/. 3D

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A simulation of the mutual orbital motion of a point and a stick of equal masses in three dimensions, described as the 3D /.-body problem, is presented herein. This simulation uses forces and torques arrived at by exact integration over the length of the stick to produce equations of motion which describe both the orbits of the point and the stick as well as the independent revolution of the stick. Angular momentum and total energy are calculated for the simulation and their conservation is used as a check to insure that the system produces physically plausible results. Several simple cases of the system were tested to make sure they act as expected. More complex simulations were tested to examine facets of of the 3D /.-body problem such as the transfer of angular momentum between the orbital and rotational motion. Simulations were also created to examine how the different initial spin velocities, ω_ϕ and ω_θ, interact to affect the evolution of the system.

I. INTRODUCTION

Problems in orbital mechanics have been popular for a long time. The most simplistic version of this is the 2-body problem, wherein the orbit of two spheres in space is examined. The solution to the 2-body problem has been precisely known for hundreds of years. More complex versions of this problem are the 3-body and n-body problem, which have three or more spheres in space. Through long and extensive study these problems have been found to be infinitely complex, and as such can only be exactly solved in certain special cases.

More recently Frank King [1] has done work on a problem which is an intermediate between the 2-body and 3-body problems. He studied the orbits of a point mass and a line segment mass confined to a two dimensional plane, a situation he refers to as the /.-body problem. The forces and torques involved in this problem were precisely integrable, which made it possible to find exact solutions for all different versions of this system.

The problem discussed in this paper is an extension of King's work in the form of the /.-body problem in three dimensions. In order to study this problem I used Mathematica to set up a computer simulation which will solve the equations of motion for different systems defined by different initial positions and velocities, for a certain amount of time into the future. In order to insure that the simulation gives physically plausible results the total energy and angular momentum of the system will be calculated to see if they are conserved. Using this simulation I have examined a number of facets of this type of system. One specific point of interest was whether angular momentum can be transferred between the orbital motion of the system, and the rotational motion of the stick.

Space programs are no longer solely focused on the largest bodies in the system, which are all spherical. They are also looking at asteroids and smaller moons which are distinctly non-sphereical. One example of this was in early 2001 when NASA had the NEAR spacecraft orbit and eventually land on the asteroid Eros, which has a somewhat linear shape. Thus despite seeming like a very abstract sort of problem, this research is a step on the way to applicable knowledge.

II. THEORY

In order to simulate the mutual orbital motion of a point and a stick we calculated the force between the two objects and the torque on the stick. However we first needed to set up our system in a logical manner as is seen in Fig. 1. We defined the stick to have a length of $2R$ and a linear density of $\lambda = m_s/2R$, where m_s is the mass of the stick which is equal to m_p the mass of the point. We also defined the stick to have a moment of inertia $I_{cm} = m_s R^2/3$. Our system sits in an inertial reference frame with the point located by the vector $\vec{r}_p$ and the center of the stick located by the vector $\vec{r}_s$ in the

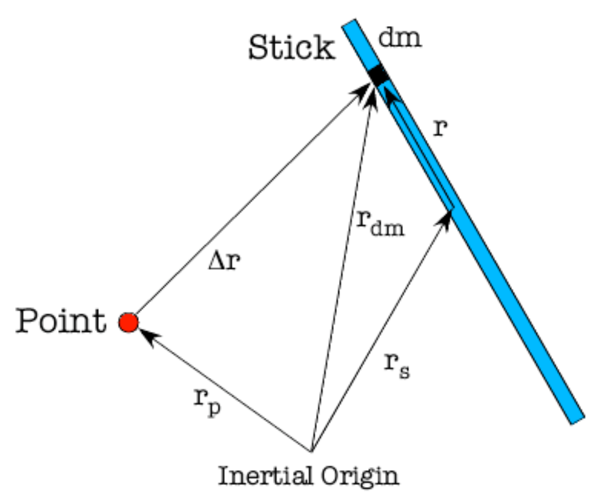

FIG. 1: This is the schematic layout of my system showing the point, the line, and all of the relevant vectors for calculating the forces and torques of the system.

