A Computational Simulation of the Untying of Knots

Joshua S. Martin

Physics Department, The College of Wooster, Wooster, Ohio 44691

May 2, 2002

Data from modeling four distinct knots were collected by varying the friction of the cords and the applied force. The decay constants associated with the fits of the data from the reduced time and the applied force were found. This data showed that the square knot has the highest decay coefficient of all the knots with right-handed sheet bend having the lowest. All knots but the thief knot showed a linear increase of the time as friction was increased. The right-handed sheet bend had the smallest slope indicating that it was least affected by a change of kinetic friction. When all the data is considered, the right-handed sheet bend is probably the best overall knot compared to the other three knots simulated.

INTRODUCTION

In the world today with its glut of joining products such as glue, epoxy, staples, and tape, joining two things together using only cord seems almost primitive. Still knots have survived and are used in everyday life. There have even been several unsuccessful attempts to determine the best way to join two cords together. The methods tried in References [1] and [2] seem to be unscientific with the variables in tying the knot compounding the problem. Usually the classification of a good knot is determined by how well the knot holds, how the knot affects the breaking point of the rope, how easy the knot is tied, and finally how easy the knot is to untie after a load has been applied to the knot. Even with these criteria, what knot is best used to join two cords is still unclear and seems to be more dependent on the situation and the cords used to tie the knot.

The scientific study of knots did not even begin to take off until after Leonard Euler had invented graph theory. It was 35 years latter with a paper by Alexandre-Theophile Vandermonde that knot theory even began but it was not until Cal Friedrich Gauss' work on linking numbers and magnetic fields that knot theory really took off[4]. The linking number that Gauss derived is given by

$$I = \oint\oint [(x - x')^2 + (y - y')^2 + (z - z')^2]^{-3/2} [(x - x')(dydz' - dzdy') + (y - y')(dzdx' - dxdz') + (z - z')(dxdy' - dydx')] \quad (1)$$

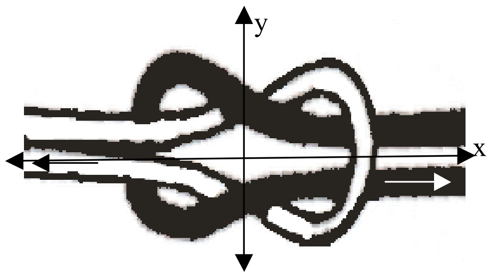
were x, y, z are the coordinates of a given point on the first line and x', y', z' are points on a second line with $I = 4\pi n$ were n is the linking number[5]. The linking number was suppose to distinguish between knots no matter how similar or different they looked.

Gauss's work did not answer the underlying question of knot theory; namely, that of distinguishing knots apart and identifying knots that are the same. This pursuit almost seems silly to anyone familiar with the actual use of knots but actually holds intense subtlety. To a mathematician, two knots are the same if and only if the knots can be manipulated using the Reidemeister moves[4] until the two have the same form. Gauss' linking number can only distinguish knots to the same extent that the number of sides of a polygon distinguishes between polygons. Huang and Lai try and classify various flexible and ideal knots by using the writhe of the knot and other traditional methods [9]. The other major difference between the mathematical knots and real knots is that mathematical knots are made of one continuous cord. Still even with all the work being done, there does not appear to be a theory that can distinguish and identify knots no matter how different or similar the knots appear.

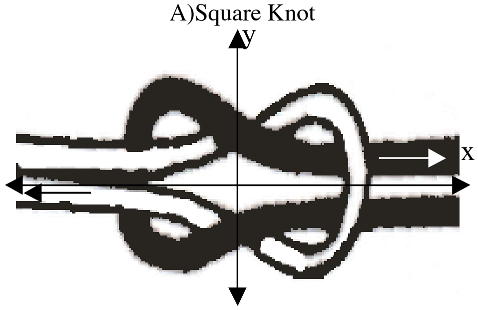
Since Gauss' time, many papers have been written that tied various pieces of physics, biology, and mathematics together. A current interest is in how strands of polymers and DNA will form knots in solution and how these knots affect the strength of the polymer[6]. In the early 1970's, theorists looking at quantum field theories discovered a surprising connection between the two fields by using nonperturbative effects[5]. Others have even studied knot-like configurations in various physical scenarios such as early Universe cosmology, elementary particles, magnetic materials and fluid dynamics as shown in References [7] and [8].

