A Mechanical Model to Investigate Nonlinear Phase Transitions

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A mechanical toy based on the previous work of Mancuso 1999 was built to investigate nonlinear phase transitions.[1] The exact apparatus used by Mancuso was unavailable for purchase, so a similar apparatus was constructed. The motion of two balls rolling in a circular hoop was investigated as the hoop was rotated about its central axis and an offset axis (0.2 R). It was found that the balls rolled to two stable equilibrium positions when the hoop was rotated above a critical angular velocity (approx 1.1 Hz). About the central axis, the equilibrium positions were symmetric but about the offset axis the equilibrium positions were asymmetric. As the angular velocity was decreased from above the critical value back to 0, a first-order phase transition was partially observed.

INTRODUCTION

Mechanical models are often useful for explaining or illustrating a phenomenon that may seem counter-intuitive. The case of balls rolling inside a hoop that is being rotated about its central axis is well known and can be predicted accurately; however, the case of balls rolling inside a hoop being rotated about an offset axis is not so well known.

The first observable effect is that the equilibrium positions where the balls come to rest at maximum angular velocity ω_{max} are asymmetric. Also, as ω_{max} is decreased towards 0, the equilibrium position closest to the offset axis becomes unstable. The ball in this location will roll down the hoop and cross to the other side to come to rest in the stable equilibrium position next to the other ball. This is a phase transition: the ball leaves an equilibrium position that has become unstable and arrives at a new stable equilibrium position. As ω decreases to 0, both balls will come to rest at the bottom of the hoop.

The data collected in this experiment is best presented in the form of a bifurcation diagram which shows how an initial stable fixed point will become unstable, but creates two stable fixed points at the same time. By definition, a fixed point corresponds to a point where an object's velocity vanishes to 0. A stable fixed point is best idealized as a "sink" -- a ball placed in a stable fixed point will tend to stay there, and any perturbations will cause the ball to oscillate but return to the fixed point. An unstable fixed point is best idealized as a "saddle". A ball placed in an unstable equilibrium will also remain at that point initially, but any perturbations will cause the ball to roll away to a new stable equilibrium point. In this experiment, the initial (ω=0) stable point is at the bottom of the hoop and the unstable fixed point is at the top of hoop. When $\omega = \omega_{max}$, the top and bottom of the hoop are the unstable fixed points and the fixed points are located on the sides of the hoop.[2]

THEORY

A free-body diagram showing the forces acting on the ball as it rolls freely in the rotating hoop is shown in Figure 1. It is assumed that the hoop rotates at a constant angular velocity ω. To solve the equation of motion for this problem, the Lagrangian equation is used.

$$L = [\frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m\omega^2(R\sin\theta + A)^2]$$
$$+[mgR\cos\theta] \qquad \text{(Eq 1)}$$

In all equations, it is assumed that the mass of the ball is a point mass. Using the Euler-Lagrange equation, the Lagrangian of Equation 1 is simplified to the form

$$mR^2\ddot{\theta} = m\omega^2(R\sin\theta + A)R\cos\theta$$
$$-mgR\sin\theta \qquad \text{(Eq 2)}$$

A Stokes' damping force proportional to the first power of $\dot{\theta}$ is added, and damping is let dominate inertia since the focus of this experiment is on only the location and nature of the fixed points and not the movement about them.
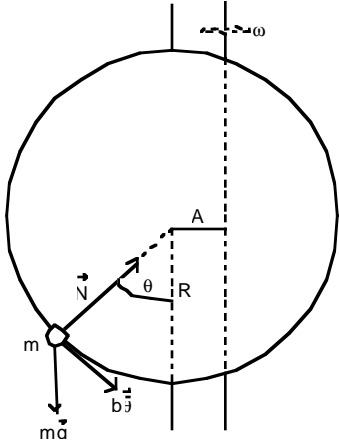

Figure 1: Free-body diagram for a ball rolling in a circular track. In this figure, the track is rotating about an offset axis (Figure taken from Ref. 1 p. 272).

To make the differential equation dimensionless, a new variable τ (dimensionless time) = t/T is introduced, where T is a "characteristic time" to be defined later. Substituting τ=t/T into Equation 2,

$$\frac{mR^2}{T^2}\frac{d^2\theta}{d\tau^2} = m\omega^2(R\sin\theta + A)R\cos\theta$$
$$-mgR\sin\theta - \frac{b}{T}\frac{d\theta}{d\tau}$$
(Eq 3)

becomes the new equation of motion. Each term in this equation has units of torque (Nm). The parameter b is a damping parameter representing the size of the retarding torque with units of Nms.

