Determining the Relation between the Power Radiated by a Black Body and its Absolute Temperature

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The purpose of this lab was to determine the order of the proportionality between temperature and radiated power from an approximate black body. A tungsten filament was heated using a direct current source. Using my experimentally observed values for current and voltage, and data provided by PASCO Scientific relating relative resistivity to temperature of my filament, an experimental replication of the original experiment by Joseph Stefan was performed in order to estimate the order of the equation of radiated power as a function of temperature. A final value of 3.724 for the power of temperature in my proportionality equation between radiated power and temperature was received, which differs from the accepted value of 4 by 6.9 %.

I. INTRODUCTION

In 1879, at the University of Vienna, Josef Stefan observed the heat transfer between two independent bodies, and noted that the heat exchange did not happen in a linear fashion [1]. He was able to experimentally deduce that the rate of the radiated heat exchange was directionally proportional to the fourth power of its temperature. The results tended to agree more with this proportionality when the system mimicked a black body, or a body that absorbs all incident radiation regardless of wavelength, then re-radiates energy in quanta characteristic to the substance's properties. Prior to this discovery, it was known that the linear model of temperature and heat transfer fell apart at higher temperatures. However, this quartic relation still seemed to come as a shock. Five years after Josef Stefan's discovery, former student of his Ludwig Boltzmann was able to confirm Stefan's hypothesis through a theoretical derivation. His derivation considered the heat radiated from a system to be fluid, however later derivations confirmed the same relation when the energy released is assumed to be in a collection of discrete quanta. This discovery came slowly, as a result of the difficulty in measuring or achieving high temperatures, and its sensitivity. However, this conclusion was a turning point in the field of physics. It is fundamental in the analysis of high temperature, radiative objects, such as celestial bodies, or even simple circuits.

II. THEORY

Electromagnetic radiation from a black body is released in discrete quanta [2]. The average energy released in a single quantum, multiplied by the density of quantum energy radiated gives the total energy density, in terms of wavelength, λ, as

$$\rho(v) = \frac{8\pi}{\lambda^4}. \tag{1}$$

Multiplying this equation by the quantity $hc/\lambda(e^x\text{-}1)$, where $x = hc/\lambda kT$, the energy per unit volume of this quanta with respect to the radiation wavelength is achieved in the following form

$$\frac{hc}{\lambda(e^{\frac{hc}{\lambda kT}} - 1)}\rho(v) = \frac{8\pi hc}{\lambda^5(e^x - 1)}. \tag{2}$$

This equation is known as the Plank radiation formula, where T refers to the temperature of the body, c to the speed of light, and k to the wave number [3]. Multiplying this formula by a factor of $0.25c$,

$$\frac{2\pi hc^2}{\lambda^5(e^x - 1)} = \frac{hc^2}{4\lambda(e^{\frac{hc}{\lambda kT}} - 1)}\rho(v) \tag{3}$$

is received.

Noting that the speed of light, c, has units of length over time, non-dimensionalizing the entire equation and considering only the units yields the relation

$$\frac{\frac{\text{Energy}_0}{\text{Volume}_0}}{\text{Length}_\lambda} \frac{\text{Length}}{\text{Time}} = \frac{\frac{\text{Energy}_0}{\text{Time}}}{\text{Length}_\lambda} \frac{1}{\text{Area}}, \tag{4}$$

where Energy$_0$ and Volume$_0$ refer to the energy and volume of one discrete quanta. Recalling that the units of power are energy per unit time, the term Energy$_0$/Time can be considered the power for a single, more or less infinitesimally small quanta. Thus, the power for the entire system can be described as

$$\frac{dP}{d\lambda} \frac{1}{a} = \frac{2\pi h c^2}{\lambda^5 (e^x - 1)}, \tag{5}$$

where a refers to the area.

Since $x = hc/(\lambda\ kT)$, $\mathrm{d}x = -hc/(\lambda^2 kT)d\lambda$. Integrating over all wavelengths

$$\frac{P}{a} = \frac{2\pi(kT)^4}{h^3 c^2} \int_0^\infty \frac{x^3}{(e^x - 1)}\, dx$$

.

is obtained.

