Chaotic Analysis of a Mechanical Oscillator With Bistable Potential by Poincare Mapping and Phase Space Analysis

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The purpose of this lab was to determine the effect that various damping and periodic driving parameters have on a Duffing mechanical oscillator with two different potential wells. This was done by experimenting with variations in the system and deriving the equation of motion for the system, $\ddot{x} + \gamma\dot{x} = \alpha x - \beta x^3 + a\sin(\omega t)$, which describes the general motion of the system. The equation of motion includes definition of the initial parameters of the system, making it possible to make reasonable predictions about the motion of the system over time. Conclusively, two stable fixed points were identified and one unstable fixed point was identified. At these points the system is able to stay at rest when under particular initial conditions and without the influence of an exterior force actively driving the system. I discovered that when the amplitude of this specific system is increased from 4.84 V to 5.27 V the range of velocities and positions also increased for the system. Also, increasing the damping parameter of the system from 2.35 cm to 1.30 cm caused the system to reach a non chaotic limit. The initial parameters of the system caused large variations in motion that included changing in velocity around the equilibrium points and changing in the position limits.

I. INTRODUCTION

Chaos occurs in a mechanical system when the motion, energy, and force that a system is subject to are constantly fluctuating, causing non-repetitive behavior over a long period of time. Even in chaotic systems it is very possible to predict the motion of the system for finite periods of time given the initial parameters and conditions of the system. However reasonable prediction of motion, energy, and force within the system becomes exponentially harder to predict as the limits of time increase to infinity. Henri Poincare was one of the first scientists to explain chaos around 1900. He postulated that the natural curvature of orbits could be described geometrically using the idea of constantly evolving state variables; state variables being periodically changing variables. Poincare was able to give further intuition into the matter by showing that reasonable approximations of these variables could be investigated at a reoccurring position [2]. At a successive point, a reasonable prediction of condition values can be made using the equation of motion to calculate from a previously known value. Other than its application to orbitals, chaotic motion can describe other nonlinear phenomenon applicable to natural systems. German electrical engineer Georg Duffing constructed one of the most widely used nonlinear motion equation in 1918, the Duffing Equation [2]. His equation takes into account the effects upon a simple harmonic oscillator when manipulated by an periodic driving force, a continuous damping or drag force, and a nonlinear term, as well as initial parameters associated to these terms. Depending on the given conditions the system may act chaotic over discrete intervals and periodic over other intervals. In 1963, Edward Lorenz noticed that when he simulated a weather pattern on separate occasions using the same model he obtained vastly different results [4]. He concluded that his initial conditions in each instance were slightly different. This idea, known as the butterfly effect, says that the behavior of a chaotic system, where values are obtained with a certain systematic randomness, changes substantially even with slight changes in initial conditions. It is important to analyze chaos in order to find patterns within a systematically random set of conditional variables that allow us to weed out and predict particular variable values as time processes. The predictions that we are able to make are useful in predicting expectations for economic patterns, weather patterns, and mechanical systems such as the oscillator studied in this experiment.

II. THEORY

It is necessary to derive the equation of motion of the system from the kinetic and potential energy equations in order to consider the the motion of the bistable mechanical oscillator and its dependence on the initial conditions of the system. Just like in any other mechanical system, finding the Lagrangian by taking the difference between the kinetic and potential energies allows us to solve for the second-order Euler-Lagrange equation to obtain Duffing's historic equation for the damped, driven nonlinear oscillator [3]. Following this logic we get

$$L = T - V \; ; \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}, \tag{1}$$

which, for this particular set of variables and initial conditions, implies

$$\ddot{x} + \gamma\dot{x} = \alpha x - \beta x^3 + a\sin(\omega t), \tag{2}$$

where x is angular displacement, α is a linear elastic constant related to the spring constants of the oscillator, β is a non-linear elastic constant related to the force the springs exert as they stretch, a is the amplitude of the force dependent on the length of the mechanical arm (see

