Modeling the Duffing Equation with an Analog Computer

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The goal was to model the Duffing equation using an analog computer composed of electrical components. This equation can be simulated using a computer but there are advantages to using an electrical circuit. To physically create a circuit that will model a nonlinear differential equation that exhibits chaotic behavior such as the Duffing equation, we used op amps to integrate the signal and multiplier chips to introduce cubic terms. Once the circuit was built we monitored the position against velocity using an oscilloscope, which produced a phase space plot of the characteristic Duffing motion and possibly chaos. We found some nonlinearity looking at the phase space plot, we also found characteristic motion of a Duffing oscillator. We were unable to find any chaos, which should be brought about through the varying of some parameters and then modeling of the Duffing equation.

I. INTRODUCTION

The Duffing equation is a nonlinear differential equation discovered by Georg Duffing, an engineer working in Germany and the U.S. in the early twentieth century. Much of his work was with vibrations, gears and engine components. In 1914 he published a book titled "Forced oscillations with variable natural frequency and their technical significance" [1]. This work made him famous with his Duffing equation as well as his work with other differential equations.

The study of chaos is a modern development, beginning in the early part of the last century. The first study of chaos began when looking at the orbits of the planets. These orbits are not always periodic, which is one of three main characteristics of chaos. Another characteristic of chaos is extreme sensitivity to initial conditions. This was first studied in weather models. Edward Lorenz found this extreme sensitivity when he was trying to predict the weather with models but a tiny change in the starting parameters resulted in vastly different results. Current computers have brought to the study of chaos a way to iterate nonlinear functions in minimal time, allowing equations like the Duffing oscillator to be studied and modeled on personal computers.

A problem with these computer models stems from the definition of chaos. By defining chaos as having extreme sensitivity to initial conditions, one would think a computer, which most associate with precision, would be a great tool to model chaos. In this instance, the true power of the computer comes from its speed of simple calculations not its precision. Each time the computer makes a simple calculation the answer is rounded off, either to the 32nd or 64th place. This may seem precise enough, but after many of iterations the round off error can become significant to the model. For most models this error has a negligible impact on the results, but looking at bifurcation diagrams this error can be visible. One way to remove this error is to model the equation using an analog computer instead of a digital computer.

What is an analog computer? It uses a combination of voltage amplitude or frequency to perform a mathematical function as the signal output. Analog computers use continuos variables for computation so there is no round off error. For this circuit we use op amps, multiplier circuits, resistors and capacitors to help model the Duffing equation. These components are not quantized in a discrete bit (0 or 1) like digital components, so there is no round off error, but there can be electrical noise. By using op amps, an analog computer still has the capability to perform mathematical functions such as differentiation and integration similar to a digital computer. We used ideas from a previous project at the College of Wooster by Clinton Braganza on a jerk circuit and a paper by J.C. Sprott which gave many examples of chaotic circuits as a guide to what we desired in the design of a circuit [2, 3].

By manipulating op amps and multipliers a circuit to model the Duffing oscillator was possible. The design for the circuit we built came from an independent study project attempting to find similar results. Noah Johnston successfully created a Duffing circuit but was unable to find chaos. Using parts of the circuit he designed along with new op amps we hoped to find the Duffing oscillator, chaos and a bifurcation diagram. If all of that worked we hoped to look at some of the more complex behaviors of the Duffing oscillator which were investigated by M. J. Brennan and others [6, 7].

II. THEORY

One of the most basic concepts in all of physics is the spring. For entry level physics it is just another problem in a long set of homework problems, but it can also provide a model for a multitude of other phenomena. A spring is a harmonic oscillator, which means upon displacement the spring will experience a force towards its equilibrium position. This can be modeled by the simple equation,

$$m\ddot{x} = F = -kx, \tag{1}$$

where F is the restoring force, k is a constant, x describes the displacement from equilibrium and $\ddot{x}$ describe acceleration. This constant k can sometimes be called a spring constant and varies based on the material and configuration of the spring. Along with modeling springs, simple harmonic oscillators are seen in pendulums, circuits, atoms, radios and any other physical object that has a stable equilibrium position.

In the world outside of physics books there is typically more than a single force acting on a mass attached to a spring. Including viscous friction as a second force creates a damped harmonic oscillator. Instead of just the restoring force of the spring there is now a non-conservative force of friction acting on the mass. This frictional force is dependent on the velocity of the mass. Taking Eq. 1 and adding viscous friction we now have

$$m\ddot{x} = F = -kx - b\dot{x}, \tag{2}$$

with b as a coefficient of friction and $\dot{x}$ representing the velocity. Although more complicated than the simple harmonic oscillator, this equation for a damped harmonic oscillator can be solved exactly. With exact solutions easily available, this equation describes many systems well but does not lend itself to intense study.

