. A spin-1/2 particle has two basis vectors: spin up along z-axis $|\uparrow\rangle$ corresponding to the particle with $s_z = -\hbar/2$, and spin down along z-axis $|\downarrow\rangle$ corresponding to the particle with $s_z = \hbar/2$. Since this particle has two possible states - "spin up" $|\uparrow\rangle$ and "spin down" $|\downarrow\rangle$, the state of the particle can be written as the superposition of two basis states as

$$|\psi\rangle = \alpha |\uparrow\rangle + \beta |\downarrow\rangle = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \qquad (5)$$

Since there are only two basis states for a spin-1/2 particle, we will be only dealing with two dimensional Hilbert space. Representing the actual physical state of the particle, $|\psi\rangle$ is normalized as

$$\langle\psi|\psi\rangle = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha\alpha^* + \beta\beta^* \equiv 1 \qquad (6)$$

From the definition of state operator, we can construct a two dimensional state operator

$$\hat{\rho} = |\psi\rangle \langle\psi| = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} = \begin{bmatrix} Hilbert\alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{bmatrix}. \qquad (7)$$

From Eq. 6, we have the trace of the state operator

$$Tr(\hat{\rho}) = \alpha\alpha^* + \beta\beta^* = 1. \qquad (8)$$

Moreover, if we construct the matrix form of the adjoint operator of the state operator

$$\hat{\rho}^\dagger = \begin{bmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{bmatrix}^\dagger = \begin{bmatrix} \alpha^*\alpha & \beta^*\alpha \\ \alpha^*\beta & \beta^*\beta \end{bmatrix}, \qquad (9)$$

and compare it with Eq. 8, we can conclude that

$$\hat{\rho}\dagger = \hat{\rho}. \qquad (10)$$

Eq. 10 proves that state operator $\hat{\rho}$ is a hermitian operator.

Applying Eq. 6 to the square of $\hat{\rho}$ shows

$$\hat{\rho}^2 = |\psi\rangle \langle\psi| |\psi\rangle \langle\psi| = |\psi\rangle \langle\psi| = \hat{\rho}, \qquad (11)$$

which indicates that $\hat{\rho}$ is a projection operator. In the following steps, we will apply the properties of two-dimensional state operator $\hat{\rho}$ above to the spin of the

spin-1/2 particle. As an observable, spin is represented by spin operator

$$\hat{S} = \begin{bmatrix} \hat{S}_x \\ \hat{S}_y \\ \hat{S}_z \end{bmatrix}, \qquad (12)$$

where $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ are component operators [11]. Component operators are given by D. J. Griffiths [4] as

$$\hat{S}_x = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \hat{S}_y = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \hat{S}_x = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (13)$$

.

Based on the form of Eq. 13, J.Gamble [7] suggested to use identity matrix and Pauli matrices to form a complete basis for an arbitrary operator $\hat{A}$ in two dimensional Hilbert space

$$\hat{A} = r_I \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{r_x}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{r_y}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \frac{r_z}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} 2r_I + r_z & r_x - ir_y \\ r_x + ir_y & 2r_I - r_z \end{bmatrix}, \qquad (14)$$

where r_I, r_x, r_y, and r_z are complex scalars.

Applying the result from Eq. 8 to the arbitrary state operator $\hat{A}$, we have

$$1 \equiv Tr(\hat{A})$$
$$= \frac{1}{2}(2r_I + r_z + 2r_I - r_z) \qquad (15)$$
$$r_I = \frac{1}{2}.$$

Applying Eq. 10 to the arbitrary state operator $\hat{A}$, we have

$$\hat{A} = \hat{A}^\dagger$$
$$\frac{1}{2} \begin{bmatrix} 2r_I + r_z & r_x - ir_y \\ r_x + ir_y & 2r_I - r_z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2r_I^* + r_z^* & r_x^* + ir_y \\ r_x^* - ir_y & 2r_I^* - r_z^* \end{bmatrix} \quad (16)$$

This equation gives us a set of equations

$$\begin{aligned} r_x &= r_x^*, \\ r_y &= r_y^*, \\ r_z &= r_z^*, \\ r_I &= r_I^* \end{aligned} \qquad (17)$$

Eq. 17 shows the equality between a number and its conjugate, which means that complex scalars r_x, r_y, r_z, and r_I are in fact real numbers.