form

$$\vec{r}_p = \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix}, \vec{r}_s = \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix}, \quad (1)$$

both of which originate at the inertial origin. In order to define the direction that the stick is pointing we are using the Eulerian angles ϕ and θ in the body frame of the stick [2] instead of polar spherical angles. The Eulerian angle ϕ is the angle that the inertial reference frame is rotated about the z-axis, and the Eulerian angle θ is the angle the reference frame is rotated about the new x-axis. At the end of the rotations the stick will lie exactly along the z-axis of the new body reference frame. Using rotation matrices [2] we moved these angles into the inertial reference frame so we could define a vector along the stick from its center to a mass element dm as

$$\begin{aligned}\vec{r} &= r \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} r\sin\theta\sin\phi \\ -r\sin\theta\cos\phi \\ r\cos\theta \end{pmatrix},\end{aligned} \quad (2)$$

where r is the distance to the mass element dm. Next we set up a vector connecting the origin to the mass element dm by

$$\vec{r}_{dm} = \vec{r}_s + \vec{r} = \begin{pmatrix} x_s + r\sin\theta\sin\phi \\ y_s - r\sin\theta\cos\phi \\ z_s + r\cos\theta \end{pmatrix}, \quad (3)$$

and use this to calculate the vector from the point to dm as follows

$$\Delta\vec{r} = \vec{r}_p - \vec{r}_{dm} = \begin{pmatrix} x_p - x_s - r\sin\theta\sin\phi \\ y_p - y_s + r\sin\theta\cos\phi \\ z_p - z_s - r\cos\theta \end{pmatrix}. \quad (4)$$

Using these vectors we will also calculate $\Delta r = \sqrt{\Delta\vec{r}^2}$ and $\Delta\hat{r} = \frac{\Delta\vec{r}}{\Delta r}$.

Now that our system is fully described we calculate the forces and torques. The force on the stick due to the point is calculated by the integral

$$\vec{F}_s = \int_{-R}^{R} \frac{Gm_p\lambda}{\Delta r^2} \Delta\hat{r} dr, \quad (5)$$

which is analogous to the calculation performed by King [1] for the two dimensional version of the same problem. Newton's third law tells us that the force on the point due to the stick, $\vec{F}_p$, is equal and opposite to the force $\vec{F}_s$ calculated above. The torque on the stick is also calculated analogously to King with the integral

$$\vec{\tau}_s = \int_{-R}^{R} \frac{Gm_p\lambda}{\Delta r^2} \vec{r} \times \Delta\hat{r} dr. \quad (6)$$

Both the force and the torque were integrated exactly using Mathematica to have explicitly real values in vector form. However as these results are highly complicated they will not be included here.

The next step in simulating the orbital motion of the point stick system is calculating the equations of motion for our two bodies. First we calculate the translational motion of the stick. To do this we use Newton's second law, $\vec{F} = m\vec{a}$, and calculate each component of the motion separately as follows:

$$\begin{aligned}\vec{F}_s(x,t) &= m_s \frac{d^2 x_s}{dt^2}, \\ \vec{F}_s(y,t) &= m_s \frac{d^2 y_s}{dt^2}, \\ \vec{F}_s(z,t) &= m_s \frac{d^2 z_s}{dt^2}.\end{aligned} \quad (7)$$

Similarly, the translational equations of motion for the point are:

$$\begin{aligned}-\vec{F}_s(x,t) &= m_p \frac{d^2 x_p}{dt^2}, \\ -\vec{F}_s(y,t) &= m_p \frac{d^2 y_p}{dt^2}, \\ -\vec{F}_s(z,t) &= m_p \frac{d^2 z_p}{dt^2}.\end{aligned} \quad (8)$$

To calculate the rotational equations of motion for the stick we move from the inertial reference frame into the body frame of the stick, wherein the stick always lies along the 3-axis. To translate the torque $\vec{\tau_s}$ from the inertial frame into a body frame torque $\vec{\tau_{sb}}$ we multiply $\vec{\tau_s}$ by rotation matrices [2], made valid by the fact that we are using Eulerian angles, as follows:

$$\vec{\tau}_{sb} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{\tau}_s. \quad (9)$$

In the body frame we know that [2]

$$\vec{\tau}_{sb} = \frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} \quad (10)$$

where $\vec{L}$ is the angular momentum of the stick and $\vec{\omega}$ is the angular velocity of the stick. Since we are in the body frame we also know that $L_i = I_i\omega_i$, and by inserting this into Eq. 10 we can determine the three components of the torque to be [2]

$$\begin{aligned}\vec{\tau}_1 &= I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3, \\ \vec{\tau}_2 &= I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1, \\ \vec{\tau}_3 &= I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2.\end{aligned} \quad (11)$$