Although much scientific work has been done to be able to distinguish between knots, very few scientific studies have been done to analyze the mechanics and strength of various knots[1]. This seem a bit strange considering how many books have been written about tying, using and the history of various knots as shown in References [2], [3] and [10]. The mechanics of various knots is incredibly important since some knots will not hold under particular conditions. When a knot will or will not hold seems to be dependent on several conditions including the topology, the chirality, and the application of the forces to the knot.

A simple example of how the application of force effects how well the knot holds is by comparing the square knot to the thief knot. The square knot (knot A in Figure (1)) is a decent knot that can be tied quickly, untied easily, and will hold a decent load, but only when the knot is pulled in a symmetric manner across the y-axis as shown in part A in Figure (1) then the knot will tighten and hold decently well. If on the other hand the square knot is pulled symmetrically across the y=-x line then the knot will slip and untie itself. When the square knot is subjected to this type of pull it is called a thief knot as shown in part B in Figure (1).



A)Square Knot



B)Thief Knot

FIG 1: A)Square knot and B) Thief knot with the arrows indicating where the force is applied to both knots.

A particularly good example of how chirality effects how a knot holds is by studying the right-handed and left-handed configurations of the common sheet bend as shown in Figure (2). The right-handed version of this knot is particularly suited to being tied in different diameter cords and its use on board large ships to join the various size lengths of large cords or sheets is how it came by its name. The left-handed version is supposedly not as strong and is more prone to slipping and untying then the right-handed version[2].

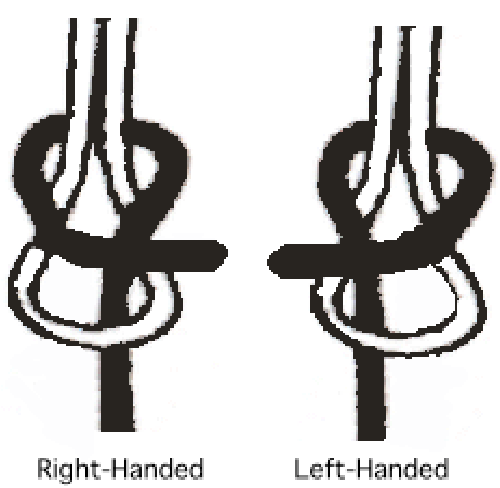


FIG 2 Right-Handed and Left-Handed configurations of the Sheet Bend

SIMULATON

The cords used to create the knots were modeled as single atoms held together by springs. These springs had a force equation of the form:

$$F = \square \frac{xk}{(x^2 \square 1)^2} \quad (1)$$

which shows the normal Hook's Law force for small displacements from equilibrium but also a nonlinear force for large displacements from equilibrium. The normal force caused by the two cords being incident was modeled as an obstacle force, F_o, of the form

$$F_o = \frac{0.5 F_s d}{d(d \square r)^{3/2}} \quad (2)$$

where d is the distance of the string from the atom in the string, F_s is the force from the object, and r is the radius of the object. This force was summed up over the entire string.

A frictional force was implemented between the two cords. Although the algorithm seemed to be correct for calculating the force, the strings were never seen to stop once in motion. This does not immediately suggest that the algorithm is wrong but suggests that once the knot begins to slide apart it will continue to slide. The other possibility is that the normal force caused by the strength of the object is too low resulting in the friction being too low to damp out the velocity of the ropes.

Once the Simulation was able to model the Square knot and Thief knot with reasonable accuracy, the right-handed and the left-handed sheet bends were introduced into the simulation. This involved adjusting the free end of the cord to the proper position for the knot.