Dividing Equation 3 by the quantity mgR gives the equation

$$[\frac{R}{gT^2}]\frac{d^2\theta}{d\tau^2} = [\frac{\omega^2 R}{g}](\sin\theta + [\frac{A}{R}])\cos\theta$$
$$-\sin\theta - [\frac{b}{mgRT}]\frac{d\theta}{d\tau}$$
(Eq 4)

The "characteristic time" T is now chosen to be
$$T = b/mgR$$
(Eq. 5)
The condition that damping dominates inertia can be represented by the case

$$\frac{m^2 gR^3}{b^2} \ll 1$$
(Eq. 6)

This condition will make the $\frac{d^2\theta}{d\tau^2}$ term negligible and remove it from Equation 4.

The dimensionless parameters β and α are now chosen so that

$$\beta = \frac{\omega^2 R}{g} \text{ and } \alpha = \frac{A}{R}$$
(Eq. 7)

Substituting Equations 5, 6, and 7 into Equation 4, the dimensionless equation of motion with damping dominating inertia becomes

$$\frac{d\theta}{d\tau} = \beta(\sin\theta + \alpha)\cos\theta - \sin\theta$$
(Eq. 8)

By setting $\frac{d\theta}{dt}$=0 and manipulating Equation 8,

$$\alpha = \frac{1}{\beta}\tan\theta - \sin\theta$$
(Eq. 9)

can be obtained. A plot of α vs. θ will give a graph of the number, location, and nature of the fixed points. Plotting β vs. θ will produce a bifurcation diagram for the hoop rotated about its central axis.

EXPERIMENT

The motion of the balls was analyzed as the hoop was rotated about both its central axis and an offset axis (0.2R). Figure 2 below shows a schematic of the experimental apparatus set up in the upright position to study the motion of the balls as the hoop was rotated about its central axis. To aid in the determination of the location of the equilibrium positions on the hoop, lines were drawn on the hoop in 22.5 degree intervals starting at θ=0 (bottom of the hoop) up to 90 degrees.
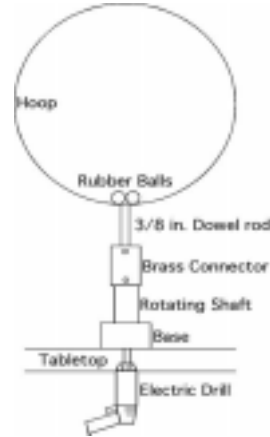

Figure 2: A schematic of the experimental apparatus used to study the motion of the balls as the hoop was rotated about its central axis.

To provide a constant angular velocity ω, a DeWalt 3/8" V.S.R. drill was locked in the on position and connected to a Powerstat variable autotransformer. The ω could then be varied by changing the voltage of the autotransformer. When the hoop was rotated about its central axis, both balls were placed in the hoop and the

autotransformer connected to the drill was turned to approximately V=20. This corresponds to an ω of approximately 7 rad/s. Both balls would then leave the equilibrium point at the bottom of the hoop (θ=0) and roll to new symmetric equilibrium positions on the sides of the hoop.

When the hoop is rotated about an offset axis, a "flapper" mechanism is necessary to hold the balls at initial positions until the hoop has reached its critical ω. When the hoop is rotating at ω_{max}, the balls roll into two new asymmetric equilibrium positions: θ_- and θ_+. θ_- is designated as the equilibrium position closest to the offset axis, θ_+ is the equilibrium position farthest from the offset axis. It was found that the first ball reached an equilibrium position at θ_+ with an ω of approximately 1.1 revolutions/s (approx. 7 rad/s) corresponding to an autotransformer setting of approximately 20. The second ball reached an equilibrium position at θ_- with an ω of approximately 2.4 revolutions/s (approx 15 rad/s) corresponding to an autotransformer setting of approximately 25.

When ω was decreased from ω_{max} to 0, the equilibrium position at θ_- becomes unstable at approx. 16 rad/s. The ball in this position will then leave θ_- and attempt to roll to the equilibrium position at θ_+. This is a phase transition: a ball leaving an equilibrium position that has become unstable and moving to a stable equilibrium position. However, the channel in the hoop was too shallow and the ball was always ejected at approx. θ=0, so the phase transition was never observed to completion.