Figure 1, label 10), ω is the frequency of the force given by 2π of the period T that it take the arm to make one rotation, and finally γ is the damping coefficient proportional to the velocity of the system. In this particular experiment I manipulated a, ω, and γ to change the behavior of the system [1]. Here we can also say that if there was no damping parameter γ, our equation of motion would look like

$$\ddot{x} = \alpha x - \beta x^3 + a \sin(\omega t), \tag{3}$$

and the natural frequency of the system would we of the form

$$\omega_o = \sqrt{\frac{\beta x^2 + \alpha}{m}} \tag{4}$$

where the frequency of the system is going to be chaotic even in the absence of damping since β is affected by the position of the non inform mass of the disc and and randomized position of x. Under particular conditions the system can behave predictably or chaotically. Chaotic behavior displays that even slight manipulation of the initial conditions can lead to drastically different results. When the system is underdamped the system will satisfy ω_o¿γ, where we will see the system act chaotically, as in most cases explored in this experiment suggest. When the system is overdamped the system will satisfy ω_o=γ, where we will see the system reach a limit of motion as time progresses. We can see this sort of motion in Figure 2 in the middle graph. When the system is critically damped the system will satisfy $\omega_o < \gamma$, where we will see the damping parameter outweigh the elasticity of the system and the driving force will be too strong [2]. This final case was not explored to its fullest extent, when the motion of the system flatlines to fixed point over a small period of time, even at large initial conditions of potential and velocity. We do see semblance of this in Figure 2 in the bottom graph since the motion is restricted over time.

Initially looking at the position versus time of the nonlinear oscillator helps to understand the motion of the system. Looking at Figure 2 we can see that under certain parameters and time constraints the system can act periodically, chaotically, or some combination of both. In certain instances we see period doubling, where a motion is repeated upon consecutive intervals, where the oscillator only moves close to one fixed point.

Next, to understand the energy related to the system we can look at the potential energy curve as displayed in Figure 3. The general form of the potential equation is

$$V = -\alpha \frac{x^2}{2} + \beta \frac{x^4}{4}, \tag{5}$$

where V is a function of position x and α and β are parameters.

We can study the behavior of chaotic motion under two separate conditions, over a discrete intervals of time or over a continuous period of time. Over a continuous interval we talk about the phase space of the system where we relate velocity and position over a related time interval. As shown in Figure 3 we observe an eight-shaped curve, called a separatrix, representing the phase space relationship between velocity and position. Each phase begins and ends at a similar position on the graph and continually revolves. The phases create a concentric pattern and no successive phase is identical. If we changed the sampling of the phases in Figure 3 we would see different spacing and behaviors between phases. Each phase gives us the range of velocities and positions over continuous, but periodic intervals of time. We can easily see where the fixed points of the system are located. At the