The solution this equation gives us the Stefan-Boltzmann law for black bodies:

$$P = a\frac{2\pi^5 k^4}{15 h^3 c^2} T^4. \tag{7}$$

The constants $(2\pi^5 k^4)/(15h^3 c^2)$ can be combined in to a single variable σ. Because the area a is derived from the assumption of an ideal black body, for experiments sake, it makes sense to add a correction, or an emissivity factor, ϵ, to take in to account imperfect black bodies, where $\epsilon = 1$ for an ideal body, and less than one for a non-ideal system. The correction applied to another area term, A, so that the new approximated energy is described as $a = A\epsilon$.

The final equation is received

$$P = \epsilon \sigma A T^4 \tag{8}$$

for the power radiated from my black body.

A tungsten filament in a bulb is considered a non-ideal black body system, or a *gray body*, where ϵ is less than one. The power of the electric circuit is classically

$$P_{elec} = IV, \tag{9}$$

where I and V refer to my current and voltage values respectively. However, this non-ideal circuit must be described in terms of my radiated power and power lost by conduction, minus a term for the power lost due to the radiation absorbed from my ambient temperature. One then receives the equality

$$P_{elec} = P_{radiated} + P_{conducted} - P_{absorbed}, \tag{10}$$

such that

$$P_{elec} = \epsilon \sigma A T_f^4 + \kappa l(T_f - T_\alpha) - \epsilon \sigma A T_\alpha^4, \tag{11}$$

where T_f refers to my filament temperature, T_α to my ambient temperature, κ to the conductivity of my filament, and l as a characteristic length scale. Because my filament has the potential to reach temperatures as high as 3000° C, T_α^4 is arbitrarily small compared to T_f^4, so that it can be ignored in the final analysis.

In order to prove that the power radiated by the system is more or less proportional to the fourth power of my temperature, as in Equation 6, one can calculate the power radiated from my filament at different temperatures from a series of currents and voltages.

One can estimate the radiated power from the classical power calculation seen in in Equation 7. Considering the general equation of P as a function of T_f without the small correction factor, the approximation can be made that

$$P \approx \epsilon\sigma A T_f^4 + \kappa l(T_f - T_\alpha). \tag{12}$$

Multiplying both sides by the linear term, the equation can be rearranged to read

$$\frac{\epsilon\sigma A T_f^4}{\kappa l(T_f - T_\alpha)} = \frac{P}{\kappa l(T_f - T_\alpha)} - 1. \tag{13}$$

Noting that my linear term is less than 1, making $P/\kappa l(T_f - T_\alpha) > P$,

$$\frac{P}{\kappa l(T_f - T_\alpha)} - 1 \approx \frac{P}{\kappa l(T_f - T_\alpha)} \tag{14}$$

such that

$$P \approx \epsilon\sigma A T^4. \tag{15}$$

Though this approximation is not ideal, it is considered for the sake of analysis; taking the natural logarithm of both sides allows the utilization of logarithm rules in a convenient way.

$$\ln(P) = \ln(\epsilon\sigma A T^m) = 4\ln(T) + \ln(\epsilon\sigma A). \tag{16}$$

It is shown then that the graph of the natural logarithm of the power versus the natural logarithm of the temperature should yield a slope of approximately 4, agreeing with Equation 6 and exhibiting the power relation between temperature and power.

III.   PROCEDURE

The Stefan-Boltzmann lamp containing my tungsten filament was connected to a direct current power supply, and then attached to an ammeter and voltmeter, as seen in Figure 1. Voltage data were collected at values between 0.5 and 16 V, and respective current data were recorded at values between 2 and 3 μA. There was approximately one minute between each data point collection where there was no current running through my filament, allowing it to cool off, ensuring that the data were recorded at approximately room temperature.

The current and voltage was then increased by a factor of 10^{-6}. The current values ranged between two and three amps, and the respective voltage values were less than 11 volts. Similar data was collected as taken at room temperature, however, this time the intention was to heat my filament, so the power supply was not turned off between data point collection. Data collection was done over a series of five separate trials in order to gain ideal results.

IV.   DATA AND RESULTS

The graph of the data collected at room temperature can be seen in Figure 2. The uncertainty values of $\pm$ 0.01 A for current and $\pm$ 0.01 V for voltage were experimentally observed, and most likely due to either an oscillatory power supply, or imperfect measurements from my voltmeter and ammeter.