There are more than two forces which can act upon a mass attached to a spring. Already with the spring force and the frictional force we have the damped harmonic oscillator. Here the system may not conserve energy but one can usually assume energy is flowing out of the system. Adding energy to the system can be described as driven harmonic motion. With equations we can represent this with a sinusoidal driving force, $F_0 cos(\omega t)$,

$$m\ddot{x} = F = -kx - b\dot{x} - F_0 cos(\omega t + \phi), \tag{3}$$

where F_0 is the maximum driving force and ω is the driving frequency and ϕ as the driving phase. Again there is an exact solution so the results of this equation alone are not exciting. Another component needs to be added for the equation to model a system that is not exactly solvable.

The Duffing equation can be described as a driven harmonic oscillator with a cubic component,

$$\ddot{x} + \delta\dot{x} + \beta x^3 \pm {\omega_0}^2 x = \gamma cos(\omega t + \phi), \tag{4}$$

with $\delta = \frac{b}{m}$ as the friction coefficient, ${\omega_0}^2 = \frac{k}{m}$ and β are constants, and γ is related to the maximum driving force, $\gamma = \frac{F_0}{m}$. The cubic term provides the double well potential found in the oscillator. Duffing's equation can be used to model an elastic beam deflected between two magnets as shown in Fig. 1 [8]. Since this is a nonlinear partial differential equation there is no exact solution. One way to examine the properties of the Duffing oscillator is to simplify the equation slightly. The easiest way to look at the equation is to remove the driving force and write the equation as a two dimensional differential equation

$$\dot{x} = y, \dot{y} = x - x^3 - \delta y. \tag{5}$$

With this notation the two dimensional flow can be analyzed more easily and graphed on an xy plot. To begin the analysis we look for fixed points. These fixed points happen when $\dot{x} = y = 0$. Taking $y = 0$ and substituting into Eq. 5,

$$\dot{y} = x(1 - x^2), \tag{6}$$

which leaves fixed points at $(1, 0), (-1, 0)$ and the origin, $(0, 0)$. Using these values we can find the Jacobian matrix, eigenvalues and eigenvectors [9]. Going through this process leads us to see the origin as a saddle fixed point and the two other fixed points along the y axis as stable spiral repellers [4]. This leads to a homoclinic orbit as show in with my Mathematica simulation in Fig. 2. Although this gives interesting insight into some of the properties of the Duffing oscillator, we are trying to model the full Duffing equation so we need other methods of analysis.

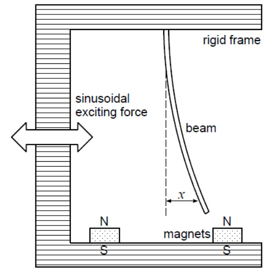

FIG. 1: A physical system which can be described using the Duffing equation.

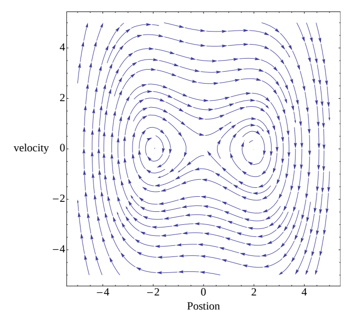

FIG. 2: An phase space plot (position vs. velocity) of the unforced Duffing oscillator.

Instead of simplifying the equation by removing the driving force we can look at the full Duffing equation with a numerical differential equation solver. This was done in Mathematica with the function NDSolve. Without having to linearize the equation and solve for fixed points, we can easily see the different properties of the Duffing Oscillator. Fig. 3 shows the phase space plot of the Duffing equation at different parameter values. Varying the parameters of the Duffing oscillator one can see a rich behavior with the addition of the driving force. In addition to viewing the phase space plot one can look for the strange attractor.

The Duffing oscillator exhibits chaos. One of the easiest ways to illustrate this is through the observation of

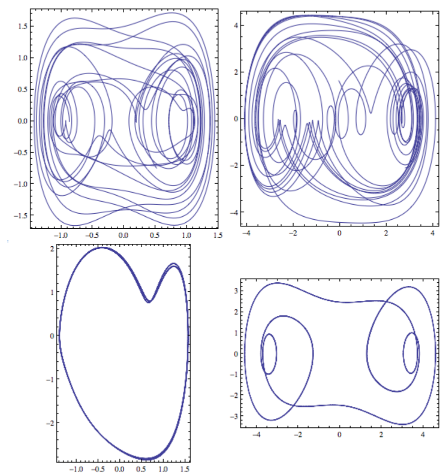

FIG. 3: Phase space plots at different parameters. The top left shows the driven version of Fig. 2, the top right a more chaotic set of parameters. The bottom graphs show periodic orbits, the left being asymmetrical and the right being mostly symmetrical.