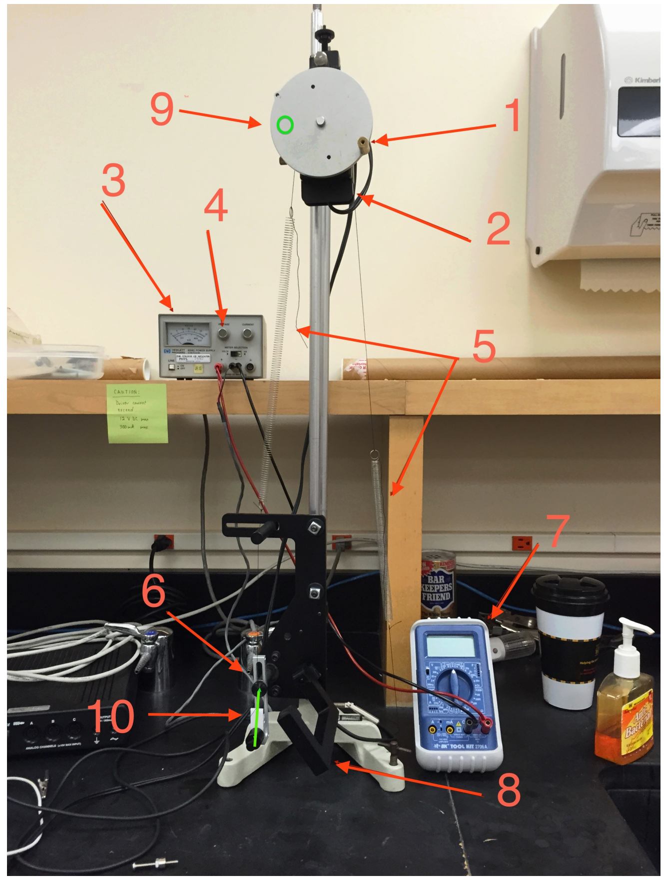

FIG. 1: Displayed is a comprehensive diagram of the bistable mechanical oscillator used in this experiment. Each key component of the system is annotated by a number. The mass attached to the metal wheel (1) is free to rotate about a Pasco SmartPulley (2). The springs (5) that manipulate the motion of the system are attached to a string that can rotate the pulley in two directions. Next, there is a power supply (3) with a variable voltage (4) that can be measured by the voltmeter (7). The Pasco Mechanical Oscillator (6) is driven by the power supply and has a variable amplitude (10). The mechanical arm (10) of the Oscillator passes through the Pasco Photogate (8), in order to measure the frequency. Finally, there is a damping magnet (9) behind the metal disc that can move closer to the disc to increase the damping effect.

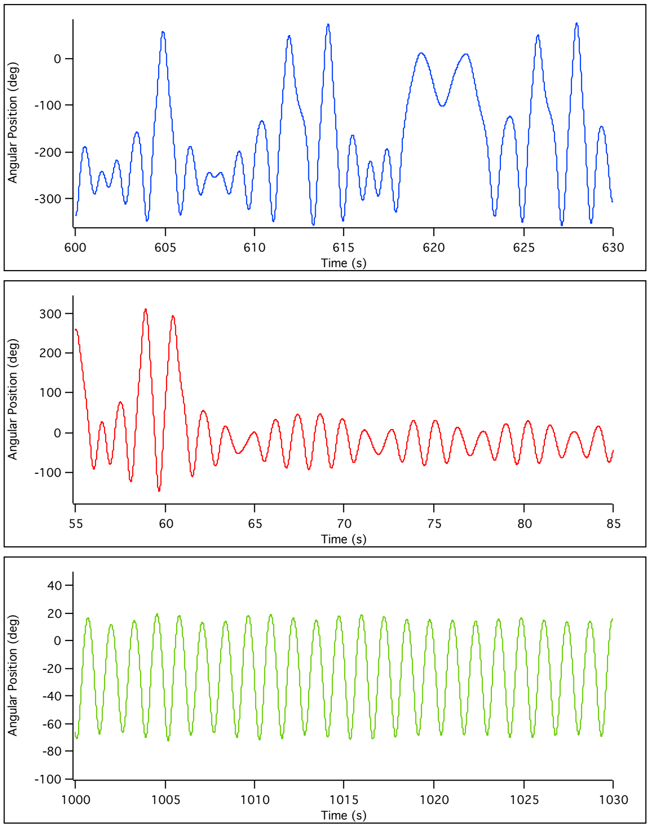

FIG. 2: The graphs depicted are position vs. time plots for different initial parameters of four separate data runs. The parameters being manipulated correspond to the driving frequency, driving amplitude, and damping position. The top plot (blue) has initial parameter of 3.40 cm amplitude, 5.27V frequency, and 1.30 cm damping position. The middle plot (red) has parameters of 4.15 cm amplitude, 5.27V frequency, and 2.35 cm damping position. Finally, the bottom plot (green) has parameters of 4.15 cm Amplitude, 4.84V frequency, and 1.30 cm damping position.