I used the relation $R = V/I$, where R refers to the resistance, to find R_{ref} given by the slope, which turned out to be 0.297 Ω.

Dividing the values of resistance taken at higher temperatures, R_T by R_{ref} gives the relative resistance, or relative resistivity. Though resistance is a calculated property, and resistivity is a fundamental property of the tungsten filament, because I am again taking a ratio, they turn out to be equal.
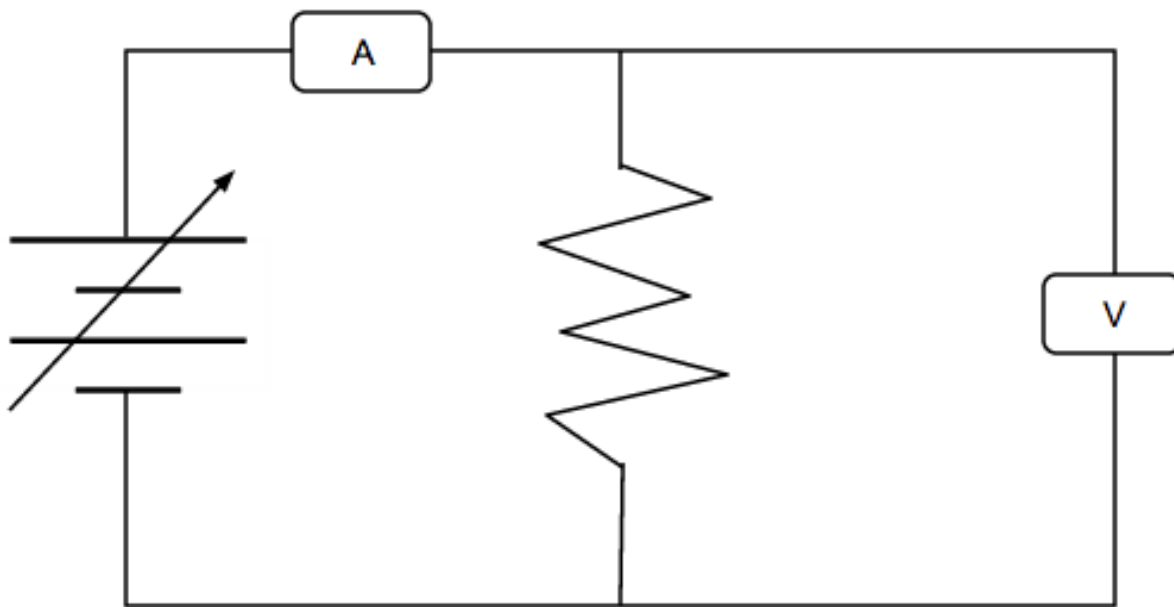
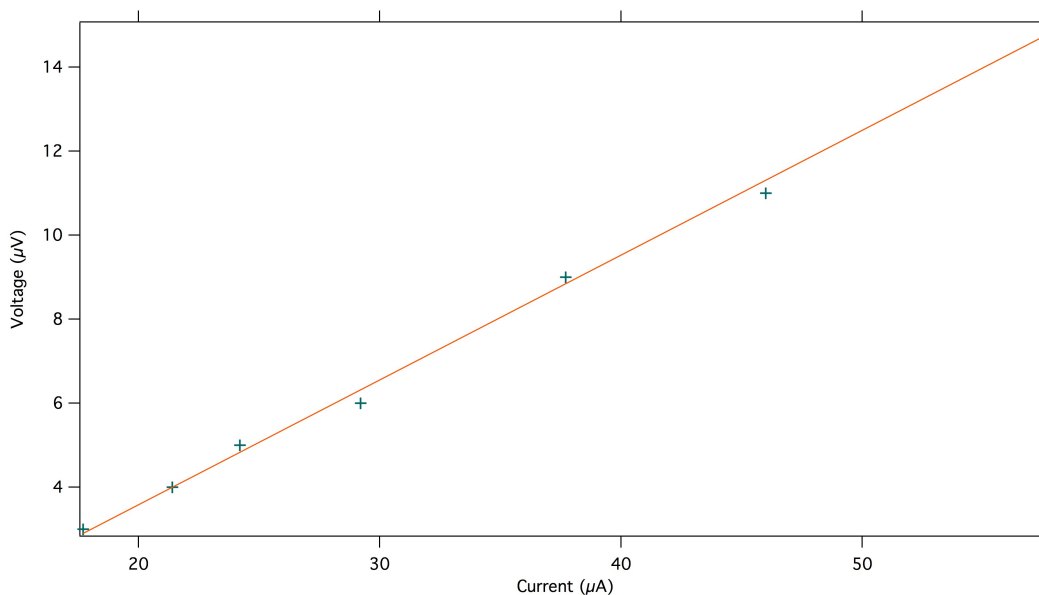FIG. 1: Stefan-Boltzmann circuit schematic replicated from FIG. 17.2 in Lab Manual [?].