However, the stick is fixed as the 3-axis in the body frame so we know that the components of the moment of inertia are $I_{cm} = I_1 = I_2$ and $I_3 = 0$, and the components of the angular velocity are: $\omega_1 = \dot{\theta}$, $\omega_2 = \dot{\phi}\sin\theta$, and

$\omega_3 = \dot{\phi}\cos\theta$. Using these values we get the following simplification of Eq. 11

$$\vec{\tau}_1 = I\ddot{\theta} - I\dot{\phi}^2 \sin\theta\cos\theta,$$
$$\vec{\tau}_2 = I\ddot{\phi}\sin\theta + 2I\dot{\phi}\dot{\theta}\cos\theta, \quad (12)$$
$$\vec{\tau}_3 = 0.$$

Thus the equations of motion for the whole system are described in Eq. 7, Eq. 8 and Eq. 12. To designate a specific system we then set initial conditions for: the positions of the stick, (x_s, y_s, z_s), the orientation of the stick, (ϕ, θ), the translational velocities of the stick, (v_{xs}, v_{ys}, v_{zs}), and the rotational velocities of the stick, $(\omega_\phi, \omega_\theta)$. The initial conditions of the point are determined by fixing the center of mass of the system. Once we have these equations we apply Mathematica's NDSolve command to the equations and the initial conditions, and from this Mathematica numerically integrates to completely describe the evolution of the system for a set period of time.

To check if our simulation was producing physically plausible results we also calculated the total energy of the system and the total angular momentum to be sure that both of these quantities are conserved. In order to calculate the total energy of the system we began with the standard formula for gravitational potential energy,

$$U = -\frac{GMm}{r}. \quad (13)$$

For our purposes we modified this formula to account for the extended mass of the stick to obtain,

$$U = \int_{-R}^{R} \frac{Gm_p\lambda}{\Delta r} dr. \quad (14)$$

Next we calculated the kinetic energy T modifying the standard $T = \frac{1}{2}mv^2$ for translational motion and the standard $T = \frac{1}{2}I\omega^2$ for rotational motion to fit the system as follows;

$$T = \frac{1}{2}m_p v_p^2 + \frac{1}{2}m_s v_s^2 + \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2. \quad (15)$$

Thus the total energy of the system is the sum of Eq. 14 and Eq. 15. The angular momentum of our system will be the sum of the angular momentum of the point and stick orbiting each other, $\vec{L}_o$, and the angular momentum from the rotation of the stick, $\vec{L}_r$. $\vec{L}_o$ was simply calculated from the standard formula $\vec{L} = m\vec{r} \times \vec{v}$ to be,

$$\vec{L}_o = m_p\vec{r}_p \times \vec{v}_p + m_s\vec{r}_s \times \vec{v}_s. \quad (16)$$

In order to calculate $\vec{L}_r$, we used rotation matrices to get the angular velocities from the body frame into the inertial reference frame and then multiplied them by the moment of inertia to get,

$$\vec{L}_r = I \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ 0 \end{pmatrix}. \quad (17)$$

TABLE I: The set of initial conditions for all of the simulations are listed below. Note that distances here are in multiples of the stick length, 2R.

Inits	A	B	C	D1	D2	D3	D4	E
x_s	0	0	0	0	0	0	0	0
y_s	$\frac{-3}{4}$	$\frac{-3}{4}$	$\frac{-3}{4}$	$\frac{-3}{4}$	$\frac{-3}{4}$	$\frac{-3}{4}$	$\frac{-3}{4}$	$\frac{-1}{2}$
z_s	0	0	0	0	0	0	0	$\frac{1}{2}$
ϕ	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
θ	π	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
v_{xs}	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$
v_{ys}	0	0	0	0	0	0	0	0
v_{zs}	0	0	0	0	0	0	0	0
ω_ϕ	0	0	0	$\frac{\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0
ω_θ	0	0	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	$\sqrt{2}$

The angular momentum and total energy of the system are both graphed against time for each set of initial conditions examined to ensure that these quantities are conserved. It is important to remember that the three components of the angular momentum are conserved separately at different values.