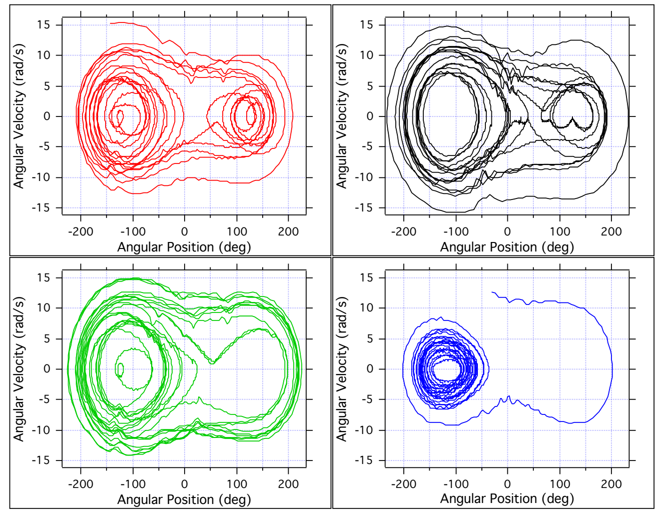

FIG. 3: The graphs depicted show phase space plots of the different initial values for four separate runs, corresponding to the driving frequency, driving amplitude, and damping position. The top left (red) has parameters of 4.15 cm amplitude, 5.27 V frequency, and 2.35 cm damping position. The top right (black) has parameters of 4.15 cm amplitude, 5.27 V frequency, and 1.30 cm damping position. The bottom right plot(green) has parameters of 4.15 cm amplitude, 4.84 V frequency, and 2.35 cm damping position. Lastly, the bottom right (blue) has initial parameter of 4.15 cm amplitude, 4.84 V frequency, and 1.30 cm damping position. The phase was plotted for each run for identical time intervals of 40 seconds. Each behavior has slightly different patterns due to the intervals of time chosen and the starting position of the oscillator. The most different plot was the blue plot which moved in a non chaotic fashion around a single fixed point within the system.

stable fixed points the velocity is converging to zero and the phase lines are moving towards a particular position. These are shown by the holes of the eights. The unstable fixed point is shown where the phase lines are deflecting away from a particular position near the center of the eight. At the unstable point the potential energy reaches a local maximum. Under particular initial conditions it is possible for the oscillator to rest at the unstable fixed point. For certain initial conditions the oscillator can also move about a single stable fixed point with a limited angular position.

The other useful type of analysis for a phase space plot is a Poincare map, over a discrete and periodic interval. The idea of a Poincare plot is that particular trends for velocity and position tend to occur over a periodic interval. If we think about the phase space as constructed concentrically outward about a center phase, you can create a cross sectional sample by cutting a line perpendicularly through all the phases at a particular junction. We can see this concept by observing the colored dotted lines intersecting through the phase spaces in Figure 3. However, it would be possible to receive a slightly different Poincare map pattern by choosing a different position of the driving force at which to record the velocity verses the position, as shown by the grey intersection in Figure 3. In this instance their would be different fractal patterns, within the Poincare map that would limit particular velocity and position combinations. Fractal patterns are continuously repeating patterns that rotate about the fixed points. Shifting the position at which the velocities and positions are recorded will rotate the fractal patterns toward the fixed points but will keep similar shape under the same initial conditions. By changing the initial conditions the fractal patterns can rotate, distort and expand, however the fixed points will still remain the most dense regions of the map. In these maps each successive point depends on the previous point, creating a type of chain reaction that can distinctively change the behavior of these maps.

III. PROCEDURE

This lab was conducted using DataStudio for data collection. The equipment used is displayed in Figure 1 with detail. In the bistable oscillator, there is a disc of nonuniform weight suspended above a flat surface and attached to a SmartPulley. The pulley and the disc are able to radially move by the way of a configuration of two strings. Both strings are attached to two springs; one is stationary and the other is attached to a mechanical oscillator. The mechanical oscillator rotates, changing the driving force upon the pulley system.The Pasco SmartPulley and the Pasco Photogate are connected to the Pasco Interface box that transmits data to the DataStudio software for Macintosh. The software records several different measurements to create the Poincare, the phase space, the position vs. time, and the potential energy diagrams. Variations of initial parameters were established for five separate runs. The changing parameters included the amplitude of the driving force (3.40 cm and 4.15 cm), period of the driving force (5.27 V and 4.84 V), and position of the damping magnet (2.35 cm and .30 cm). These parameters are summarized in Table 1.

TABLE I: This is a summary of the data runs performed in this experiment. Five data runs were collected using various parameters, as specified.

