Using toy dynamos to model the sun's magnetic field

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We attempt to shed light on the solar dynamo problem by idealizing the sun as a simple system of Faraday disk dynamos. We first investigate Bullard's simple single-disk system, showing that it is not capable of magnetic reversal. We then move to Rikitake's system of coupled Bullard dynamos and explore its characteristic chaotic current reversal. Afterwards, we address the characteristics of a damped Rikitake system, and show that we can regain at least periodic current-reversals. Finally, we discuss the possibility of modifying the Rikitake dynamo into a geometry more similar to that of the sun. To do this, we begin by moving the system coaxial, which unfortunately eliminates current reversal. We then derive a formula for calculating the mutual inductance for an arbitrary system, and explicitly calculate the mutual inductance for a disk and wire loop.

I. INTRODUCTION

Dynamos, devices that convert mechanical energy to an electrical current, are very important to today's world. Primarily used for power generation, they keep our society running. The type of dynamo this paper focuses on, however, is not the standard power-generating dynamo. Rather, we investigate a special type of system, the self-exciting dynamo. Self-exciting dynamos can, with only minute external magnetic fields, take mechanical energy and convert it to large currents and magnetic field. The importance of these dynamos is primarily scholarly, since common dynamos such as power generators are not self-exciting. However, the planets and sun do exhibit spontaneous and self-generated magnetic field reversals, which are thought to be caused by self-exciting dynamo action.

Contemporary solar physicists working on the solar dynamo problem use as many observed features of the sun in their models as can be computationally afforded in an effort to match existing data[1]. However, the actual solar features and mechanisms that cause the sun's polarity reversal are still not understood. Rather than attacking the problem magnetohydrodynamically and trying to develop a realistic model for the solar dynamo, we will focus on a proof-of-concept toy model using Faraday disks.

Simple models of this kind were first proposed for use in the geodynamo problem, beginning with Sir Edward Bullard in 1955[2]. Bullard's model is a single-disk, single-wire system, which has the advantage of being straightforward to analyze and was also the backbone of more complex models, the first of which was constructed by Tsuneji Rikitake in 1958[3]. Rikitake's model contained two Bullard dynamos coupled together.

A number of additional modifications were made to the Rikitake system, including the recent study by Dmitry Volobuev in 2006 that compares aspects of simple dynamo systems to various aspects of solar activity[4]. While he does suggest a mapping of the simple dynamo system onto the sun's zones, this process is not very refined. We make this mapping more explicit by investigating the results of changing the geometries of the disk dynamo systems in an attempt to make a more convincing case for its similarity to the solar cycle.

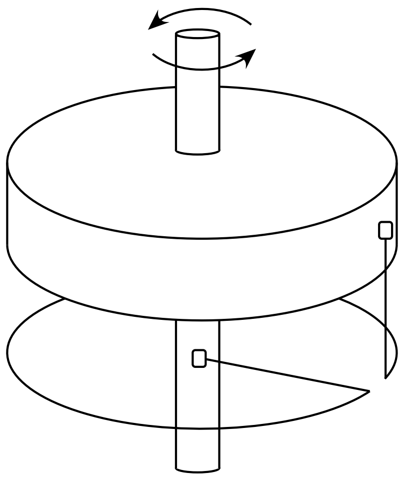

Figure 1: Bullard's simple homopolar disk dynamo. The disk has a radius a, moment of inertia $\mathcal{I}$, and angular velocity ω. Also, a constant torque τ_{ext} is applied to spin the disk[2].

II. BULLARD'S DYNAMO

The Bullard Dynamo, shown in figure 1, is self-exciting, meaning that as the angular velocity of the disk increases, it eventually reaches a critical value for which electrical inactivity becomes a non-stable equilibrium solution[2]. Thus, any stray magnetic field will cause the system to start producing both magnetic field and current. The set of coupled differential equations that describes the motion of the Bullard system is[2]

$$\begin{aligned} \omega M I &= L\frac{dI}{dt} + RI, \\ \mathcal{I}\frac{d\omega}{dt} &= \tau_{ext} - MI^2, \end{aligned} \tag{1}$$

where $2\pi M$ is the mutual inductance between the disk and wire, L is the self inductance of the wire, ω is the

