Two Dimensional Causal Dynamical Triangulation

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In this paper, a theory of quantum gravity called Causal Dynamical Triangulation (CDT) is explored. The 1+1 dimensional universe is simulated in xCode. This paper explains CDT in general and presents and explains the results of 1+1 dimensional simulations. The critical value of the reduced cosmological constant was found to be 1 (it is taken to be dimensionless).

I. INTRODUCTION

Understanding gravity at the fundamental level is key to a deeper understanding of the workings of the universe. The problem of unifying Einstein's theory of General Relativity with Quantum Field Theory is an unsolved problem at the heart of understanding how gravity works at the fundamental level. Various creative attempts have been made so far at solving the problem. Such attempts include String Theory, Loop Quantum Gravity, Hořava-Lifshitz gravity, Causal Dynamical Triangulation as well as others.

Causal Dynamical Triangulation (CDT) is a relatively new attempt developed by Jan Ambjørn, Renate Loll and others to quantize gravity. It involves decomposing space-time into 'triangular' building blocks[2]. It has so far predicted that the universe is two dimensional and nonclassical at the Planck scale (about 10^{-35} m). It has also managed to recover the classical behaviour of space-time at four dimensions.

Unification of Quantum Field Theory and General Relativity is important in order to get a more fundamental understanding of gravity. In order to use the CDT approach, the Einstein-Hilbert action of General Relativity and the path integral approach to Quantum Field Theory are extremely important[2]. We begin by introducing both concepts as well as the metric and the Einstein Field equations.

In this paper we attempt, at least briefly, to explain CDT in general and explain what we have found with our simulation.

II. BRIEF REVIEW OF GENERAL RELATIVITY

In everyday life, we are accustomed to measuring the Euclidean distance between two places (that is, the distance between two points on a flat, smooth surface). As we go throughout our daily lives, we rarely ever, if at all, measure distances on curve surfaces. We generally use what is called the Euclidean metric. In general, a metric is a function that obeys certain properties.

Let $d(x, y)$ be a metric (a distance function between points x and y). Then it obeys the following properties:

(a) $d(x, y) \geq 0$

(b) $d(x, y) = 0$ iff $x = y$

(c) $d(x, y) = d(y, x)$

(d) $d(x, y) \leq d(x, z) + d(y, z)$

From the properties above, one sees that the distance between any two points must always be either 0 or strictly positive[10]. This is in line with our every day experiences. The pythagorean theorem for a right triangle is an example of a metric. This is called the Euclidean metric[9]. The Euclidean metric is the metric of flat space. In flat space-time however, we have the Minkowski metric described by

$$ds^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{1}$$

We see right away that from the definition above, this function can give negative values so it fails the first property. The space with which this metric is associated is called a Pseudo-Riemann space, but the space that obeys $a - d$ above is called a Riemann space. The metric is invariant and is intrinsic to the space. We may write Eq. 1 in terms of matrices as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

where we are using the Einstein summation convention. This takes advantage of the fact that we may write the metric in terms of the product of some matrix and two vectors. This matrix is called a metric tensor.The metric tensor for flat space-time is:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Metric tensors obey general rules of transformation which will be explained in the context of General Relativity below. The metric tensor above is the tensor of flat four dimensional space-time, where there is no matter (or where we only have subatomic particles) and in a universe where the cosmological constant is zero. In General Relativity, to understand what goes on in another observer's frame of reference, one needs to do general coordinate transformations[9]. We typically define a general coordinate transformation as follows. Let's say the coordinates in an observer's frame of reference are $x^{\overline{\mu}}$ and we want to transform into the coordinate system of x^{α}. Then we have:

$$g_{\alpha\beta} = g_{\overline{\mu}\overline{\nu}} \frac{\partial x^{\overline{\mu}}}{\partial x^{\alpha}} \frac{\partial x^{\overline{\nu}}}{\partial x^{\beta}} \tag{3}$$

The basic idea behind the expression above is very powerful in General Relativity. This principle and the properties of tensors in fact are what allows gravity to be able to be seen as manifesting from the curvature of space. Tensors appear in the Einstein Field equations which can be derived using the Einstein-Hilbert action[2]. The Einstein-Hilbert action turns out to be very important in attempts to quantize gravity. The Einstein-Hilbert action will be presented in terms of the determinant of a matrix. In order to understand how the determinant comes in, the reader will be reminded of a few things from multivariate calculus. If we let $det[g_{\mu\nu}] = g$, then from this we have that $g = \overline{g}J^2$[9], where $\overline{g}$ is related to the observer's frame of reference and J is the Jacobian which is defined as:

$$J = det\left[\frac{\partial x^{\overline{\mu}}}{\partial x^{\alpha}}\right] \tag{4}$$

Now from standard multivariate calculus, we know that for the 4-volume $d^4\overline{x} = Jd^4x$[9]. Now, we then get the invariant volume:

$$dV = \sqrt{-\overline{g}}d^4\overline{x} = \sqrt{-\overline{g}}Jd^4x = \sqrt{-g}d^4x \tag{5}$$