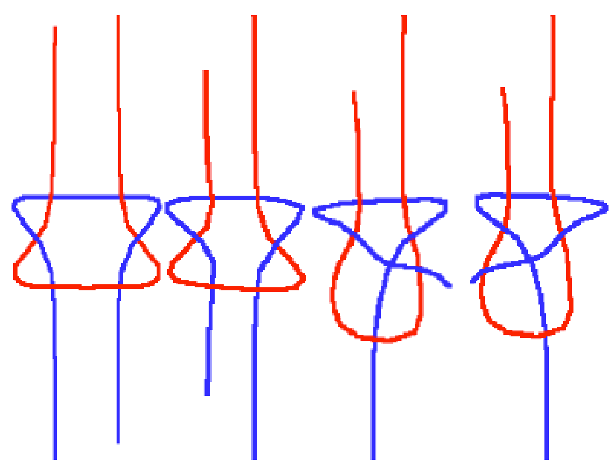


FIG 3 All four knots currently able to be simulated from left to right; Thief Knot, Square Knot, Right-handed Sheet Bend, and Left-handed Sheet Bend.

RESULTS

The final simulation behaved realistically for all knots when less then 25 units of force were applied to the long ends of the knot. Above 25 units of force, all knots but the thief knot displayed the two cords passing through each other. The thief knot “rolled over" and slid out as seen in actual knots tied in real cord. The process can be seen in Figure (4) as snap shots of the knot at various times during this process.

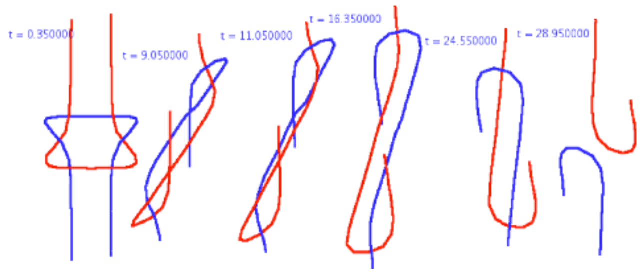


FIG 4: Six pictures from the evolution of the Thief knot as it rolls over and slides apart.

The Square knot on the other hand did not deform and loose its shape. Instead it tightened and stayed stable for a much longer time until the blue cord was actually pulled though the knot. This process can be seen in Figure (5) were the pictures are taken from a QuickTime movie of the process as was the images for the Thief knot. The first picture in the sequence shows the starting point of the knot, and each successive picture shows the upper string being pulled through the knot until the short end has been pulled though half of the knot causing the knot to rotate and loose its shape. The two strings then slide apart relatively quickly.

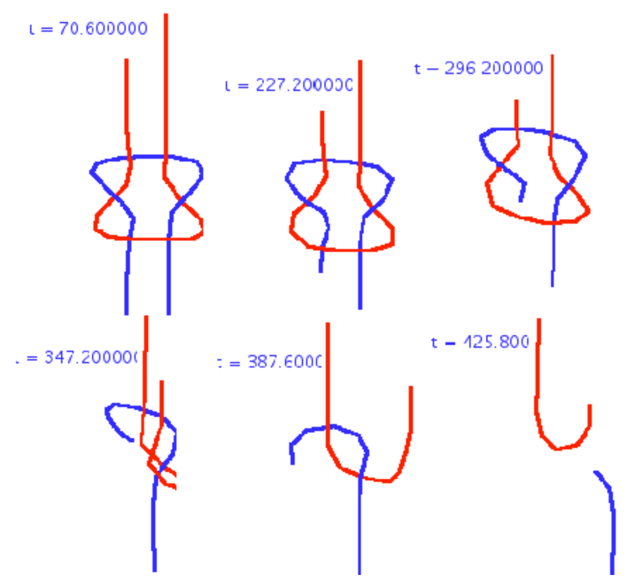


FIG 5: Six pictures of the evolution of the square knot as the shorter end of the string is pulled through the knot.

How the chirality of the knot affects how the knot slips was clearly seen with the two enatiomers of the Sheet Bend. Since the Sheet bend has a slightly different topology then the Square Knot, it went through a different mechanism. The mechanism for the Right-Handed Sheet Bend can be seen in Figure (6). The knot first tightens and then pulls the free (short) end of the blue cord out of the knot. The force applied to the long end of the blue cord then rotates the knot to the right as the short end continues to unwrap from the knot. The two strings then fall away from each other.