III. RESULTS & ANALYSIS

To begin analyzing the validity of our simulation of the 3D /.-body problem in three dimensions, we first look at a simulation of simplest possible system that can be described. This system, Sim. A, has the point and the stick both in the xy-plane with the stick in a completely vertical orientation. The initial conditions which describe this system, and every other system discussed in this paper can be found in Table I. This system should behave like two point masses orbiting each other in a plane because the center of mass of the stick lies on the plane, and the stick has no initial spin. When we check this simulation we see that the total energy varies only on the order of ten parts per trillion, and the three components of the angular momentum also vary on the order of ten parts per trillion. Variations this small are almost certainly the results of slight errors in the numerical integration, and as such these quantities can be considered to be conserved. Since all of the subsequent simulations have variations in total energy and angular momentum on approximately the same order we know that all of these simulations are at least physically plausible. The strobed representation of Sim. A in Fig. 2 shows us that this system imitates the orbits of point masses in a plane very nicely. This strobed image, as well as all of the subsequent ones show the passage of time as changing color, from blue at the beginning to red at the end. Additionally there is an equal interval of time between each pair of adjacent points and sticks. First we see that both the point and the stick stay perfectly in the xy-plane for the

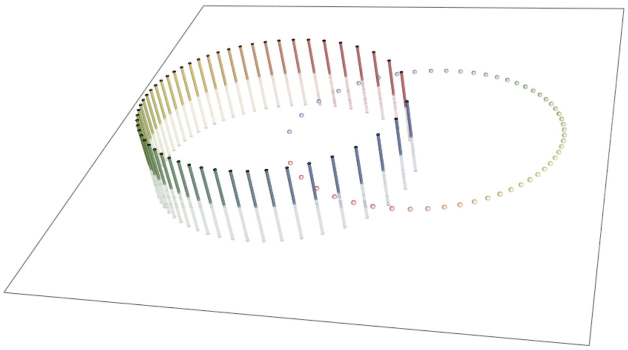

FIG. 2: A strobed representation of Sim. A, shown with a translucent xy-plane. In this figure adjacent points and adjacent sticks are separated by equal amounts of time. Additionally, the passage of time is represented by changing colors starting from blues at the beginning to reds at the end, thus a point and stick of the same color represent the same moment in time.

duration of the single orbit shown here. Next we can see that the point and the line appear to be obeying the first and second of Kepler's Laws of Planetary Motion. In Fig. 2 both objects are clearly moving in ellipses, and furthermore the point and the stick move faster when they are near to each other, and slower when they are far apart.

The next test case for our simulation is to mimic the planar case of the system as examined by King [1]. This simulation, Sim. B, will begin with the point and the stick on the xy-plane, this time with the stick in a horizontal orientation. We can see from the strobed picture of Sim. B in Fig. 3 that both the point and the stick stay exactly on the xy-plane, and additionally this system also seems to obey Kepler's Laws. Since this system had no initial rotational velocity, it is interesting to note that orbiting the point induces the stick to spin. This is due to what A.J. Maciejewski refers to as "mutual coupling between orbital and rotational motion" [3]. In essence this means that it is possible for angular momentum to be transferred from the orbital motion of the system to the rotational motion of the stick and vice versa.

Our next simulation, Sim. C, will be attempting to achieve a visible case of angular momentum being transferred from the rotational motion of the stick to the orbital motion of the system. To do this the initial set up will be very similar to Sim. B, however the stick will start with an initial rotational velocity about the imaginary line connecting the point and the center of the stick. The strobed picture in Fig. 4 of Sim. C shows the point moving slightly above the xy-plane on one side of its orbit, and slightly below on the other side of the orbit. This oscillation of the point about the xy-plane is due to a transfer of momentum from rotational to orbital motion, as the spinning stick pulls the point above and below the plane. The center of mass of the stick also moves out of the plane as a result of this, however this is much harder to observe. This effect seems to be slightly changing the orbital plane of the system.