If we compute $\hat{A}^2$ by using the results from Eq. 15 and Eq. 11, we have

$$r_x^2 + r_y^2 + r_z^2 = 1. \qquad (18)$$

Eq. 18 gives us a unit sphere in space formed by the unit vector $\hat{r} = r_x\hat{x} + r_y\hat{y} + r_z\hat{z}$, where $\hat{x}$, $\hat{y}$ and $\hat{x}$ are unit vector in x, y and z directions. Since we will be dealing with spherical coordinates when applying state operator in the following thought experiment, we will express $\hat{A}$ in terms of spherical coordinates based on Eq. 21. Expressing such a unit sphere with spherical coordinates $\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$, we have

$$\begin{aligned} r_x &= \sin\theta\cos\phi \\ r_y &= \sin\theta\sin\phi \\ r_z &= \cos\theta. \end{aligned} \qquad (19)$$

If we apply the values of r_x, r_y, and r_z to Eq. 14, we will obtain the expression of an arbitrary state operator in spherical coordinates

$$\hat{A} = \frac{1}{2} \begin{bmatrix} 1 + \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi - i\sin\theta\sin\phi & 1 - \cos\theta \end{bmatrix} \qquad (20)$$

This result from our derivation of the state operator of spin-1/2 particle matches J. Gamble's result in reference [8].

B. the Stern-Gerlach Experiment

In the Stern-Gerlach experiment, a beam of a spin-1/2 neutral silver atoms is sent through the Stern-Gerlach analyzer along the z-axis and projected onto the screen, as shown in FIg. 1. The Stern-Gerlach analyzer is composed of two pieces of nonidentical magnets that create a non-uniform magnetic field. The analyzer is oriented in a way such that the magnetic field gradient is in the y direction. If we send a series of classical bar magnets with different orientation within the x-y plane, the magnets will experience a magnetic force when passing through the magnetic field. Since the magnets are deflected before reaching the screen, they will create a pattern of the projected position shown as in FIg. 1.

Similar to the dipole magnetic field of classical magnets, a spin-1/2 neutral silver atom has a magnetic field due to the spin $\vec{S}$ of its unpaired electron, as shown in FIg. 2. The strength of its magnetic field, measured by magnetic moment $\vec{\mu_s}$ is proportional to the spin

$$\vec{\mu_s} = g\frac{q}{2m}\vec{S}, \qquad (21)$$

where g is the gyromagnetic ratio, q is the charge of the electron, and m is the mass of the electron. Since the magnetic field in the analyzer is not uniform, there will be a magnetic force exerted on the silver atom in y-direction with magnitude $\vec{\mu_s}$ is proportional to the spin

$$F_y = \frac{\partial(\vec{\mu} \cdot \vec{B})}{\partial y} = \frac{\partial(\mu_y\vec{B})}{\partial y}, \qquad (22)$$