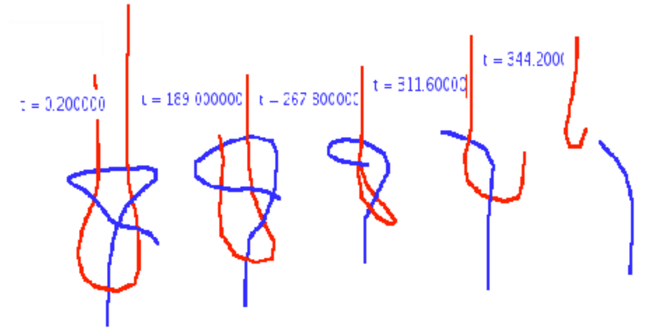


FIG 6: The progression of the Right-Handed Sheet Bend as it unties with the time given for each image.

The left-handed sheet bend went through a different process due to its chirality as shown in Figure (7). The knot first tightened slightly as the force from the long end of the blue cord rotated the knot to the left. The force on the long end of the blue cord then pulled the short end of the red cord through the knot causing the knot to spill, freeing the two cords to slide past each other.

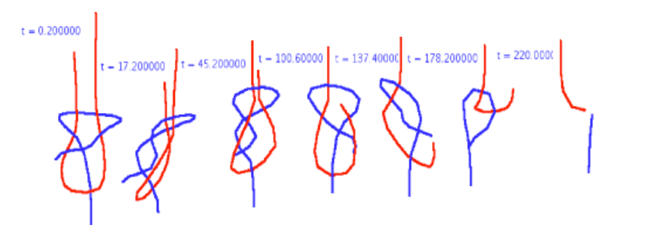


FIG 7: The progression of the Left-Handed Sheet Bend as it unties with the time given for each image.

Data from these knots was collected to show how the force on the long ends of the cords and the friction effected the untying times of the knots. As expected, the thief knot continuously had the shortest untying time between all the knots studied with the square knot having the longest. The graph of the data collected for various forces is shown in Figure (8). The graph only shows up to 25 units of force because above 25 units all but the thief knot showed the cords passing through each other. It is unclear whether this phenomenon is merely a flaw in the program or an effect caused because the string cannot break in this simulation.

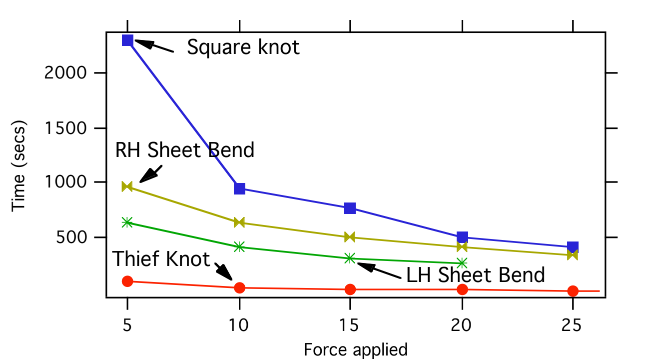


FIG 8: Graph of Force Applied Vs. Untying time for all knots.

All knots were modeled using an exponential decay for the untying time as the force on the long end was increased. The decay coefficient, □, and the error associated with □ for these knots can be found in Table (1). The error was given by *Igor Pro* when the curve was fitted to the data points.

Table 1: Table of knot decay constant with error in the fit

Knot	**Decay(□)**	**Error**
Thief	0.18	0.01
Square	0.23	0.05
RH Sheet Bend	0.13	0.02
LH Sheet Bend	0.152	0.002

Table (1) shows that the square knot has the highest decay constant leading to the conclusion that with larger forces the Square knot should come untied quicker than both versions of the sheet bend. The right-handed sheet bend showed the lowest decay constant suggesting that the knot will remain relatively stable for larger forces applied. This same conclusion can be reached by defining a reduced time, t_{red}, as

$$t_{red} = \frac{t}{t_{\max}} \qquad (3)$$

were t is the time and t_{max} is the maximum untying time for that knot. Graphing the reduced time against the force as shown in Figure (8) shows how the knots behave independently of the total time taken for a particular knot.