The next thing I examined is how the two different types of initial spin velocity of the stick interact with each other. The two initial spin velocities of stick are ω_ϕ which is the spin about the z-axis and ω_θ which is the spin about the planar axis perpendicular to the initial ϕ direction. To do this I have set up four related simulations, Sim. D1-D4, with all parameters the same except for the initial spin velocities, all which last for exactly the same amount of time. In all four simulations the magnitude of both ω_ϕ and ω_θ is equal to $\sqrt{2}/2$, the differences between the simulations is the directions of the spins. Thus Sim. D1 has both ω_ϕ and ω_θ positive, while Sim. D2 has both spins negative. Sim. D3 has only ω_ϕ as a positive spin, and Sim. D4 has only ω_θ as a positive spin. We can see in Fig. 5 that these conditions produce very different patterns of motion. In these figures we can see a number of different examples of the coupling between orbital and rotational motion. In Sim. D1 we see that the points dip below the plane on the close side of the orbit, whereas in Sim. D4 the points rise above the plane on the close side of the orbit. Thus it appears the direction of ω_θ controls where the point goes above and below the plane. However the more interesting observation about this is that in Sim. D2 and Sim. D3 the point stays almost perfectly in the plane. Thus it would appear that an initial ω_ϕ in the same direction as the orbital direction, as in Sim. D2 and Sim. D3, will prevent the point from oscillating about the xy-plane during its orbit. Conversely, an initial ω_ϕ in the opposite direction from the orbital direction, as in Sim. D1 and Sim. D4, will amplify, or at least not prevent, the

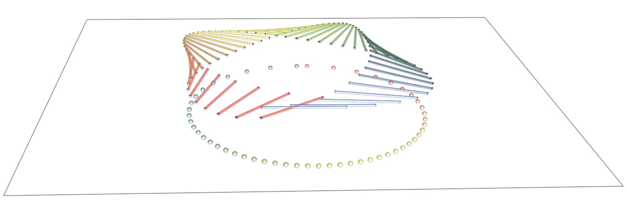

FIG. 3: A strobed representation of Sim. B, shown with a translucent xy-plane. This is a simulation of the planar case of the /.-body problem.

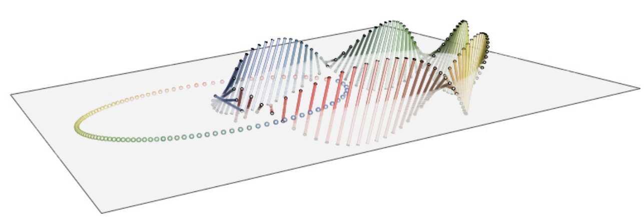

FIG. 4: A strobed representation of Sim. C, shown with a translucent xy-plane. This simulation shows that rotational motion of the stick can cause the point to move above and below the plane.

point's oscillations. Another facet of these simulations to note is that while they all last the same amount of time, they do not all terminate at the same point in their orbit. The simulations Sim. D2 and Sim. D3 conclude almost exactly one complete orbit in the given amount of time, whereas Sim. D1 and Sim. D4 terminate noticeably short of completing their first orbits. Thus it seems that an initial ω_ϕ in the same direction as the orbital motion will also cause the point and the stick to orbit about each other slightly faster, whereas an initial ω_ϕ in the opposite direction from the orbital motion will cause the orbits to be slightly slower. Since Sim. D2 and Sim. D3 complete nearly identical amounts of their orbits, and Sim. D1 and Sim. D4 also terminate in nearly identical places in their orbits, it would appear that the initial ω_θ direction has no effect on the speed of the orbit. Finally, we also see how the different initial spins cause the spiral patterns of the stick during the orbit to occur very differently.

All of the previous simulations have primarily confined the point and the stick to the near vicinity of the xy-