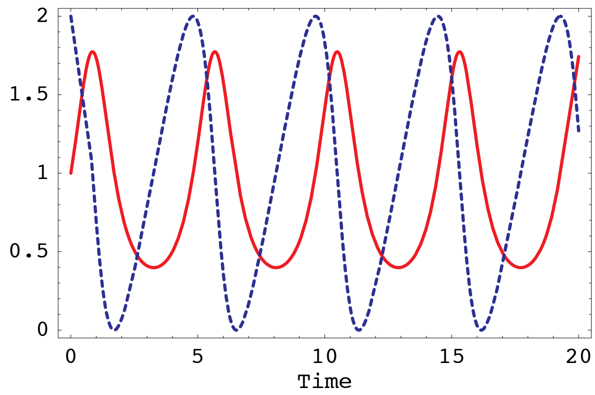

Figure 2: Numerical integration of Bullard's single disk system, plotted in arbitrary units. Pictured as a solid red curve is the current with time, and as a dashed blue curve is the angular velocity with time. The initial parameters are $\omega_0 = 2$ and $I = 1$, while the constants are $M = L = \mathcal{I} = \tau_{ext} = R = 1$, for simplicity. The system is approximately periodic, but does display nonuniform current and angular velocity, despite a constant external applied torque.

angular velocity of the disk, I is the current running through the system, $\mathcal{I}$ is the rotational inertia of the disk, and τ_{ext} is the externally applied torque driving the disk. We cannot analytically solve this system, but we can study some special cases, and it can be readily integrated numerically. Steady state solutions to the system must have $I \equiv I_0$ and $\omega \equiv \omega_0$, or both the current and angular velocity constant. Thus, considering steady states, our system of equations simplifies to

$$\begin{aligned} \omega_0 M I_0 &= 0 + R I_0, \\ 0 &= \tau_{ext} - M\,(I_0)^2\,. \end{aligned} \tag{2}$$

So, we now have a simple algebraic system, and we find $\omega_0 = R/M$ and $I_0 = \sqrt{\tau_{ext}/M}$.

Next, we numerically integrate the system to observe how a more general solution behaves. The results of this integration, shown in figure 2, display a general trend of the system, that the current never changes sign, and so the magnetic field never reverses.

To validate this assertion, we look at the phase-space plot shown in figure 3. We find a closed orbit, as we would expect for a periodic solution. Further, we notice that there exists a separatrix between the positive and negative current regions, explaining why the current never changes signs. Unfortunately, this means that the Bullard model can never explain the magnetic reversal of the Sun.

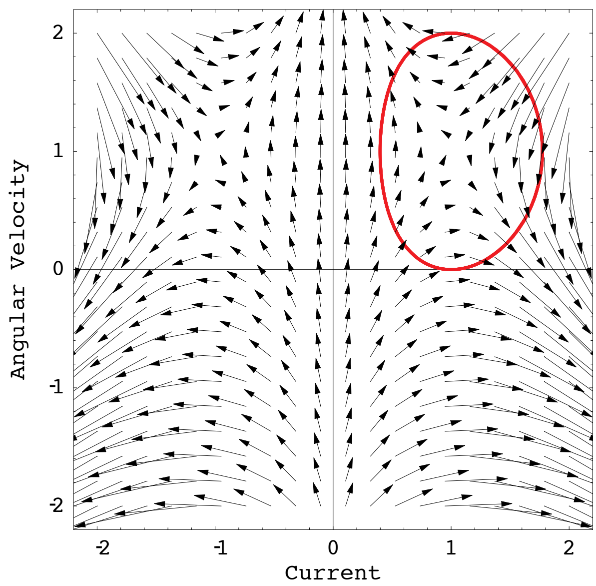

Figure 3: The same numerical integration of Bullard's single disk system done in figure 2, viewed in phase-space with arbitrary units. The vector field shown represents the initial values of the time derivatives of angular velocity and current, and the curve is the actual orbit taken by the system in figure 3. Note that there is a clear separatrix between the positive and negative current regimes, which prevents the current from changing signs.