where $\vec{B}$ is the magnetic field in the analyzer with magnitude of B, μ_y is the y component of magnetic moment. From both Eq. 21 and Eq. 22, we can conclude that the
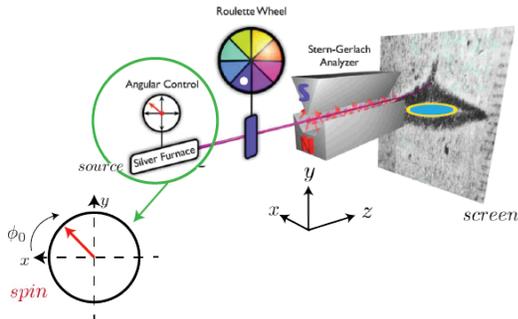
FIG. 1: The apparatus of the Bernstein experiment. The apparatus consists of a Stern-Gerlach analyzer, a roulette wheel, an angular control device and the screen. The purple line indicates the traveling silver beam. The red arrow indicates the initial spin of the silver atoms. The light blue area on the screen indicates the classical results. The light yellow area on the screen indicates the non-classical results. This figure is reproduced from the reference [7].
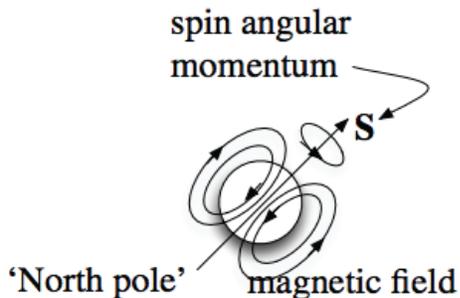

FIG. 2: The spin moment of an electron. The top arrow indicates the spin of the particle. The elliptical shaped arrows indicate the magnetic field of the particle. This figure is reproduced from the reference [8].

magnetic exerted on the silver atom is determined by the component of the spin in the magnetic field gradient direction, in our case, in the y-direction. In conclusion, since the deflection of the silver atom depends on its spin, the Stern-Gerlach analyzer can sort out the spin of the silver atom by showing the location of its landing.

C. the Bernstein Experiment

We now examine a thought experiment that is extended from the Stern-Gerlach experiment and originally revised by Herbert J. Bernstein [2]. There are two new elements added to the apparatus in the original experiment: an angular control device and a classical roulette wheel, as shown in FIg. 1. The angular control device is attached to the silver beam source to fix the initial spin of silver atoms at an arbitrary angle ϕ_0 within the XY plane. The classical roulette wheel is inserted be-

tween the angular control device and the analyzer and attached to another angular control device. Every time we spin the wheel, it will generate a number randomly chosen between 0 and 2π and assign this number to the connected angular control device. This angular control device will then add an angle ϕ_w to the initial angle of the passing silver atom. First, we will explore the case where the classical roulette wheel is not added to the apparatus. The initial condition given by the first angular control device allows us to modify the Eq. 19 at $\theta = \pi/2$ and $\phi = \phi_0$

$$
\begin{aligned}
r_x &= \cos \phi_0 \\
r_y &= \sin \phi_0 \\
r_z &= 0.
\end{aligned}
\tag{23}
$$

Applying these values of r_x, r_y and r_z to Eq. 14, we can construct the state operator for a silver atom

$$
\begin{aligned}
\hat{\rho}_a &= \frac{1}{2} \begin{bmatrix} 1 & \cos \phi_0 - i \sin \phi_0 \\ \cos \phi_0 + i \sin \phi_0 & 0 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 & e^{-i\phi_0} \\ e^{i\phi_0} & 1 \end{bmatrix}.
\end{aligned}
\tag{24}
$$

Firstly, we will construct an operator $\hat{O}$ associated with our Stern-Gerlach analyzer that has magnetic field oriented in y direction. This means that the analyzer will sort out the y component of spin. In addition, as addressed in the previous session, since the silver atom is in the superposition of two states ↑ and ↓, we can construct the state vector of a silver atom as

$$
\hat{S}_y = \frac{-i\hbar}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{i\hbar}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\tag{25}
$$