FIG. 2: Graph of voltage vs. current, with a linear fit of slope, or R_{ref}, 0.297 ± 0.007, and y-intercept -2.363 ± 0.251.

Consulting a chart provided by PASCO Scientific for the temperatures of the filament at different relative resistance values, I graphed values of relative resistance vs. temperature of the filament, using only data that was similar to the data I experimentally received, so that the temperature values graphed from my PASCO provided chart were similar to the temperature values I received. This ensured the closest fit from the PASCO data graph to my data. From a linear fit of this graph, seen in Figure 3, I received an equation from which one could estimate the temperature of the filament form my relative resistance data. From this fit, I received the equation
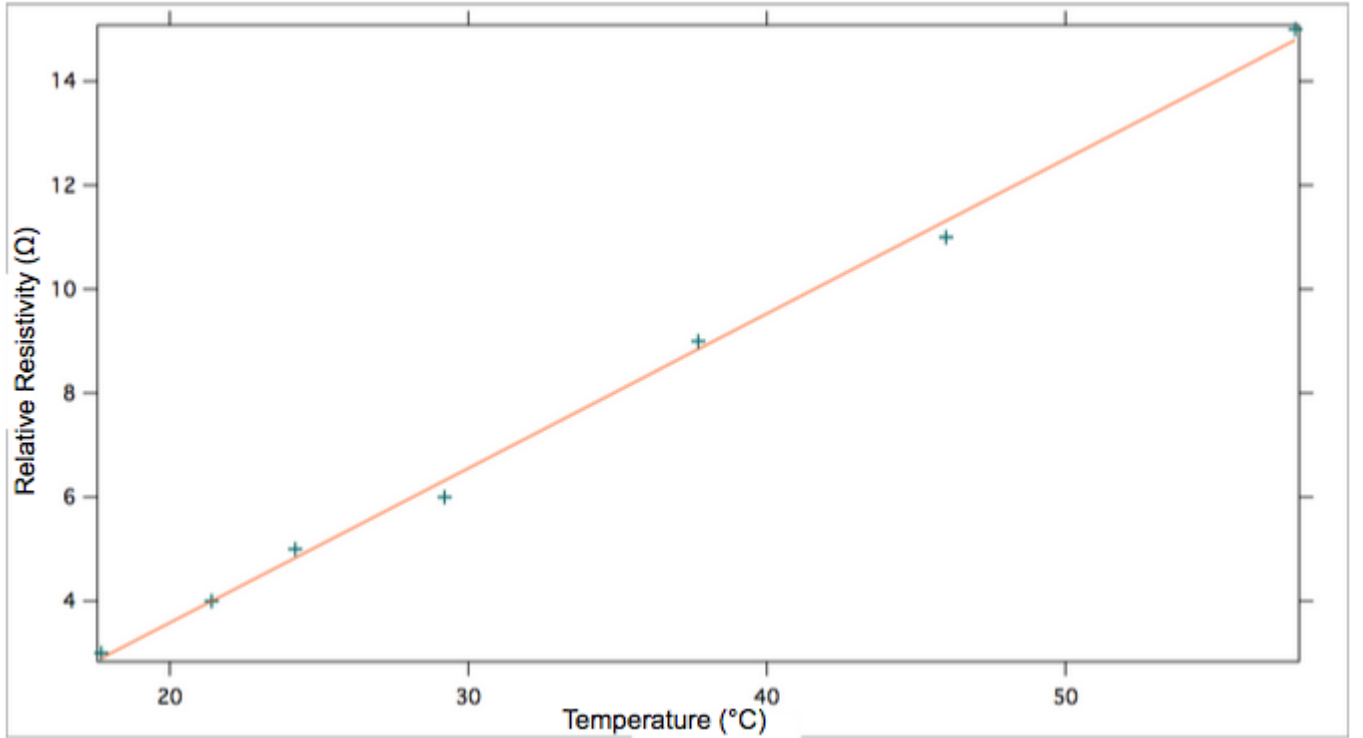
FIG. 3: Graph of relative resistivity vs. temperature, with a linear fit yielding an equation of $y = 163x + 365.3$, and an R^2 value of 0.999.