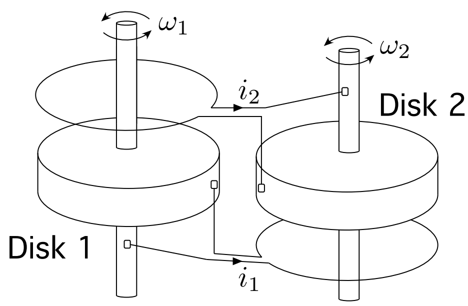

Figure 4: Rikitake's system of coupled disks. Each disk has a separate (and possibly unequal) rotational inertia $\mathcal{I}$, applied external torque τ_{ext}, mutual inductance M, and resistance R. They are each rotating at angular velocity ω, and have the current I. The quantities that apply to the first disk are denoted with a subscript "1", and the quantities pertinent to the second disk have a subscript "2" [5].

III. RIKITAKE'S DYNAMO

Rikitake's system, pictured in figure 4, consists of two Bullard dynamos coupled together such that the wire of one disk is wrapped around the other. The equations of motion are[5]

$$\begin{aligned}
\omega_1 M_1 I_2 &= L_1 \frac{dI_1}{dt} + R_1 I_1, \\
\omega_2 M_2 I_1 &= L_2 \frac{dI_2}{dt} + R_2 I_2, \\
\mathcal{I}_1 \frac{d\omega_1}{dt} &= \tau_1 - M_1 I_1 I_2, \\
\mathcal{I}_2 \frac{d\omega_2}{dt} &= \tau_2 - M_2 I_1 I_2. \qquad (3)
\end{aligned}$$

Similar to the equations of motion for the Bullard system, these equations are also nonlinear and cannot be solved exactly. Thus, we resort to numerical methods.

We integrate the Rikitake equations to obtain the solutions shown in figure 5, which exhibits current reversals. Besides just current reversal, the phase-space plots in figure 6 show that the orbits do not close, and so the system is chaotic.

The sun is thought to exhibit such chaotic reversals of magnetic polarity[6], so the Rikitake system is much more likely to be a good candidate for a toy model for the sun than the Bullard dynamo. Unfortunately, two coupled disk dynamos do not resemble the spherical geometry of the sun, and frictional forces, as we will see in the next section, tend to decrease the chaotic behavior of the system, presenting another problem.

IV. HIDE'S MODIFICATION

In 1995, Raymond Hide found a disturbing problem with the Rikitake dynamo, which dramatically impacted its feasibility as a physical model[7]. In an effort to make

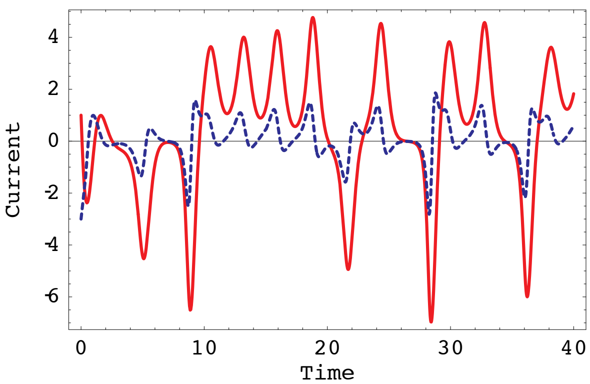

Figure 5: Numerical integration of the Rikitake System, plotted with arbitrary units. The solid red curve is I_1, and the dashed blue curve is I_2. The system starts with the initial conditions $I_1(0) = 1$, $I_2(0) = -3$, $\omega_1(0) = 4$, and $\omega_2(0) = -2$. Note that as in the Bullard integration, all the constants were set to unity for simplicity, and so the units are arbitrary.

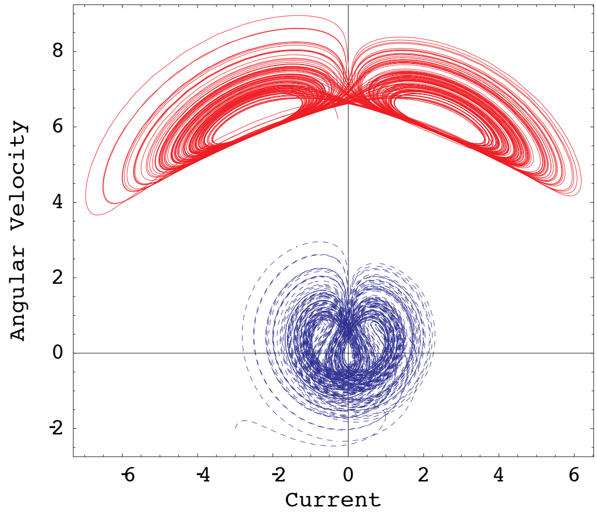

Figure 6: Phase-space plot of the first disk in the numerical integration done in figure 5, plotted with arbitrary units. The solid red curve is the first disk system, and the dashed blue curve is the second. The arbitrary time runs from $t = 0$ to $t = 600$.