We can use this to obtain an expression for the Einstein-Hilbert action in terms of the determinant of the tensor $g_{\mu\nu}$. The Einstein-Hilbert action is:

$$S_{EH} = k\int dV\left(\frac{1}{2}R - \Lambda\right) = k\int \sqrt{-g}d^4x\left(\frac{1}{2}R - \Lambda\right) \tag{6}$$

where $k = c^4/8\pi G$, R is the Ricci scalar curvature (which in terms of the Einstein-Hilbert action could be thought of the lagrangian density) and Λ is the Einstein cosmological constant (which affects the expansion of the universe)[9]. From the Einstein-Hilbert action, one can get the Einstein Field Equations in the vacuum:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0 \tag{7}$$

The cosmological constant term Λ is thought to explain the observed accelerated expansion of the universe and it is believed that if this is indeed the correct equations of the vacuum, then it means that space-time without matter is not truly flat but is slightly curved. It is also thought that the cosmological constant is responsible for quantum fluctuations of the vacuum which might have prevented the formation of a cosmological singularity (a time at which the universe had infinite density). At this point, the equations of relativity 'blow up' and a unification of Quantum Field Theory with General Relativity required to explain what goes on there.

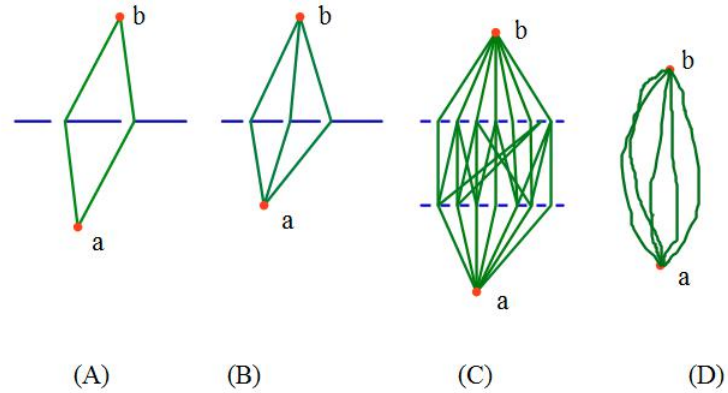

FIG. 1. Diagram showing the resulting paths of a particle traveling through a screen of multiple slits (A and B) and multiple screens with multiple slits (C and D).

III. BRIEF REVIEW OF QUANTUM FIELD THEORY

Imagine particles incident on double slits. This looks like the situation in part (A) in figure 1. Now, imagine we keep adding slits (part B) such that there are an infinite number of slits and screens (part C). Part (D) shows that after adding an infinite number of slits and screens, we have empty space. What we have are an infinite number of paths as a result[3].

Now specifically, let us consider photons incident on a screen with double slits. Let the photon travel from some point a to some point b and let

$$\mathbb{E}[a \to b] = e^{i\phi_1} + e^{i\phi_2} \tag{8}$$

be the electric field amplitude, where $\phi = \int \omega dt$. We then have the probability that the photon goes from point a to b given by:

$$\mathscr{P}[a \to b] = |\mathbb{E}[a \to b]|^2 \tag{9}$$

Now, instead of photons, imagine we have some other particle, say electrons that are incident on the double slit. We then have:

$$\mathscr{P}[a \to b] = |A[a \to b]|^2 \tag{10}$$

where A is the probability amplitude of the electron[3]. We have:

$$A[a \to b] = e^{i\phi_1} + e^{i\phi_2} \tag{11}$$

where here we have $\phi = \int \omega dt = \int L dt/\hbar = S/\hbar$. S is known as the action and as one can see, it is defined similarly as in classical mechanics with the lagrangian L. All we have done so far is use a double slit, but there is more to this thought experiment. Let us now add more slits to the screen. In fact, let us add an infinite number of slits in infinitely many screens. The total probability amplitude is a sum over the amplitude for each slit and each screen (a double sum)[3]. Let i be the index over slits and j be the index over screens, we then have:

$$A = \sum_{i,j} A_{ij} = \sum_{paths} A_{path} \tag{12}$$

where $A_{path} = e^{iS_{path}/\hbar}$[3]. From this, the quantum propagator[7] is defined. The quantum propagator for a particle moving from point a to point b is defined as:

$$A[a \to b] = \int_a^b \mathscr{D}x[t] e^{iS[x[t]]/\hbar} \tag{13}$$

where $\mathscr{D}x[t]$ is a path measure, S is the action and $\hbar$ is Planck's constant which as of now on, along with the speed of light will be defined as $c = 1$ and $\hbar = 1$. This is called the path integral[3]. The path integral approach is an approach completely developed by Richard Feynman. It can be thought of as a generalization of the action where instead of thinking of a single unique path of say a particle, we have multiple paths[3]. The path integral is a sum over all these paths[3].