Since two spin states $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenfunctions of $\hat{O}$ with eigenvalues $-i\hbar/2$ and $i\hbar/2$ respectively, we will have

$$
\begin{aligned}
\hat{O} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \frac{-i\hbar}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\hat{O} \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \frac{i\hbar}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\end{aligned}
\tag{26}
$$

which gives us the Stern-Gerlach operator

$$
\hat{O} = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.
\tag{27}
$$

In order to obtain the expectation value of the observable $\hat{O}$, we use the formula provided by Ballentine that indicates the relation between a state operator $\hat{\rho}$ and an arbitrary observable $\hat{A}$

$$
Tr(\hat{\rho}\hat{A}) = \langle \hat{A} \rangle.
\tag{28}
$$

Applying Eq. 28 to the operator $\hat{O}$ and the state operator

$\hat{\rho_a}$, we obtain the expectation value

$$\langle \hat{O} \rangle = Tr(\hat{\rho_a}\hat{O})$$
$$= \frac{\hbar}{4}Tr\left(\begin{bmatrix} 1 & e^{-i\phi_0} \\ e^{i\phi_0} & 1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \right) \quad (29)$$
$$= \frac{\hbar}{2}\sin\phi_0.$$

Secondly, we will explore the situation when the roulette wheel plays a role in the experiment. In this case, the total spin angle of the silver atom is $\phi = \phi_0 + \phi_w$. Based on Eq. 28, we know that in order to obtain the new expectation value of the observable $\hat{O}_w$, it requires us to derive a new state operator $\hat{\rho_w}$. From the definition of the expectation value, we know for a function $f[\phi]$, its expectation value is

$$\langle f[\phi] \rangle = \frac{1}{2\pi}\int_0^{2\pi} f[\phi]d\phi, \quad (30)$$

where $1/2\pi$ is the result of the integration of the probability density. Followed from this equation, we can also conclude that

$$\langle g[x]f[\phi] \rangle$$
$$= \frac{1}{2\pi}\int_{-0}^{2\pi} g[x]f[\phi]d\phi$$
$$= \frac{1}{2\pi}g[x]\int_{-0}^{2\pi} f[\phi]d\phi \quad (31)$$
$$= g\langle f[\phi] \rangle.$$

where $g[x]$ is a function that does not depend on ϕ. Eq. 27 shows that operator $\hat{O}$ does not depend on ϕ while state operator $\hat{\rho}$ does. Hence, as J. Gamble pointed out, to average $\hat{O}$ is to average $\hat{\rho}$

$$\hat{\rho_w}$$
$$= \langle \hat{\rho_a} \rangle$$
$$= \frac{1}{2\pi}\int_{-0}^{2\pi} \hat{\rho_a}d\phi. \quad (32)$$
$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Using this result of $\hat{\rho_w}$, we compute the expectation value of $\hat{O}$ with the roulette wheel

$$\langle \hat{O} \rangle_w = Tr(\hat{\rho_w}\hat{O})$$
$$= \frac{\hbar}{4}Tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \right) \quad (33)$$
$$= 0.$$

Comparing the expression of the state operators from Eq. 24 and Eq. 36, we can see after including the roulette wheel the off-diagonal elements of the matrix become zero. Furthermore, the expectation value in Eq. 37 shows
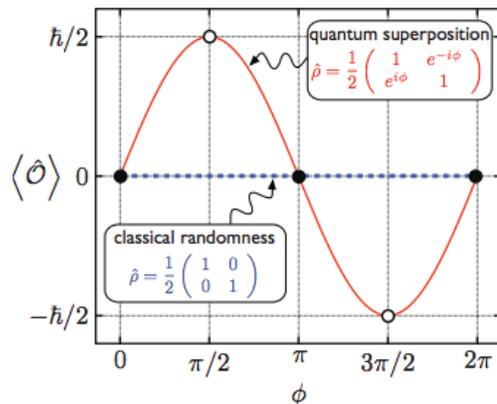

FIG. 3: The expectation value of the state operator $\hat{O}$. The dotted line indicates the expectation value $\langle \hat{O} \rangle_w$ when including the roulette wheel. The red solid line indicates the expectation value $\langle \hat{O} \rangle$ when not including the roulette wheel. The initial angle ϕ_0 varies from 0 to 2π. This figure is reproduced from the reference [7].

that it no longer depends on the initial angle ϕ_0 and yields a constant value of zero, as shown in FIg. 3.