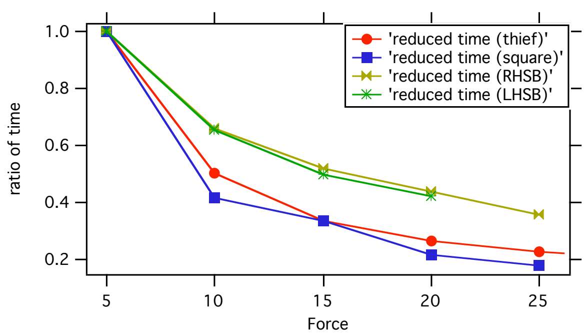


FIG 9:: Graph of the Reduced Time Vs the Applied Force.

The data for how the friction of the ropes effects the knots was found by varying the kinetic friction coefficient in the program to produce data that will shed some light on how the various knots will behave in strings with different amounts of kinetic friction. The data was collected by setting the static friction to 0.8 and changing the kinetic friction. The first run was done with a kinetic friction of 0.0 and the

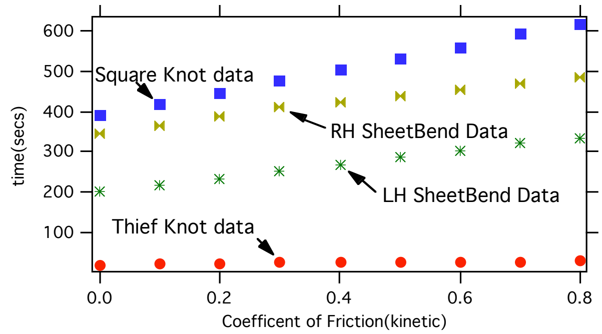


FIG 10 Graph of Untying time Vs Kinetic Friction Coefficient

last run had a kinetic friction of 0.8 with friction being increased by 0.1 between runs. The lower end kinetic frictions would correspond to synthetic fibers like nylon while the higher end coefficients would correspond to natural fibers such as hemp or cotton. Figure (10) shows that all the knots had an increase of untying time as the kinetic coefficient was increased. On a regular time scale, it is difficult to compare how

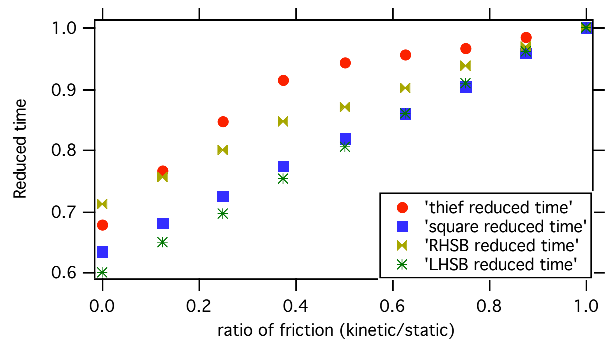


FIG 11: The graph of the Reduced Time Vs the ratio of the coefficients of friction for all simulated knots.

the knots behave so the same reduced time was computed as shown in Equation (3) was used as a comparison. The plot of the reduced time against the ratio of the kinetic over the static coefficients of friction is shown in Figure (11). Figure (11) shows that all knots but the Thief knot showed a linear increase in untying time as the ratio of the friction was increased. The actual slopes of the linear increase can be seen in Table (2) along with the error associated

Table 2: Table of knots to the slope and error in slope of the linear increase of the untying time as kinetic friction was increased.

Knot	**Slope**	**Error**
Square	0.365	0.003
RH Sheet Bend	0.28	0.01
LH Sheet Bend	0.409	0.005

with the slope. Although the Square knot has the over all highest untying time the reduced time shows that the friction effects the thief knot and the right-handed sheet bend the least. The thief knot showed a exponential increase as shown in Figure (12).