IV. REGGE CALCULUS

Regge Calculus is a formalism which involves decomposing space-times that are solutions to the Einstein Field Equations into building blocks called simplices[9]. A simplex is a generalization of a triangle[4] shown in figure 2. This decomposition of space-time into triangular building blocks is called a triangulation. Triangulating a space involves gluing together the simplices in a specific manner[2]. The simplices are glued together such that the curvature is restricted to regions of space-time[2]. The triangulation of space-time in classical Regge Calculus uses simplices with a specific and consistent edge length[2]. The triangles are flat and thus locally have a curvature of zero. At a single vertex, multiple triangles meet and it is with respect to this common vertex that curvature is localized[5]. In figure 3 we see a vertex with a certain number of triangles. The angle δ is a result of the curvature of the space[6] at that location being triangulated. From this, one can obtain the so called Regge action[2] after making the following substitution:

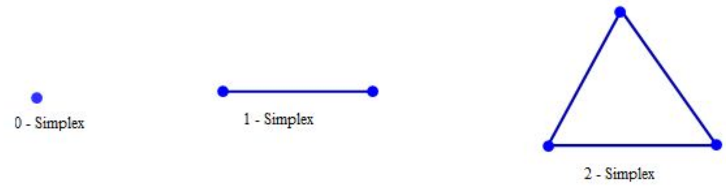

FIG. 2. Diagram showing n-simplices, where n is the number of dimensions.

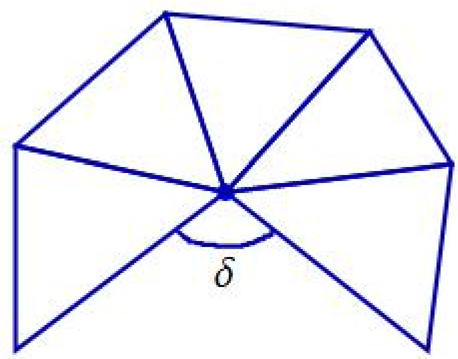

FIG. 3. Triangles sharing a common vertex

$$\frac{1}{2}\int d^n x \sqrt{-g} R \to \sum_{j \in T} \delta_j \tag{14}$$

$$\int d^n x \sqrt{-g} \to \sum_{j \in T} V_j \tag{15}$$

We are summing over the deficit angles δ_j and the volume V_j [2]of the region around each vertex associated with each deficit angle. The Regge action (discretized version of the Einstein-Hilbert action)[2] is thus:

$$S_R[l_j^2] = \sum_{j \in T} (k\delta_j - V_j \lambda) \tag{16}$$

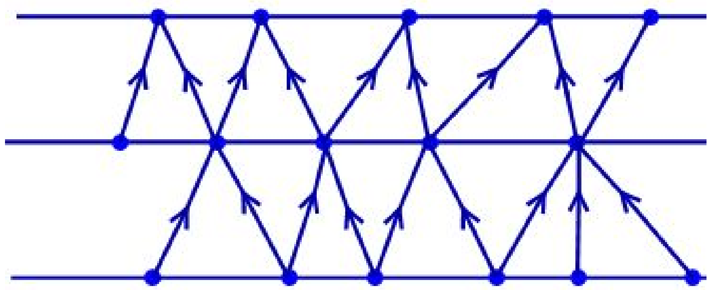

FIG. 4. Diagram showing an example of a triangulation of a 1+1 dimensional universe.

where $\lambda = k\Lambda$ and $k = 1/8\pi G$ and V_j is the volume of the region obtained by counting the triangles.

This means of studying General Relativity has been successful for some time now and this idea has been adopted to quantize gravity in previous approaches, particularly the the theory of Euclidean Dynamical Triangulation (EDT). This approach to quantizing gravity was developed and explored in the 1990s. It accepted all possible triangulations of space-time. This highly democratic nature of the theory appeared to be responsible for its failure. The theory failed to reproduce back the 3+1 dimensional classical universe that we currently observe when it was supposed to. It instead obtained an infinite dimensional universe as space-time condensed on a few vertices. Some simulations also lead to a 2 dimensional universe as space-time polymerized into 1+1 dimensional branches. After some time, a new theory using somewhat a similar idea was developed. Simulations have reproduced the classical 3+1 dimensional universe and it has predicted that the universe is 1+1 dimensional at the Planck scale and is fractal at the quantum level. This theory, called Causal Dynamical Triangulation is the focus of this paper and it will be explained in the next section. The 1+1 dimensional simulation that we are developing will be explained in the next few sections of the paper.

V. CAUSAL DYNAMICAL TRIANGULATION

Causal Dynamical Triangulation (CDT) quantum gravity is different from EDT quantum gravity in various ways. One of the main assumptions of CDT is the existence of microcausality. What this means is that causality is no longer considered just as a global property of space-time, but something that is local. It implements this assumption by making the simplices Lorentzian[2]. That is, the simplices no longer just have space-like edges as in EDT, but they also have time-like edges as well. The simplices are then glued such that their time-like edges are always future pointing. An example of a resulting space is shown in figure 4. The triangles have the same squared spatial edge lengths l_s^2, but their time-like edge lengths squared is $l_t^2 = -l_s^2$.