This indicates that no matter what angle the silver atom was initially assigned, half of the measurements we obtained will always be spin up and the other half spin down, even if the spin is not in the same direction as the magnetic field gradient. Due to the classical randomness, 50% of getting the $\hbar/2$ eventually averages out the 50% of getting $-\hbar/2$. On the other hand, the expectation value in Eq. 29 indicates its dependency of the initial angle ϕ_0, as shown in FIg. 3. Moreover, it shows that the particle is indeed in a superposition of two states: one at initial condition of $\phi_0 = \pi/2$, where we only measured $\hbar/2$ for a spin up, and another at $\phi_0 = 3\pi/2$, where we only measured $-\hbar/2$ for the spin down. When $\phi_0 = 0$, π and 2π, the measurements are the same as the case including the wheel, since the spin is polarized in the x-direction that is perpendicular to the magnetic field gradient. Based on such comparison, we can conclude that the interaction between the classical roulette wheel and the original apparatus causes the loss of quantum information. This is implied by the disappearance of the off-diagonal elements in their state operator.

IV. BROWNIAN MOTION OF A PARTICLE IN A FINITE SQUARE WELL

In order to emphasize the connection between the disappearance of the off-diagonal elements of the density matrix and the loss of quantum information, we generated a model based on a physical system in thermal equilibrium with a statistical gas [7].

Considering a quantum particle (a system) with mass m_s surrounded by environment particles with mass m_e,
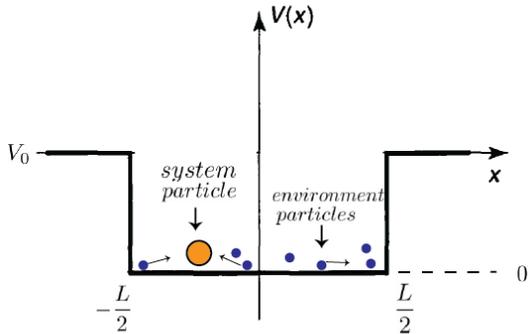
FIG. 4: The Brownian motion of a quantum particle in a finite square well. The orange circle indicates the system particle in the well. The blue circles indicate the moving environment particles with the arrows indicating their moving direction. The potential well is reproduced from the reference [4].

as shown in FIg. 4, the quantum particle continuously experiences collisions with the environment particles. The motion of the system in such an environment is called Brownian motion. The mass of the system is significantly larger than the mass of the environment particles $m_e/m_s \ll 1$. The environment particles, in this case, acts similarly to the classical roulette wheel in the last session. By interacting with the quantum systems, they both cause the leakage of the quantum information and turn it into a classical mixture over time.

The equation of motion for such a system is derived by J.Gamble as

$$\frac{\partial}{\partial \rho_s} t = -2(x-y)^2 \frac{\Gamma m_e kT}{\hbar^2} \rho_s, \qquad (34)$$

where Γ is the frequency of the collision between the environment particles and the system, T is the temperature of the environment, and k is the Boltzmann's constant. We apply such a result to the case of a system in finite square well. As shown in FIg. 4, a particle is placed in a finite square potential well

$$V(x) = \begin{cases} 0, & \text{for } -L/2 \leqslant \text{x} \geqslant L/2, \\ V_0, & \text{for } |\text{x}| > L/2, \end{cases}$$

where V_0 is a positive constant, L is the width of the well, and $V(x)$ is the potential as a function of position x. By solving the Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V\psi = E\psi, \qquad (35)$$

we have the solution as a piecewise function

$$\psi(x) = \begin{cases} g\ e^{(cx)}, & \text{for } |\text{x}| < -L/2 \\ a\sin dx + b\cos dx & \text{for } -L/2 \leqslant \text{x} \geqslant L/2 \\ h\ e^{(-cx)}, & \text{for } |\text{x}| > L/2 \end{cases}$$