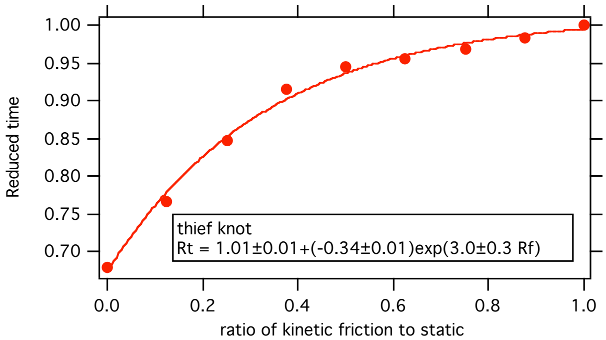


FIG 12: Graph of the Reduced time to the ratio of coefficients of friction for the Thief knot.

This exponential increase of the thief knot as the friction was increased suggests that a little bit of friction between the cords will slow down the process but above a kinetic friction coefficient of 0.3, the friction does not effect the knot in any major way. This seems plausible because the thief knot is notorious for always sliding out no matter what rope it is tied in.

CONCLUSION

The data collected from varying the friction of the cords and the applied force showed that the square knot had the greatest untying time of all the knots with the thief knot having the smallest untying time. The decay constants associated with the fits of the data from the reduced time and the applied force allowed a more equal comparison of the knots. This data showed that the square knot has the highest decay coefficient of all the knots with right-handed sheet bend having the lowest. This suggests that for greater forces the right-handed sheet bend will hold better then the square knot. It is surprising to notice that the order of the decay coefficient from highest to lowest is: the square knot, thief knot, the left-handed sheet bend and the right-handed sheet bend.

The data collected for the friction showed that the thief knot is least affected by decreasing the friction but only up to 0.3 were it dropped almost as drastically as the square knot. All knots but the thief knot showed a linear increase of the time as friction was increased. The right-handed sheet bend had the smallest slope showing that it was least affected by a change of kinetic friction.

With all the data collected, it can be seen that the right-handed sheet bend is probably the better overall knot compared to the other three knots simulated, even though the square knot had the longest untying times for all experiments. Thief knot showed the shortest untying times even though when the reduced time was used it compared favorable with the other knots. The difference between the enatomers of the sheet bend was clearly visible in all the data sets with the right-handed sheet bend performing better in all cases then the left-handed sheet bend. This fits with information given about the stability of those two knots in Reference [2].

Even though the knots were simulated in a decent manner there are still improvements that could be made to the simulation. One improvement would be to prevent the cords from passing through each other as seen in all but the thief knot when forces of greater then 25 units. The other major improvement would be to simulate the string breaking as shown in studies of the overhand knot and figure eight knot in Reference [11]. Since it is known that most knots will decrease the strength of the cord by up to 50% [1], this would produce more realistic knot behavior.

ACKNOWLEDGMENTS

This project would not have been possible with out Dr. Lindner's and Dave Miller's help with the coding.

[1] J.C. Turner and P van de Griend (Eds.), History and Science of Knots, (World Scientific, New Jersey 1996).

[2] Clifford W. Ashley The Ashley Book of Knots, (Doubleday, New York 1944).

[3] Roger E. Miles, Symmetric Bends; How to Join Two Lengths of Cord, (World Scientific, New Jersey 1995).

[4]Józef H. Przytycki, *Chaos, solutions & Fractals,* 9 (4/5) 531-545, (1998).

[5]Allen C. Hirshfeld, *American Journal of Physics*, 66 (12) 1060-1066, (December 1998).

[6]A. Marco Saitta, Paul D. Soper, E. Wasserman, & Michael L. Klein. *Letters to Nature,* 399, 46-48 (May 1999).

[7]Antti J. Niemi. *Physical Review D*, 61, 125006-1-125006-4, (May 2000).

[8]Vsovold Katritch, Jan Bednar, Didier Michoud, Robert G. Scharein, Jacques Dubochet & Andrzej Stasiak. *Letters to Nature* 384, 142-145 (November 1996).

[9]Juin-Yan Huang and Pik-Yin Lai. *Physical Review E*, 63, 024506-1-021506-6, (January 2001).

[10] Derek I. Avery, The New Encyclopedia of Knots, (Brockhampton Press, London 1997).

[11]Piotr Pieranski, Sandor Kasas, Giovanni Dietler, Jacques Duochet and Andrzej Stasiak, *New Journal of Physics*, 3, 10.1-10.13 (June 2001).