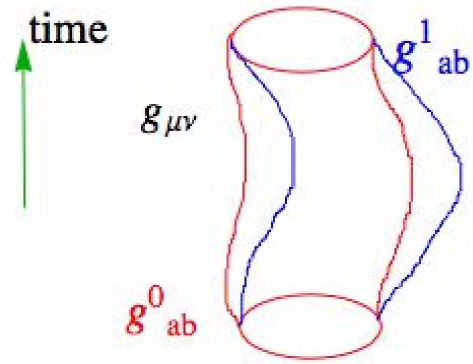

FIG. 5. Diagram showing history of geometries.

Now, in CDT, the Path integral is done over geometric histories[9] rather than over paths. The propagator developed earlier can be modified to do a sum over the history of geometries. One does this by using the Einstein-Hilbert action[9]. The modification is:

$$A[g^0_{ab} \to g^1_{ab}] = \int \mathscr{D}g_{ab} e^{iS_{EH}[g]} \tag{17}$$

What happens here is that the propagator takes into consideration different possible geometries beginning with an initial geometry g^0_{ab} to a final geometry g^1_{ab} (latin indices represent spatial geometries and greek indices represent space-time)[9]. This is shown in figure 5. In CDT, this integral has to be discretized and this is done by making the integral into a sum which gives us:

$$\int \mathscr{D}g_{ab} e^{iS_{EH}[g]} \to \sum_{T\in\mathscr{T}} \frac{1}{C(T)} e^{iS_R(T)} \tag{18}$$

where $1/C(T)$ is the measure on the space of triangulations, $C(T)$ is the size of the automorphism group[2] of the triangulation T and $\mathscr{T}$ is the space of all triangulations[2]. This makes understanding the geometries from a combinatorial perspective possible thus making the problem workable using computers. In computer simulations, the configuration space of all triangulations is sampled to obtain an approximation to the path integral[2]. The discretization is however not yet complete because we need to handle the imaginary part of the sum. This part is handled using what is called a Wick rotation[2] which will be explained further in the next section.

A. Wick Rotation

The Wick rotation takes a Lorentzian geometry to its associating Euclidean geometry[2]. This is done by making the substitution $-l_t^2 \to l_s^2$[8] for the time-like edges. As a result, the Regge action corresponding to a Lorentzian triangulation is replaced by the corresponding Euclidean action[2]. This means that we make the following substitution:

$$iS_R[T_l] \to -S_E = iS_R[T_E] \tag{19}$$

where S_E is the Euclidean action, T_l is the Lorentzian triangulation and T_E is the Euclidean triangulation[2]. In effect, this approach converts the path integral into a partition function and we can handle the space of triangulations the way we would handle a system in statistical mechanics[2]. The result is that we get:

$$Z = \sum_{T \in \mathscr{T}} e^{-S_E[T]} \tag{20}$$

We set $\frac{1}{C(T)} = 1$ because of the observation that general observations in critical phenomena tells us the choice of measure does not affect the continuum limit of the theory[2]. With this problem now in a statistical mechanics form, one can now implement the Metropolis algorithm to do Monte Carlo simulations. This will be explained in more detail in the section that explains the simulation.

In 1+1 dimensional CDT, we need the 2 dimensional Einstein-Hilbert action[9]. This is just:

$$\frac{1}{2}k \int d^2x \sqrt{-g}R = \sum_{j \in T} kV_j\delta_j = 2\pi\chi \tag{21}$$

where χ is called the Euler Characteristic[4]. We also have:

$$k \int d^2x \sqrt{-g}\Lambda = \sum_{j \in T} \lambda V_j = k_2\lambda N_2 \tag{22}$$

where $\lambda = k\Lambda$ and N_2 is the number of triangles in a given triangulation. The Euler Characteristic is known as a topological invariant[4]. What this means is that when we compute its value, this number will remain the same for any space that is topologically the same as the space for which we computed the number initially.

From the above equations, we have the Euclidean action in terms of the number of Lorentzian triangles in a triangulation. This gives:

$$S_E[T_l] = 2\pi\chi - k_2\lambda N_2 \tag{23}$$

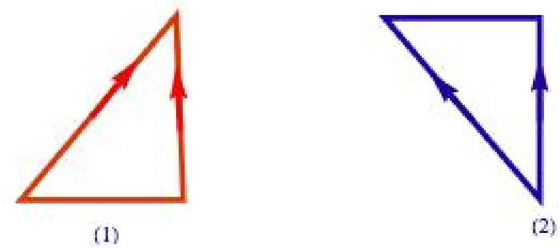

FIG. 6. Diagram showing two types of triangles - up pointing (type 1) and down pointing (type 2). The time-like edges are distinguished from the space-like edges with the arrows shown. Time is taken to go from bottom to top.