Run	Drv. Amp. (cm)	Freq. (V)	Dmp. Pos. (cm)
1	4.15	4.84	2.35
2	4.15	4.84	1.30
3	4.15	5.27	2.35
4	4.15	5.27	1.30
5	3.40	5.27	1.30

Data was collected in 1:30hr increments. In order to create the position verses time diagram the interface simply tracked the angular position of the smart pulley, at 0.01s per measurement. The phase space plot gave the angular velocity verses the angular position and was measured over a continuous interval. In contrast, the Poincare map plotted the angular velocity verses the angular position at the discrete interval of time when the mechanical oscillator crossed the path of the Photogate. Finally, the potential energy curve plotted the potential energy of the system as the angular position of the oscillator changed. All the data collected was exported to Igor Pro for analysis and manipulation. All figures were constructed using Igor Pro, excluding Figure 1 and Table 1.

IV. DATA AND ANALYSIS

Figure 2 gives the angular positions verses time graphs. The last graph shows us an example of, not chaotic, but periodic motion when the oscillator is constrained to a small interval with a repeating pattern of motion. This is the simplest case where position over time is predictable. The first graph displayed has a larger frequency and a smaller amplitude with a damping magnet. We see that the system oscillates around the two fixed points with occasional oscillations between the fixed points and frequent consecutive oscillations around a single fixed point. The second graph shows a transition between chaotic and periodic motion. This occurs because the damping driving force drives the system when it is at various positions and directions for the first 60s, but it eventually reached a predictable pattern when the motion of the oscillator was in a small interval.

The next set of plots, Figure 3, shows the phase spaces of four different parameter sets. The red and black phase spaces have different damping parameters as well as the green and blue phase spaces. The black and blue have different driving frequencies as well as the red and the green. First, we see as the damping parameter increases there is more amplitude of motion around the fixed points shown from red to black. In other words the oscillations become larger since the damping parameter forces the system to take on a smaller range of velocities and positions. Between the green and the blue we see that as the damping increases the motion of the system tends to center around a single fixed point with a smaller range of velocities and positions. This same occurrence happens between the black and the blue when the frequency is decreased. It is observed that a particular fixed point, represented by the larger circular pattern on the left, created a larger range of position and velocity, with the exception of the bottom right phase plot, which had a discrete range of motion, considering it was non chaotic.

In Figure 4 we see the Poincare maps for 4 separate parameter sets. The black and the blue plots now differ in the amplitude of the driving force. We see that the range of velocities and positions expands slightly as the amplitude increased. There also seems to be a shift in the fractal patterns as the amplitude increased. The difference shows more empty spaces where the velocity and the position are not expected to be, within the range of values for the blue map. In sum the green map covers a wider range of values then the blue, which has lesser amplitude. The red and the black differ in damping parameters. As the damping increases we see that there is more empty space near the fixed points even though the patterns seem to be very similar in shape. This was a similar observation in the phase plot for a similar change in damping.

To give a sense for how the Poincare plots are developed over a period of time refer to Figure 5, showing the progression of the point values for a particular parameter set. Figure 6 shows the Poincare plots for a system with damping and without damping. This shows that the bottom right plot, that includes damping, reaches a more discrete range of values for position and velocity, based on the complexity of the fractal patterns and the smaller dispersion of points. The green plot covers a wide range