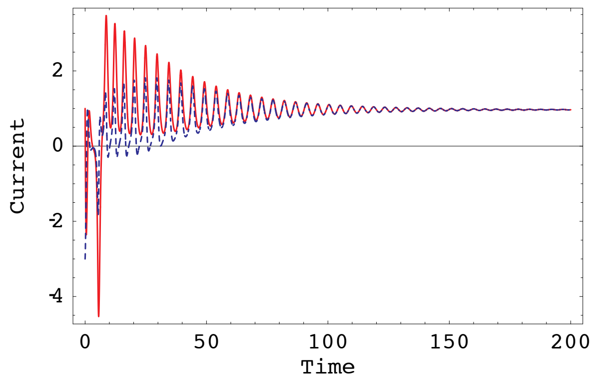

Figure 7: Numerical integration of the Hide system, plotted in arbitrary units. The solid red curve is the current in the first disk, and the dashed blue curve is the current in the second. The initial parameters were the same as in figure 5, and the damping terms were $k_1 = k_2 = 0.06$. The system does exhibit chaotic reversals initially, but oscillations are quickly damped out.

the system more physical, Hide included damping terms in the two Rikitake torque equations. Hide's equations were

$$\begin{aligned}
\omega_1 M_1 I_2 &= L_1 \frac{dI_1}{dt} + R_1 I_1, \\
\omega_2 M_2 I_1 &= L_2 \frac{dI_2}{dt} + R_2 I_2, \\
\mathcal{I}_1 \frac{d\omega_1}{dt} &= \tau_1 - M_1 I_1 I_2 - k_1 \omega_1, \\
\mathcal{I}_2 \frac{d\omega_2}{dt} &= \tau_2 - M_2 I_1 I_2 - k_2 \omega_2, \qquad (4)
\end{aligned}$$

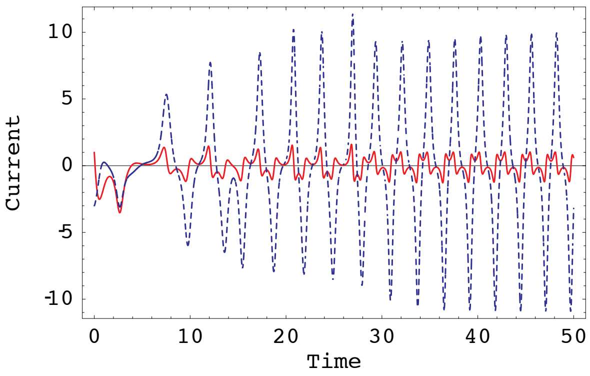

Figure 8: Numerical integration of the Hide system with asymmetric torques, plotted in arbitrary units. The solid red curve is the current in the first disk, and the dashed blue curve is the current in the second. The initial parameters were the same as in figure 7, except that $\tau_2 = 3$ rather than unity.

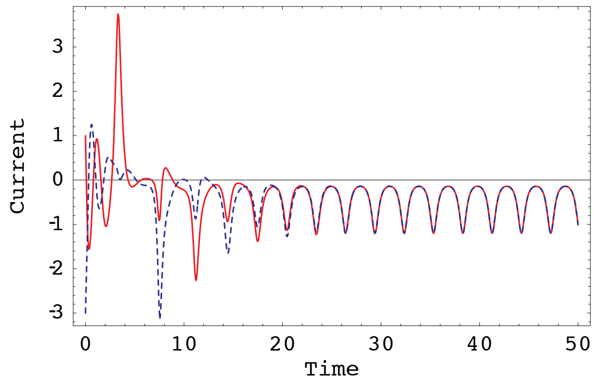

Figure 9: Numerical integration of the coaxial system, plotted in arbitrary units. The solid red curve is I_{13}, and the dashed blue curve is I_{24}. The initial conditions were $I_{13} = 1$, $I_{24} = -3$, $\omega_2 = -2$, and $\omega_3 = 2$. The constants were all set to unity. The system does exhibit chaotic reversals initially, but after a short time all current fluctuations are constrained to be negative.

which are identical to the Rikitake equations except for the damping terms. When integrated numerically, these yield substantially different results from the Rikitake equations, as shown in figure 7. From this plot, we see that the system spends a short amount of time oscillating and then decays to a steady-state solution. This has problematic consequences for the Rikitake system being used as a model for a chaotically reversing system such as the sun, since most dynamical models would have to include at least a small damping term.