the strange attractor at various time increments. Attractors are sets that a dynamical system will evolve toward, and a strange attractor is an attractor with a non integer or fractal dimension. The easiest way to view the strange attractor present in the Duffing oscillator is to look at the Poincare section. A Poincare section of Poincare map is a view of a section of multiple periodic orbits. Now with a section of the periodic orbits if we observe these over time we can see the occurrence of a strange attractor. To do this we use the same parameters only varying t of the cosine driving function. A few of the Poincare maps are displayed in Fig. 4. In animation, the strange at-

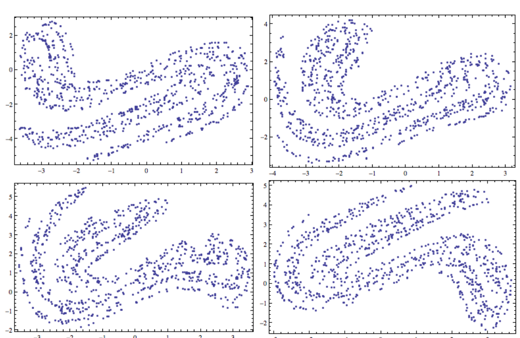

FIG. 4: A set of poincare maps with the same set of parameters only varying t from 2.00 (top left) to 2.03 (bottom right).

tractor rotates and folds over itself as it progress from $t = 0 \to 2\pi$. It is possible to create a Poincare section on an oscilloscope by finding the point when the trajectory cuts through a certain plane and then plotting that plane at various frequencies [10]. This process is easier on digital computers which can easily simulate the strange attractor present in the Duffing oscillator, so it is not our main goal when using the analog circuit.

Another quality of the Duffing oscillator is hysteresis. Hysteresis is when the output can change depending on what was previously done to the system [11]. An example would be increasing one parameter to a value of 5. If hysteresis occurred then by restarting the system and coming down from a value of 10 to a value of 5 the system would not be in the same state at the same set of parameters. This is not a goal of the project but we will recognize it if it does happen.

III. PROCEDURE

To create the Duffing circuit we first needed the correct components. Using 355 op amps and 633 multipliers I tested each component in a simple circuit by itself to make sure the parts were working properly. To get the input signal to follow Duffing's equation for the oscillator, the signal needed to be integrated twice and cubed. Using the circuit schematic from Noah Johnston in Fig. 5 we began to build the circuit. To begin with we used a

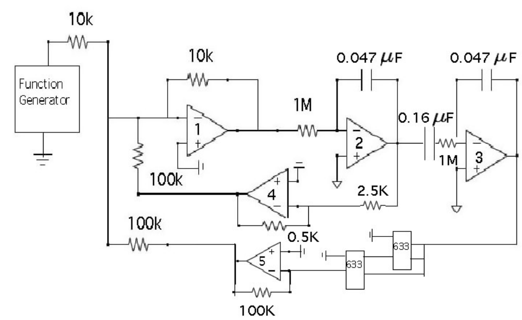

FIG. 5: A schematic of the final circuit used for the last results given with a Duffing oscillator.

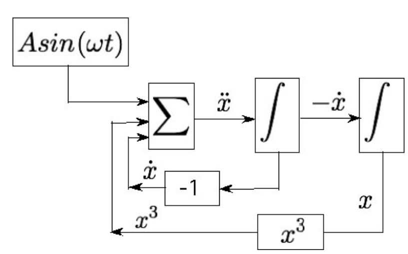

FIG. 6: An equivalent schematic showing the correlation between the circuit and the Duffing equation. Op amp 1 is the summer, while op amps 2 and 3 are integrators and op amp 4 inverts the signal.

Hewlett Packard 33120A function generator for fine control of amplitude and driving frequency. Next came a summing circuit (op amp 1) to add the input signal, the inverted signal and the cubed signal. The output of the summing circuit is taken to be $\ddot{x}$, which is sent through an integrator (op amp 2). Now the signal represents $\dot{x}$, this was done with integration because integration is more stable then differentiation in op amps. Here the signal is split off to an inverting op amp (op amp 4) and another integrating op amp (op amp 3) to give x. The last part of the circuit is the cubic term. To cube, instead of using op amps we used two 633 analog multipliers in succession and a non inverting op amp to trim the output of the multipliers. To read the output signal we used a 2 channel Tetronix TDS 2024B digital oscilloscope. Fig. 6 shows the circuit chunked as parts in the Duffing equation, Eq. 4.