where $c = \sqrt{2m_s(V-E)}/2$, $d = \sqrt{2m_s E}/2$, E is the energy of the system particle, and g, a, b, h are constants. We applied the continuity and smoothness of the wave function at $x = -L/2$ and $x = -L/2$ as boundary conditions in order to solve the coefficients numerically. Using *Mathematica*, we obtained the eigenvalues and eigenfunctions for several allowed energy levels.

In order to obtain the state operator of such a system, we use the formula [7]

$$\rho_s(x, y, 0) = \psi(x)\psi^*(y) \qquad (36)$$

to obtain the (x, y) element of density matrix. We then applied this result to the solution of Eq. 34

$$\rho_s(x, y, t) = \rho_s(x, y, 0) e^{-(2\Gamma m_e kT/\hbar^2)(x-y)^2 t}, \qquad (37)$$

for the element of density matrix at time t.

The evolvement of the state operator (in density matrix form) of system particle at the third excited state is shown in Fig. 5. From this figure, we can see the disappearance of the off-diagonal elements of the state operator as the collision continues over time in a finite square well.

V. CONCLUSION

Though quantum system and classical system have some fundamental differences. The leakage of quantum information from a quantum system to the environment (a classical system) can eventually lead to the transition from one to another. Through introducing the ideas of Hilbert space, the state of quantum system, superposition, and the state operator, we were able to examine some of the simple quantum systems, such as the Stern-Gerlach experiment and the infinite square well, and conclude a connection between the disappearance of off-diagonal element of the density matrix and the emergence of the classical system. Both the thought experiments and the results from the model support the conclusion.

[1] W. H. Zurek, *Los Alamos Science*, **27**(2002).

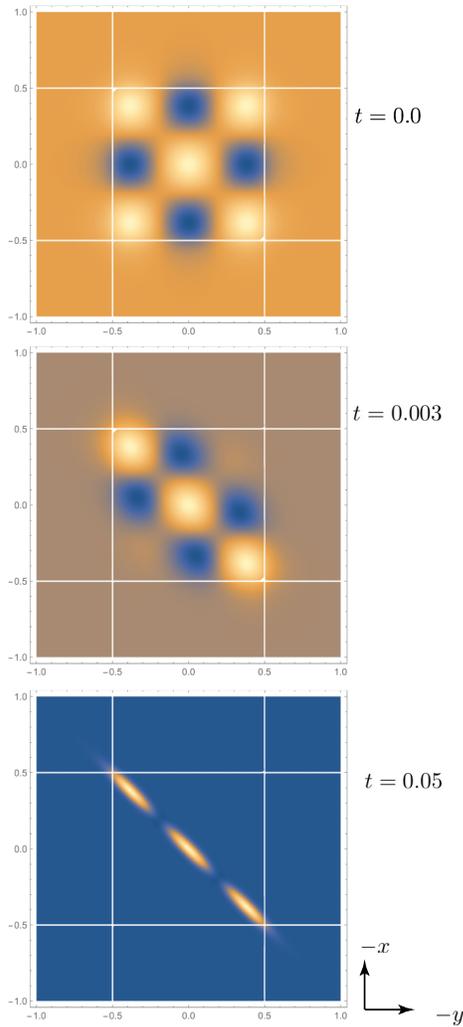[2] J. K. Gamble, *Foundations of Quantum Decoherence,*

FIG. 5: The matrix form of the state operator of a particle at the third excited state in finite square well. The top, middle and bottom graph present the state operator ρ_s at time $t = 0$, $t = 0.003$ and $t = 0.05$. Each bright yellow region region indicates an element of the density matrix with positive value. Each bright yellow dark blue region indicates an element of the density matrix with negative value.

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