$$f(x) = 163x + 365.3. \tag{17}$$

where the function f(x) refers to the temperature as a function of the relative resistance, x, which was used to calculate a new data set for temperature.

I then used the classical statement for power as seen in Equation 9 to determine the power radiated from my relative resistivity, using the values for current and voltage from the heated filament. The natural logarithm of my new power data was then graphed against the natural logarithm of the temperature data, and can be seen in Figure 4.

From Equation 13, one can see that the slope in this case approximates the order of the function of power with respect to temperature, and is approximately 3.724.

V.   UNCERTAINTY AND ERROR ANALYSIS

The uncertainty values for the current and voltage were determined by observing oscillations in my ammeter and voltmeter data from my power supply. The uncertainty for voltage, δV, was 0.01 V, and uncertainty for current, δI, and came to be ± 0.01 A. This value was used to calculate the power uncertainty, δP, and the uncertainty values used in my natural log graph in Figure 4.

I gain an uncertainty propagation for power from the equation

$$\delta P = P\sqrt{\left(\frac{\delta I}{I}\right)^2 + \left(\frac{\delta V}{V}\right)^2} = P\sqrt{\left(\frac{0.01}{I}\right)^2 + \left(\frac{0.01}{V}\right)^2}. \tag{18}$$

Inserting the δP value in to the natural log for my uncertainty values for Figure 4 poses a problem, since the natural log operation will give a large result for small uncertainty values. Instead, one must consider the derivative of the natural log of power values,

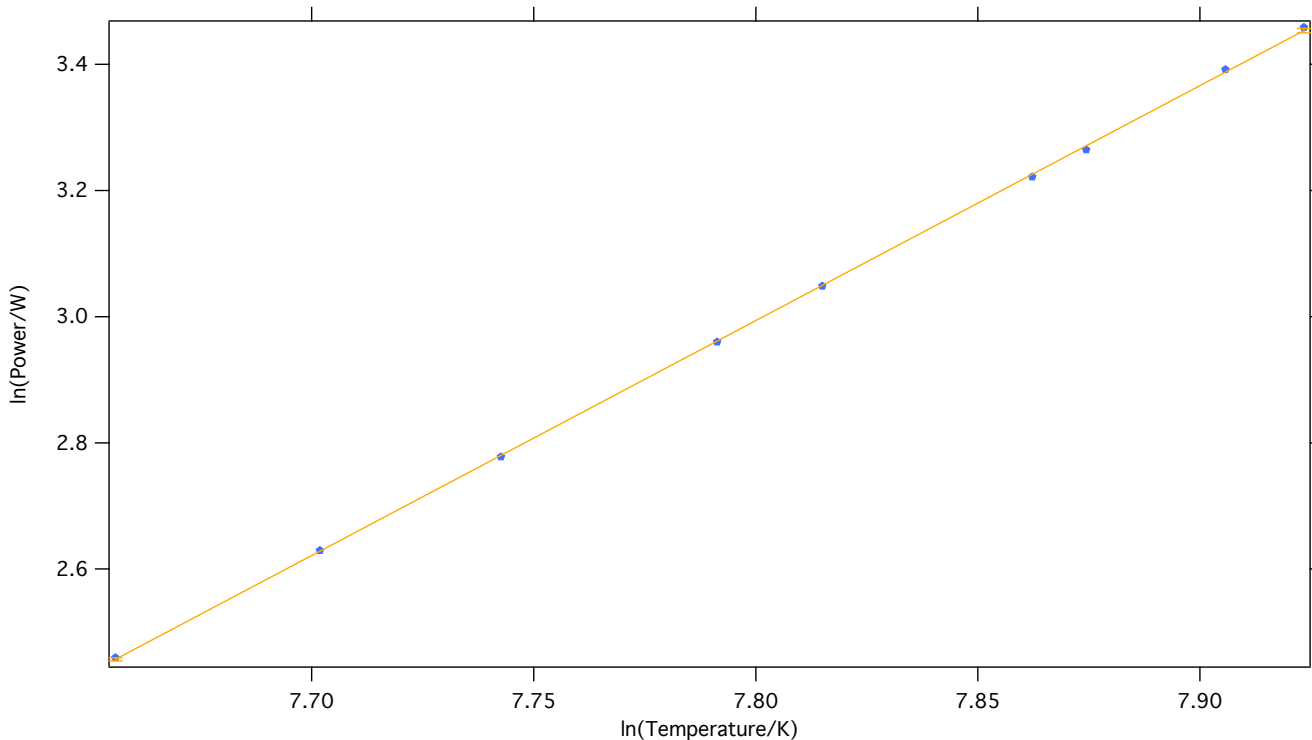$$\delta \ln(P) = \frac{d\ln(P)}{dx} = \frac{1}{P}. \tag{19}$$