Since our topology is fixed in this simulation ($S^1 \times S^1$), then the term $2\pi\chi$ is constant because χ is a topological invariant[2]. This term would not be constant if we allow the topology to vary[2]. The partition function then becomes:

$$Z = \sum_{T \in \mathscr{T}} e^{2\pi\chi - k_2\lambda N_2} = \xi \sum_{T \in \mathscr{T}} e^{-k_2\lambda N_2} \tag{24}$$

where $\xi = e^{2\pi\chi}$ is just a constant in the simulation[9]. There is a critical value λ_c such that at a certain amount above this value, the partition function converges[3] and at a certain amount below this value, the partition function diverges[3]. This value seems to vary and this will be found for this simulation.

VI. SIMULATION

In this simulation, we only looked at the 1+1 dimensional universe. In this simulation, the 1+1 dimensional universe is decomposed into 2 dimensional simplices - one with two time-like edges and a space-like edge[9]. These are shown in figure 6. In the simulation, the data

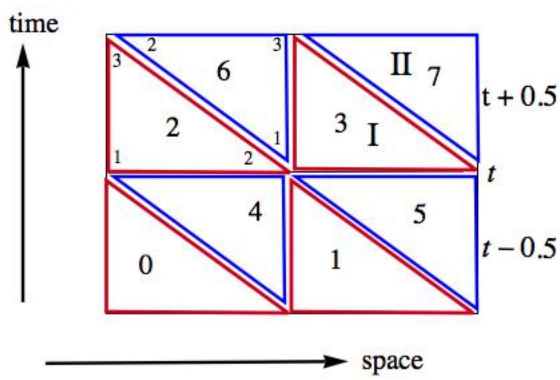

FIG. 7. Diagram showing an example of a triangulation with labelled triangles. Type I triangle vertices are numbered beginning from left to right and up, while type II triangle vertices are labelled from down to upper left to upper right.

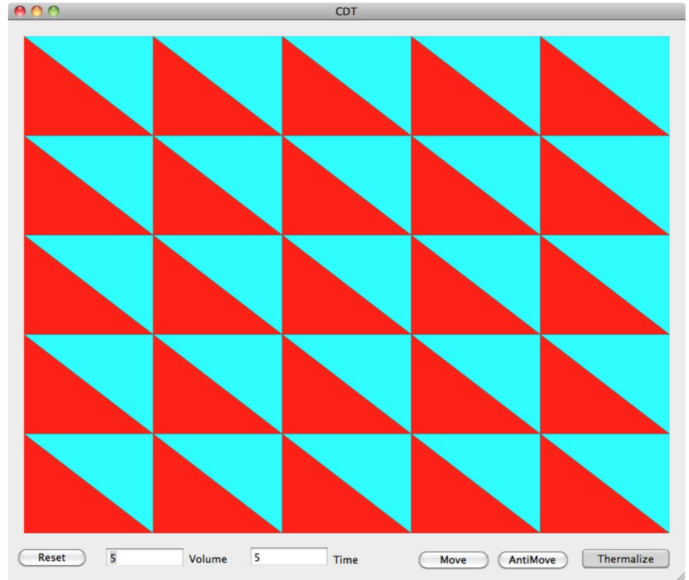

FIG. 8. Figure showing 1+1 dimensional universe before any move or anitmove was made.

structure stores this kind of information which turns out to be very useful to implement the simulation. The up pointing triangles have their spatial edges on a past time slice and the down pointing triangles have their spatial edges on a future time slice. Those time-like edges are always future pointing as shown in the diagram. In this simulation, periodic boundary conditions were chosen. What this means is that the final time-slice and the first time-slice are connected so that every vertex of the beginning time-slice are identified with every vertex on the final time-slice. The toroidal topology $S^1 \times S^1$ was used to achieve the periodic boundary conditions.

In our simulation, we used a data structure that stores certain information about each triangle in the triangulation. This information is stored in an array (called a Hash Table). In this array, the following information about the triangulation is stored: type of triangle (that is, if it is up pointing or down pointing), the time slice which it is located, the vertices as well as the neighbors of the triangles. Figure 7 shows an illustration. Using a labeling scheme very much like the one in figure 7, we were able to store information about the triangles in a triangulation. Each triangle was given a key starting from 0 to $n-1$. The type of the triangle (which is either type I or type II) was also stored. The neighbors for the different triangles were also stored. The neighbors were determined by looking at the triangle (the key) opposite a given vertex of another triangle. Take for example, the triangulation in figure 7. Look at the triangle labelled 3 in the figure. This triangle is within time slice labelled $t + 0.5$ and it is of type I. Vertex 1 for this triangle has the triangle labelled 7 opposite to it. Similarly, vertex 2 has the triangle labelled 6 opposite it and vertex 3 has the triangle labelled 5 opposite it. An array for the triangle labelled 3 would thus look like $T3 = \{I, t + 0.5, 1, 2, 3, 7, 6, 5\}$. In general, the array structure for any triangle takes the form $Tn = \{type, time, p1, p2, p3, n1, n2, n3\}$ where $p1, p2, p3$ are the vertices of the triangle and $n1, n2, n3$ are the neighbor of vertex 1, neighbor of vertex 2 and neighbor of vertex 3 respectively. This entire structure is composed strictly of integers. The integer assignments are in line with the idea of reducing the problem to a counting problem. This data structure is then taken advantage of to do the combinatorial moves to split a vertex and add two triangles and antimoves to remove two triangles. The moves are done by randomly picking a vertex and splitting it, adding two triangles in the gap left behind where ever the vertex was split. The antimoves involve randomly picking vertices and deleting the triangles associated with the vertex. This is the same thing as doing a random walk through space-time. When the moves and antimoves are done, the arrays are repeatedly updated for every move and antimove. The universe is then made to evolve by repeatedly doing moves and antimoves rapidly and the size of the universe is measured every time over every 100 such combinatorial move.