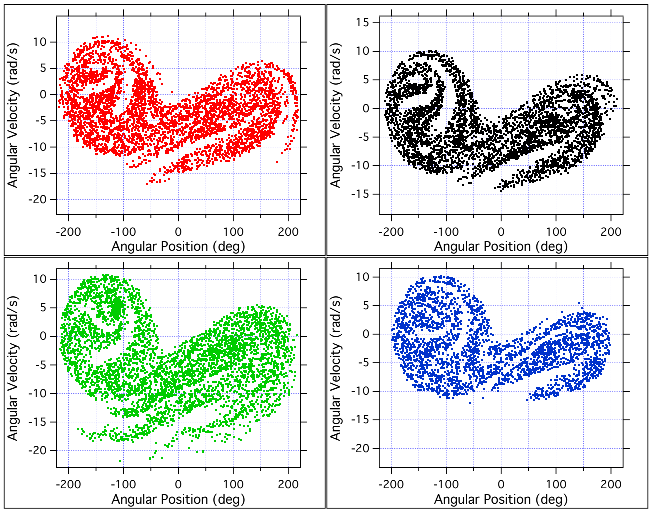

FIG. 4: The graphs depicted show Poincare plots for the different initial values of four separate data runs, corresponding to the driving frequency, driving amplitude, and damping position. The top left (red) has parameters of 4.15 cm amplitude, 5.27 V frequency, and 2.35 cm damping position. The top right (black) has parameters of 4.15 cm amplitude, 5.27 V frequency, and 1.30 cm damping position. The bottom right plot(green) has parameters of 4.15 cm Amplitude, 4.84 V frequency, and 2.35 cm damping position. Lastly, the bottom right (blue) has initial parameter of 3.40 cm amplitude, 5.27 V frequency, and 1.30 cm damping position. The fractal patterns tend to move around the known stable fixed points and have slight distortion under changing parameters. Each plot contains 7 500 data points which varies in time scale for the two different frequency values.

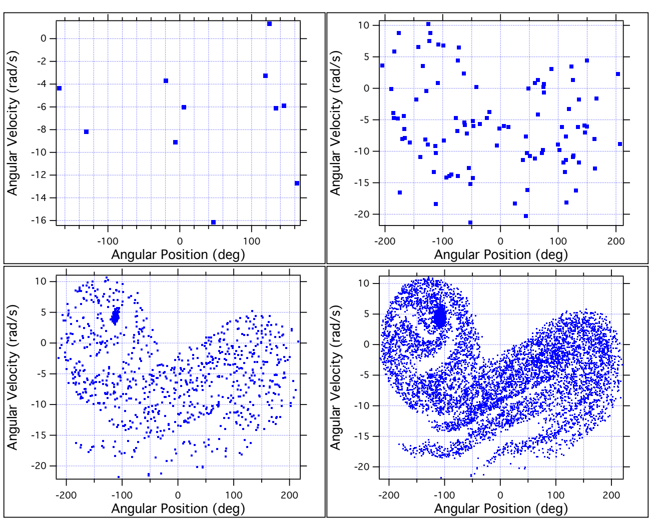

FIG. 5: This shows the succession of the Poincare map at 10 points, 100 points, 1000 points, and 10 000 points. The parameters for this data set are; 4.15 cm Amplitude, 4.84 V frequency, 2.35 cm damping position

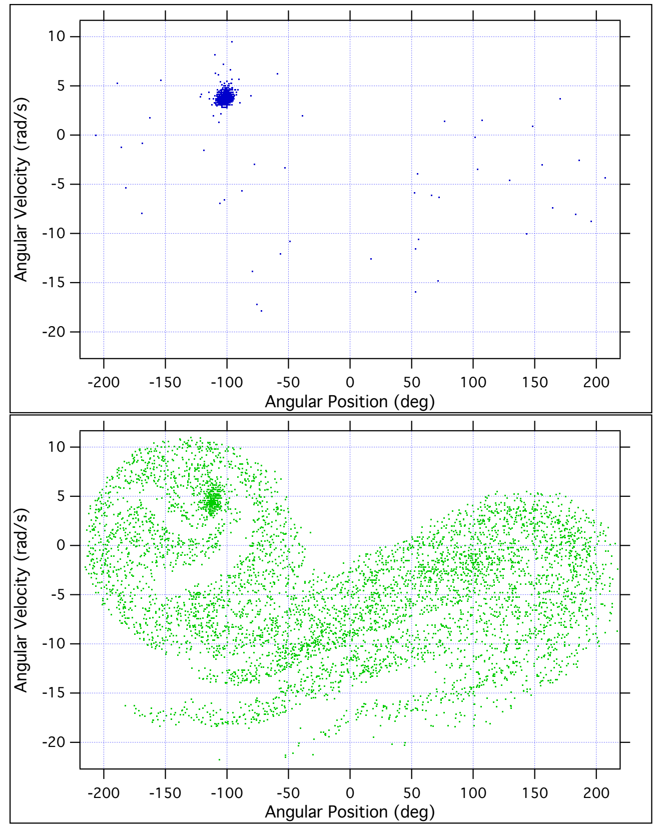

FIG. 6: The graphs depicted show Poincare plots of two different initial values for damping position.The bottom plot (green) has parameters of 4.15 cm amplitude, 4.84 V frequency, and 2.35 cm damping position. The top plot (blue) has initial parameter of 4.15 cm amplitude, 4.84 V frequency, and 1.30 cm damping position. The bottom was plotted over 7 500 data points while the top was plotted over only 1000 data points because it had a very small range of velocity and position values as time passed.

of values but we can see that the most dense populations occur at stable fixed points with a probable velocity at the fixed points.