FIG. 4: Graph of the natural logarithm of power vs. the natural logarithm of temperature, with a linear fit of slope 3.724 ± 0.01 and y-intercept of -26.05 ± 0.12. Both power and temperature as seen on my axes have been non-dementionalized due to the natural log function.

The values for my δP uncertainty averaged to around 0.08 watts, so that $\delta P/0.08$ W ≈ 1. Multiplying the natural log uncertainty propagation by this experimentally determined factor yields

$$\delta\ln(P) \approx \frac{\delta P}{0.08P}, \tag{20}$$

which I used for my point uncertainties in Figure 4. The uncertainties were smaller than the values I received for the natural log of the power by an order of magnitude of 10^{-2}.

The experimentally derived value for the order of the equation of power as a function of temperature was 3.724. The percent error was calculated by the following function:

$$\text{Percent Difference} = \frac{\left|\text{Experimental Value - Accepted Value}\right|}{\text{Accepted Value}}100\%. \tag{21}$$

$$\text{Percent Difference} = \frac{\left|3.724 \text{ - } 4\right|}{4}100\% = 6.9\%. \tag{22}$$

Because I received low uncertainty values, and had a pretty direct correlation with the data in the graph in Figure 4, one can deduce that the majority of my error was not due to a mis-calculation, or a single faulty data point, but rather a uniform error throughout my data collection. Referring back to Equation 8, it may be valid to say that some of the power radiated was lost or absorbed in to the ambient surroundings, such that my T_α^4 wasn't entirely negligible. Because my system was not ideal, and not done in a perfect vacuum, energy was also lost to the surroundings.

It is also likely that some of the error came from my T_{ref} calculation. It could be that the bulb heated slightly during calculation, so that the filament was not in thermal equilibrium with the room temperature, or that the bulb was not given adequate time to cool off. Error could have also came from a uniform miscallibration of my ammeter or voltmeter.

VI.   CONCLUSION

The purpose of this lab was to determine the order of the equation of power radiated from an approximate black body as a function of its temperature. It is ideal to use a black body system, as it ideally absorbs all incident radiation. The discovery of the Stefan-Boltzmann power and temperature relation was revolutionary in the field of quantum mechanics. In order to improve my results, one may attempt to measure the power absorbed by the ambient surroundings, and factor that in to my total power radiated results. However, because we used a filament in a circuit, the system lost less energy to its surroundings than if I were heat the filament with some outside source of energy. After five trials, I received a data set that yielded a percent error of 6.9 %, with a value of 3.724 for my exponential proportionality, differing from the accepted value of 4 by only 0.1 order of magnitude. The error was uniform and most likely experimental, due to an idealization of my non-ideal system.

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[1] John Crepeau; University of Idaho, Dept. Of Mechanical Engineering. *Accessed January 30th, 2015* `http://webpages.uidaho.edu/~crepeau/ht2009-88060.pdf`.
[2] HyperPhysics, *Accessed January 30th, 2015.* `http://hyperphysics.phy-astr.gsu.edu/hbase/mod6.html(IIIed.)`
[3] HyperPhysics, *Accessed January 30th, 2015.* `http://hyperphysics.phy-astr.gsu.edu/hbase/thermo/stefan2.html`).