In figure 8, we have the universe before any move or antimove and in figure 11 we have the universe after the moves and anitmoves are done together by the computer.

In figure 9, we see the result after doing a move on the triangulated 1+1 dimensional universe. It is visible that more triangles are added to the universe after a move is made. Two triangles are added after a move. They are added after a vertex is randomly selected and split. In fact, after each move, two triangles are added (one of type I and another of type II). The other kind of combinatorial move done on the universe is called an antimove. This

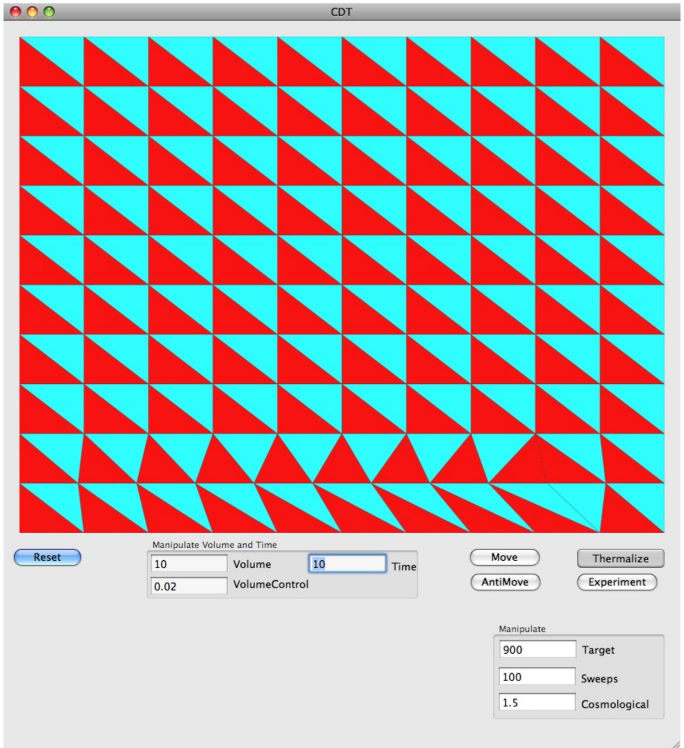

FIG. 9. Figure showing result after a Combinatorial move was made.

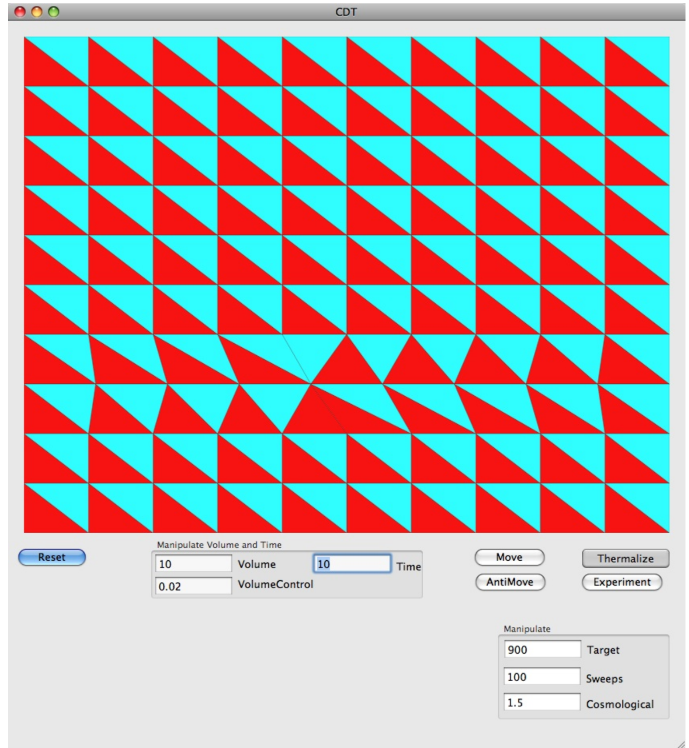

FIG. 10. Figure showing result after a Combinatorial anti-move was made.

is shown in figure 10. An antimove deletes two triangles associated with a randomly selected vertex.