Figure 8 displays a potential energy plot and best fit for a large amplitude low frequency system without damping. We see that the expected potential energy equation does not hold true because the double potential wells were not exactly equal in position and the spring constants of the oscillator were not exactly equal. We do however get a sense for where the two stable and the unstable points should occur at. Setting the derivative of the potential function equal to zero and solving for the minimum and local maximum values gives us a rough idea of where the critical points of the system occur. The potential function for every parameter set was approximately equal since the fixed points did not change.

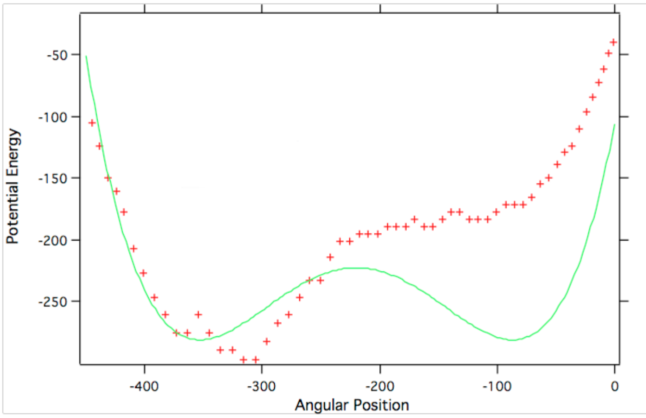

FIG. 7: This graph displays the potential plot for the parameters; 4.15 cm amplitude, 4.84 V frequency, and 2.35 cm damping position. The figure also includes an approximation fit that shows the stable and unstable fixed points when the slope of the fit line approaches zero. The green potential curve-fit has the form of $V(x) = -k_1(x-x_o)^2+k_2(x-x_o)^4+k_o$, where k_o=-223$\pm$ 6, k_1= (6.6$\pm$ 0.8)$\times$ 10^{-3}, k_2= (1.9$\pm$ 0.2)$\times$ 10^{-7}, *and* x_o= -220. The value k_o corresponds to a shift in the y-axis and the value x_o corresponds to a shift in the x-axis related to the position of the unstable fixed point.

V. CONCLUSIONS

Albert Einstein once said, "As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality [3]." This statement give us a sense of what chaotic motion tells us, which is that we can explore different forms of the data in a system such as the one we explored in this experiment, but we cannot rigorously define the motion in a mathematical sense. By using data analysis, such as the phase plot (continuous velocity vs. position), the Poincare plot (periodic velocity vs. position), and the potential energy plot (energy vs. position), we can make reasonable predictions about how a system will act by observing the patterns of motion around the fixed points. The motion of the bistable oscillator that was analyzed in this experiment gave us some sense of the relative patterns that occur when the parameters of the system are changed. We know that the non-linear changing of the elastic constant β and the linear elastic constant α directly affect the magnitude of the potential energy for the system at various angular positions of x. Also we found that the natural frequency ω_o could be compared to the damping parameter γ in order to explain the conditions of the system. This comparison gave a reasonable explanation for the non chaotic behavior of the oscillator of amplitude 4.15 cm, frequency of 4.84 V, and 1.30 damping position. The motion of the system was not chaotic over all parameter sets. Subtle changes in the parameters gave us noticeably different behavior. There were distinctive distortions of the Poincare plots as the parameters changed that allowed us to make inferences on how the parameters changed the motion of the system.

[1] The College of Wooster, Department of Physics, *Junior Independent Study Lab Manual*, **57-60**, (Spring 2015).

[2] N. Mann, *Classical Mechanics Lecture 28 Notes: The Damped Driven Simple Harmonic Oscillator*, **104-9** (Fall 2014).

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