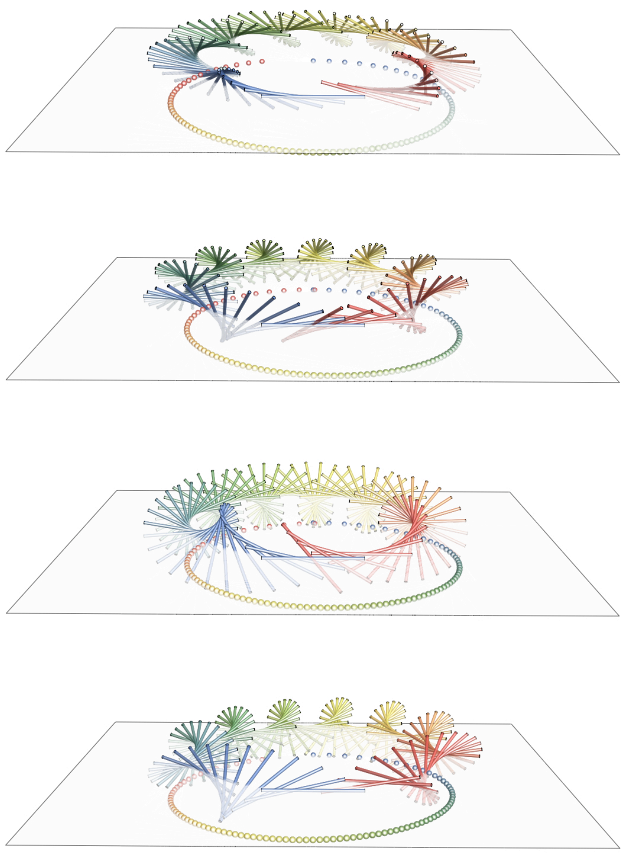

FIG. 5: Above, from top to bottom, are strobed representations of Sim. D1(+,+), Sim. D2(-,-), Sim. D3(-,+), Sim. D4(+,-). These four simulations show effects of the directions of the initial spin velocities ω_ϕ and ω_θ on the evolution of the system.

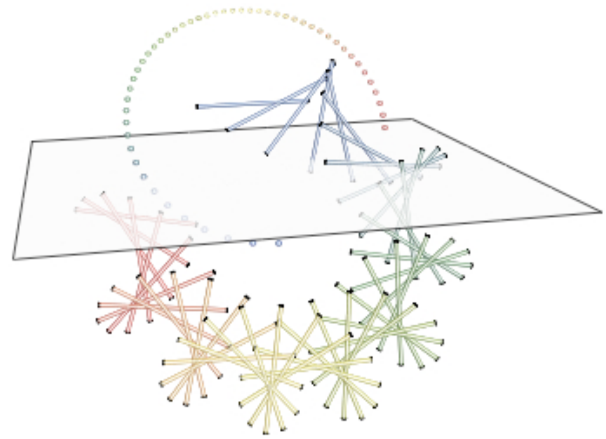

FIG. 6: A strobed representation of Sim. E, which is not initially confined to the plane. Note that the orbital motion appears similar to previous simulations with a different orbital plane.

plane, in order to make effects such as the oscillation of the point due to the rotation of the stick easier to study. Our last simulation, Sim. E, is mostly to show what can happen to the system when the point and the line are not initially confined to the plane. In Fig. 6 we can see that Sim. E looks very similar to some of the previous simulations, except for the fact that the point and the stick are moving in an orbital plane which is significantly tilted with respect to the xy-plane.

IV. CONCLUSIONS & FUTURE WORK

The first thing to ask is whether this simulation program really is a legitimate representation of the physical world. The fact that total energy and angular momentum are conserved, as well as how nicely our test cases conformed to expectations would seem to indicate that it physically reasonable, at least in the cases tested here. There are a few problem areas, most notably the fact that if the point and the stick ever come into contact the simulation gives them seemingly random trajectories. Thus it might be interesting to extend this simulation to include the physics of collisions. Another issue is that program refuses to integrate the equations of motion if the stick is initially set to be vertical by $\theta = 0$ instead of $\theta = \pi$. One final issue is that if the line traced by the initial orientation of the stick goes through the point, the system will either not integrate or produce a result in which the total system energy is not conserved.

From our simulations we were able to observe several facets of the 3D /.-body problem. It appears that angular momentum can be transferred between the orbital motion of the system and the rotational motion of the stick in a number of complex ways. Examples of this transfer of angular momentum include the stick acquiring rotational momentum from the orbit, and the point being forced to oscillate about the xy-plane due to the rotation of the stick. Furthermore it is interesting to note that the impact of the initial ω_ϕ of the stick has several different effects depending on whether or not it it is in the same or opposite direction as the orbital motion.

There are several interesting extensions of this work which could be done in the future. One possibility would be to use this simulation to acquire numerical data relating to these systems which could be used to determine more information about the 3D /.-body problem, such as whether it exhibits chaotic motion over long periods of time. It would also be interesting to perform this simulation with varying masses, stick lengths, and gravitational constants. One could also create the system with both attracting and repelling forces so as to study the electromagnetic 3D /.-body problem. A final possibility would be to replace the one dimensional stick with a higher dimensional figure such as a circle or alternatively replace the point with another stick. Most of these extensions should be useful in learning more about the problems, arising from non-spherical celestial bodies, which the space program is currently dealing with.

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[2] S.T. Thornton and J.B. Marion, *Classical Dynamics of Particles and Systems* (Thomson Learning, Inc., 2004).
[3] A.J. Maciejewski, *Celestial Mech. Dyn. Astr.*, **63**, 1 (1995).