After the moves and antimoves started working properly, we then implemented the Metropolis algorithm. The Metropolis Algorithm does a selective sampling of the vertices[1]. This selection is done based on importance[2]. The importance of a sample is determined based on a probability based on its Boltzmann weight (which is e^{-S_E} in this case)[1]. This is important in the Monte Carlo integration[1]. In general, in a Monte Carlo simulation, one finds what is called the Monte Carlo average of N measurements of some observable $\mathscr{O}$[9] determined by:

$$< \mathscr{O} >= \frac{\sum\limits_{T\in\mathscr{T}} \mathscr{O}[T]e^{-S_E[T]}}{Z} \approx \frac{1}{N}\sum_{n=1}^{N}\mathscr{O}[T_n] \tag{25}$$

Now, with the Metropolis Algorithm, we use a detailed balance condition[9]. This occurs when a random walk through our configuration space reaches an equilibrium. This equilibrium is reached when:

$$\mathscr{P}[T_n]\mathscr{P}[T_n \to T_{n+1}] = \mathscr{P}[T_{n+1}]\mathscr{P}[T_{n+1} \to T_n] \tag{26}$$

where $\mathscr{P}[T_n]$ is the probability of getting the nth triangulation and $\mathscr{P}[T_{n+1}]$ would be the probability of getting the $n+1$ triangulation[2]. So we have:

$$\frac{\mathscr{P}[T_n \to T_{n+1}]}{\mathscr{P}[T_{n+1} \to T_n]} = \frac{\mathscr{P}[T_{n+1}]}{\mathscr{P}[T_n]} = \frac{e^{-S_E[T_{n+1}]}}{e^{-S_E[T_n]}} = e^{-\Delta S_E} \tag{27}$$

What the Metropolis Algorithm does is accept or reject one of the Monte Carlo moves based on the change in the action ΔS_E[9]. This is done using the following criteria:

$$\mathscr{P}[T_n \to T_{n+1}] = \begin{cases} e^{-\Delta S_E} & , \Delta S_E > 0 \\ 1 & , \Delta S_E \leq 0 \end{cases}$$

The volume of the universe was controlled with the addition of a term δS_E to the action[2]. We have:

$$\delta S = \epsilon \delta N = \epsilon |N_2 - V| \tag{28}$$

where V is the volume of the universe. We then make the substitution:

$$S \to S + \delta S = \lambda N_2 + \epsilon \delta N \tag{29}$$

The addition of this extra term to the action was used to suppress transitions to extreme volumes[9].

After the Metropolis algorithm was implemented, multiple simulations were ran to find how many data points were best suitable to use for our calculations. After this was done, we ran eight simulations at a spatial volume of 100 and we went up to 100 time steps. We then used the data to find a critical value for the reduced cosmological constant. The critical value is where the fluctuations are stable[9]. After this value was found, we computed the average size of the 1+1 dimensional universe.

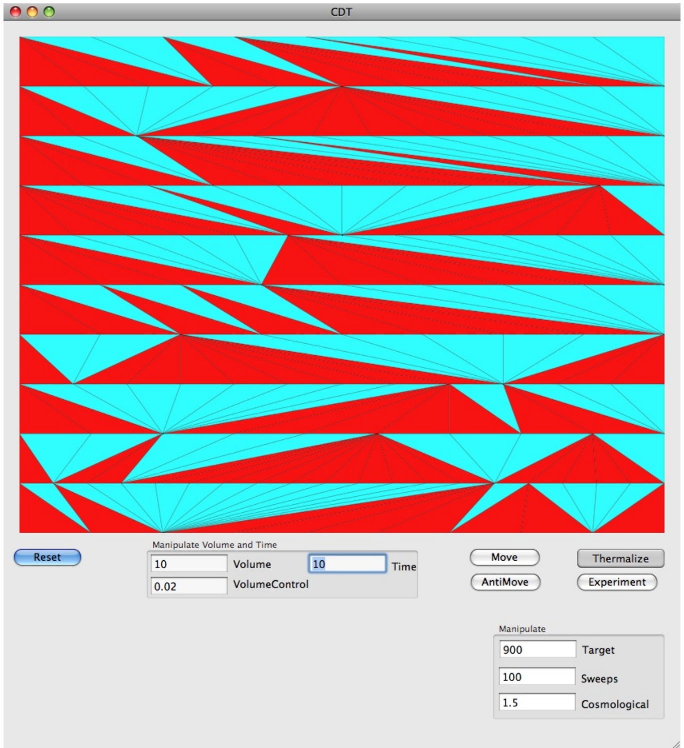

FIG. 11. Figure showing 1+1 dimensional universe after move and anitmove are done together.