However, we found that inserting an asymmetry into the torques caused the system to once again exhibit reversals, as shown in figure 8. Unfortunately, the system is periodic rather than chaotic, so we have overcome the damping effect but at the cost of some of the chaotic tendencies of system.

V. CHANGING THE GEOMETRY

We now turn our attention to altering the geometry of the Rikitake system to make a more physically reasonable model for the sun. To do this, we will first move the disks to be coaxial. This will complicate our work substantially, because now each of the disks and wire loops can potentially interact with every other element through a plethora of mutual inductances. After that, we will begin laying the framework for more general geometries. Until now, we have been simply setting all the mutual inductance terms to unity to observe solutions of the resulting differential equations qualitatively. However, now we must calculate them directly so that altering the geometry of the system will have a physically realistic impact on its behavior.

A. The Coaxial geometry

We imagine moving the two disks over top of each other, as shown in figure 10. The original Rikitake equations only take into account the mutual inductance between the first disk and second wire and the second disk and first wire. However, now each element is acted on by three other elements. To start, we rewrite the Rikitake equations (eq. 3) in a more suggestive form using the labeling scheme in figure 10. They are

$$\begin{aligned}
\omega_3 M_{34} I_{24} &= L_1 \frac{dI_{13}}{dt} + R_{13} I_{13}, \\
\omega_2 M_{12} I_{13} &= L_4 \frac{dI_{24}}{dt} + R_{24} I_{24}, \\
\mathcal{I}_2 \frac{d\omega_2}{dt} &= \tau_2 - M_{12} I_{13} I_{24}, \\
\mathcal{I}_3 \frac{d\omega_3}{dt} &= \tau_3 - M_{34} I_{13} I_{24}.
\end{aligned} \tag{5}$$

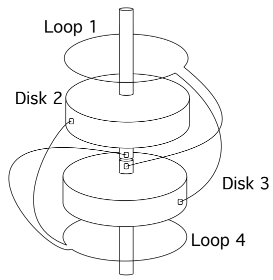

Figure 10: A coaxial version of the Rikitake dynamo. Note that the two axes do not touch, so the only difference between this and the Rikitake system is the two disks' proximity to each other.

Now, we add the new interactions due to the mutual inductances of all the components. By symmetry of the system, we extend the Rikitake equations to be

$$
\begin{aligned}
L_1 \frac{dI_{13}}{dt} + R_{13} I_{13} &= \omega_3 M_{13} I_{13} + \omega_3 M_{23} I_{24} + \omega_3 M_{34} I_{24}, \\
L_4 \frac{dI_{24}}{dt} + R_{24} I_{24} &= \omega_2 M_{12} I_{13} + \omega_2 M_{23} I_{13} + \omega_2 M_{24} I_{24}, \\
\mathcal{I}_2 \frac{d\omega_2}{dt} &= \tau_2 - M_{12} I_{13} I_{24} - M_{23} I_{13} I_{24} \\
&\quad - M_{24} I_{24} I_{24}, \\
\mathcal{I}_3 \frac{d\omega_3}{dt} &= \tau_3 - M_{34} I_{13} I_{24} - M_{23} I_{13} I_{24} \\
&\quad - M_{13} I_{13} I_{13}.
\end{aligned}
\tag{6}
$$

We numerically integrate these equations. Sample results from this integration are shown in figure 9. Unfortunately, the system shows no apparent sustained chaotic current reversals. At first, some reversals are present, but the system quickly settles into periodic oscillation. Thus far, we have not found a suitable set of parameters to attain reversals, as we could in the Hide dynamo. To proceed further, we need to alter the system in some way by using a different geometry. However, to improve our results, as well as making them more physically realistic, we need a better grasp on the mutual inductance terms appearing in our equations.