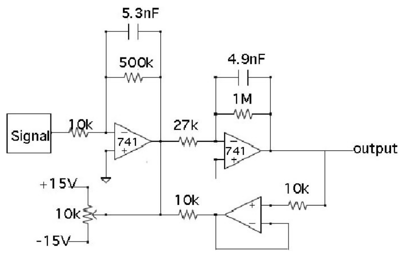

FIG. 7: A phase shifter with a 741 op amps which gives a constant voltage output while changing the phase.

The last component that was created was a constant amplitude phase shifter. If the circuit were working properly this component could be hooked up to adjust the phase of the input signal. This phase change would be the ϕ in Eq. 4. Fig. 7 gives the schematic of the phase shifter. The resistors would need to be changed in order to use the shifter depending on the range of frequencies the signal is attempting to vary.

IV. RESULTS

The goal of this project was to create a Duffing circuit with enough resolution to get dramatic results and observe chaos. To do this we built a circuit with two integrating components to model the derivative components in Eq. 4 along with two multipliers for the cubic term and a phase shifter to vary ϕ. For this plot we took the X component from the output of the first integrator and the Y component from the output of the second integrator. These outputs corresponded to a phase space plot. A chaotic phase space plot of the Duffing oscillator with different parameters is shown in Fig. 3. We found a more plausible result for on oscillator, Fig. 8.

The clearest oscilloscope image we saw is shown in Fig. 9(left). We tried to match my simulation of the Duffing oscillator in Mathematica by scanning over a large parameter space. The circuit we work with had no linear term and no phase shift (time constraint). Using function manipulators the closest plots we could find are shown in Fig. 9(right). There are differences between the phase space plots but at the same time they are clearly the

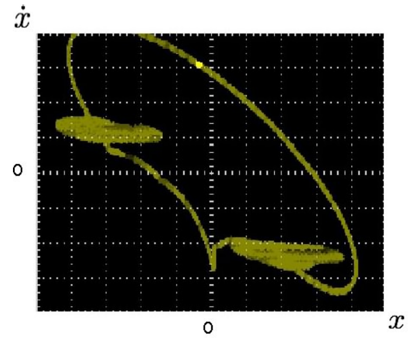

FIG. 8: One of the first plots which modeled an oscillator but still had an interesting and unknown error along the trace.

same function.

There were a multitude of problems constructing the circuit; the main ones were hooking up the multipliers and large DC offsets. To combat the large DC offset we inserted small capacitors after integrating op amps to try and minimize the DC offset. Although this worked to some degree there were still large DC offsets in some parts of the circuit. This could be due to the op amps or multipliers needing to be trimmed for offset voltages. Eventually we took data on the oscilloscope in XY mode in AC mode so we did not have to worry about the DC offset.

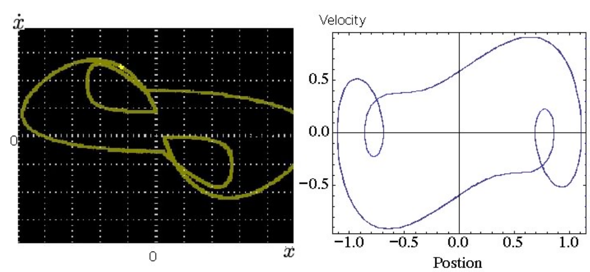

FIG. 9: The best trace from the circuit on the left and the simulation with Mathematica on the right attempting to show the similarities, $\delta = 0.72, \gamma = 1.66, \omega 0 = 0, \omega = 0.48, \beta = 4.8$.

With more time and the use of trimpots I think this circuit could be used to do some analysis of the Duffing oscillator.CIrcuit theory could be used to find the resistors and capacitors that equate to the parameters in Fig. 9 (right). If this could be done successfully then varying the components values would be interesting to compare with a computer simulation. If there was also a way to modify the circuit so a Poincare section could be produced, as in Fig. 4, that would be interesting. There was not enough time to get the phase shifter working with the newest version of the circuit.

V. CONCLUSION

The goal of this project was to create an analog simulation of the Duffing oscillator. This was done with an electrical circuit that took advantage of the mathematical function capability of op amps and multiplier chips when used in a electrical circuit. A nice correlation between the simulation on Mathematica and the analog circuit we created. Although building a working circuit took longer then expected, we were still able to see the analog Duffing oscillator. In the future with the correct circuit much more time could be spent working to find the parameters and match them to a simulation. A bifurcation plot on an oscilloscope would also be an interesting result. Also one article suggested a Poincare section could be visible on an oscilloscope trace.

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