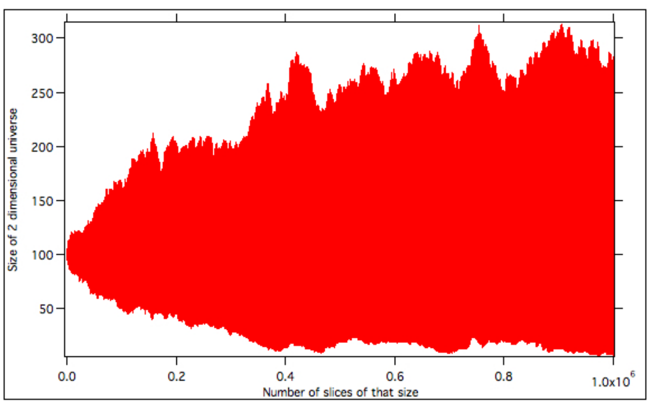

FIG. 12. Figure showing the size of the universe versus the number of slices at a certain size.Here, the reduced cosmological constant was 1.2.

VII. RESULTS AND ANALYSIS

The results we obtained yielded significant fluctuations. This already hints that the value of the actual cosmological constant must be quite small. Figure 12 shows the fluctuations in the size of the universe at a specific cosmological constant.
Notice here how the fluctuations start from some point then diverge away from the point and as it goes further away from that point, it seems to diverge less. The part closer to this source is due to certain artificial conditions

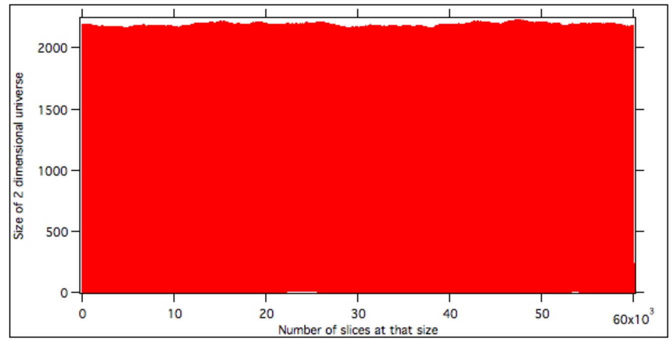

FIG. 13. Figure showing stable fluctuations of the size of the 2 dimensional universe at a specific reduced cosmological constant. This is a data reduction of the data represented by figure 12.

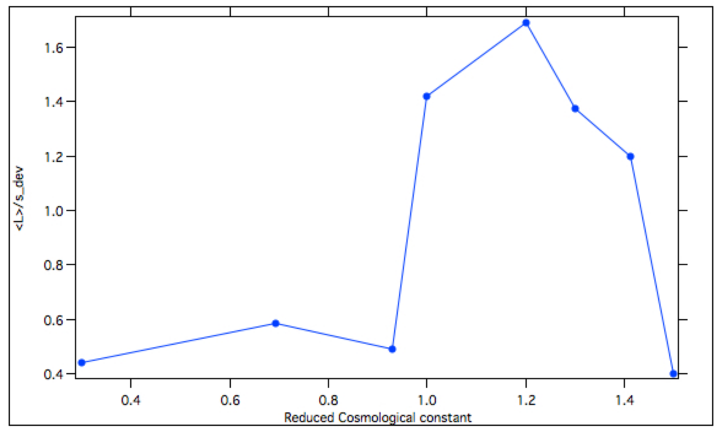

FIG. 14. Figure showing the ratio of the average size of the universe and the standard deviation of the size of the universe versus the reduced cosmological constant.

within the simulation. As the fluctuations continue however, for a stable universe, the fluctuations stop diverging and stabilize. It is this region of stability (Fig. 13) that is considered useful data and hence is what is used to figure out the critical value of the reduced cosmological constant.

The best critical value for the reduced cosmological constant we found in our simulations was 1. We obtain this critical value by noting from analytical results in CDT in 1+1 dimensions that $\mu_L/\sigma_L = \sqrt{2}$ where μ_L is the average size of the universe and σ_L is the standard deviation of the size of the universe. Using this result as a guide, we ran our simulations at different values for the reduced cosmological constant. Figure 14 shows the results. Notice that after the peak which occurred at a reduced cosmological constant of 1.2, there was a fall in the ratio. The ratio rises to some point then falls, more data would most certainly give a much better idea as to what is really happening with regards to how the ratio and hence the fluctuations vary with the cosmological constant and so more work needs to be done on that.

VIII. CONCLUSIONS

These results imply large quantum fluctuations in the 2 dimensional universe which in turn implies a small value for Einstein's cosmological constant. We obtain a critical value for the cosmological constant to be 1 and the best dimensionless value for the average size of the universe is 104. If we let the length of each triangle be a(remember they all have the same spatial edge length), then the physical average size of the universe based on these results is thus just $104a$. Overall, the 2 dimensional universe's size has a spread $\sigma_L = 73$. There is still much work to be done. We need to understand how much varying the cosmological constant would vary the fluctuations, and we need to understand how the theory behaves in higher dimensions. These are just a few of the things we need to explore in the future.

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