B. Mutual inductance

Until now, we have been assuming a mutual inductance for a given system and investigating the properties of the resulting differential equations. However, to accurately gauge the effectiveness of different geometries, we need a method for calculating it explicitly. Our mutual inductance $2\pi M$ is[8]

$$
2\pi M := \frac{\Phi}{I} = \frac{1}{I} \int_a \vec{B} \cdot d\vec{a}, \tag{7}
$$

where we integrate over an area a. But we know by the Biot-Savart law that

$$
\vec{B} = \mu_0 \oint_l \frac{d\vec{l} \times \hat{u}}{4\pi u^2}, \tag{8}
$$

where we integrate around the closed loop l, and $\vec{u}$ runs from source point to field point. Thus, in general, our mutual inductance is

$$
2\pi M = \frac{\mu_0}{4\pi} \int_a \oint_l \frac{d\vec{l} \times \hat{u}}{u^2} \cdot d\vec{a}. \tag{9}
$$

We now consider a special case especially important to the dynamo problem: an infinitesimally thin, parallel disk and wire loop, separated by distance h. We take the radius of the wire loop to be R and the radius of the disk to be R_0. We pick our origin to be the center of the disk, and we use polar cylindrical coordinates, letting a point in the disk be located by the coordinates ρ, ϕ, and $z = 0$, and a point in the loop be located by $\rho = R$, ϕ', and $z = -h$. Through direct computation, we find

$$
\begin{aligned}
2\pi M &= \frac{\mu_0}{4\pi} \int_{\phi=0}^{2\pi} \int_{\phi'=0}^{2\pi} \int_{\rho=0}^{R_0} \frac{\rho R d\phi d\phi' d\rho \left(-\rho \cos(\phi - \phi') + R\right)}{\left(\rho^2 + R^2 - 2R\rho \cos(\phi - \phi') + h^2\right)^{3/2}}, \\
&= \frac{R\mu_0}{4\pi} \int_{\phi=0}^{2\pi} \int_{\phi'=0}^{2\pi} \int_{\psi=0}^{R_0/R} d\phi d\phi' d\psi \frac{\psi \left(1 - \psi \cos(\phi - \phi')\right)}{\left(\psi^2 + -2\psi \cos(\phi - \phi') + (h/R)^2\right)^{3/2}},
\end{aligned}
\tag{10}
$$

where we have introduced the dimensionless parameter $\psi := \rho/R$.

Now that we can readily compute the mutual inductance for a system of two disks, we could conceivably change the geometry of the system and reflect the change mathematically in our equations of motion. However, to provide a complete physical description of the system, we also need the self inductances of the components. Unfortunately, we were not able to perform the self-inductance calculations, and we leave their computation for future work.

VI. DISCUSSION AND FUTURE WORK

We began by briefly reviewing the solar dynamo problem, and we noted that systems of Faraday disk dynamos might be able to qualitatively model some features of the sun. We showed that Bullard's dynamo was inadequate as a model of the sun, since the solar dynamo reverses polarity chaotically.

We then noted that Rikitake's model does exhibit the chaotic magnetic reversals we need to model the sun. However, Hide provided a physical objection for the validity of the Rikitake system as a model. We addressed this concern, and showed that we could force current reversals in a damped Rikitake system by making the torques driving the disks unequal, but in doing so we lost the chaotic nature of the reversals.

Next, we attempted to change the geometry of the system to strengthen the analogy between our system and the solar dynamo. To do this, we brought the two disks of the Rikitake system into coaxial alignment. We worked out the new equations of motion for this system, and numerically integrated them.

The results indicated that the system settled into oscillatory but non-reversing currents. To understand this more, we would have to alter the geometry of the system in an exact way. To facilitate this, we worked out the mutual inductance for an arbitrary system, and we solved a specific example of a parallel disk and wire loop.

There is much left to be done on this project. Stability calculations could be performed on the various models. This would address whether or not the coaxial system truly will never reverse. Further, the mutual inductance calculation should be applied to the models to get a more physical sense their behavior, rather than using arbitrary mutual inductances.

Also, the self-induction of the loops still needs to be calculated to complete a full model. Finally, once the mutual and self inductances are known and applied, new geometries such as cylinders and spheres could be attempted, which would bear a much closer geometric resemblance to the sun.

VII. ACKNOWLEDGMENTS

I would like to thank Dr. John Lindner for his extensive help and invaluable assistance in this project.

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