ANALYSIS AND INTERPRETATION

All data were collected using a Canon ZR10 digital video camera capturing video at a shutter speed of 1/8000. A firewire-equipped Macintosh G4 cube running iMovie was used to import the video files for analysis. Apple Computer's iMovie v. 2.1.1 was then used to determine ω, θ_-, and θ_+ as the hoop was rotated about the offset axis.

To construct the bifurcation diagrams shown in Figures 3 and 4, the quantities "corrected θ_-" and "corrected θ_+" were introduced. The bifurcation diagram requires that the initial position of both balls be θ=0; however the flapper holds both balls at initial positions of 25 and 32 degrees, respectively. The quantity corrected θ_- was given as -(32-θ_-) and corrected θ_+ was given by (25-θ_+). This sets the initial

position of both balls to 0 and gives corrected θ_- a negative value as θ_- increases and corrected θ_+ a positive value as θ_+ increases as a function of ω.
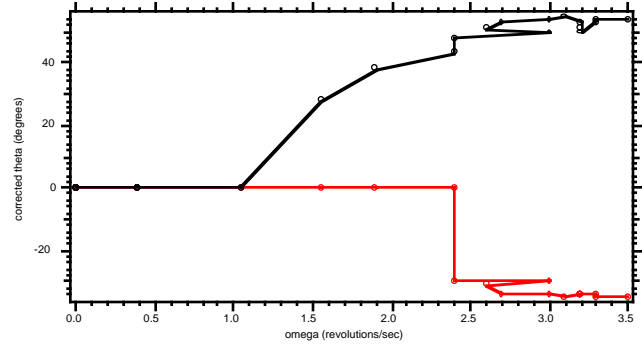
Figure 3: The bifurcation diagram constructed using the data of ω, corrected θ_- and corrected θ_+. ω is plotted from 0 to ω_{max} on the x-axis. Corrected theta is plotted from -35 to 55 on the y-axis. θ_- reaches a maximum displacement of -34° and θ_+ reaches a maximum displacement of 54°.
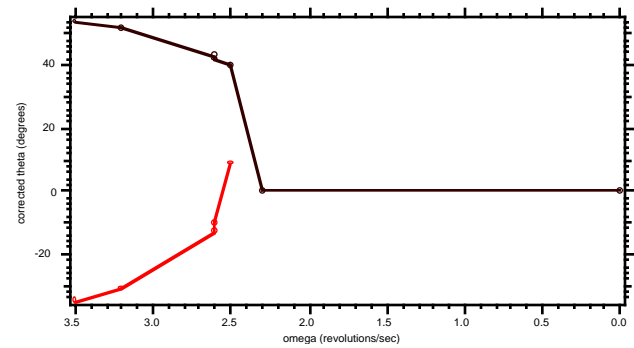
Figure 4: The bifurcation diagram constructed using the data of ω, corrected θ_- and corrected θ_+. Corrected theta is plotted on the same scale as in Figure 3. The phase transition starts in θ_- when ω=2.4. At ω=2.5, the ball exits the apparatus and the phase transition is not observed to completion. It is assumed that the balls would rest at the stable equilibrium at θ_+=25° and approach θ=0 together as ω approaches 0.

The bifurcation diagrams above can be used to explain the general behavior of the system. It can be seen that both balls start at θ=0 (due to the expressions for corrected θ_+ and corrected θ_-). As ω is increased from 0, the ball on the side of the hoop farthest from the offset axis will break away first (ω=1.1 rev/s) and be on its way to the stable equilibrium point θ_+ when the ball closest to the offset axis will break away (ω=2.4 rev/s) and roll towards the stable equilibrium point θ_-. At ω_{max}, both balls are in their respective asymmetric stable fixed points on opposite sides of the hoop.

Decreasing the angular velocity from ω_{max} to 0, it can be seen that θ_- becomes unstable first (at a lower ω). The ball will then leave θ_- and undergo a phase transition to move to the new stable point at θ_+. However, when the ball reaches approximately $\theta=0$ ($\theta=9°$, the ball crosses over slightly) it falls out of the track and terminates the phase transition. Based on the overall appearance of the bifurcation diagram and literature, it can be assumed that the ball would, ideally, continue the phase transition and come to rest beside the other ball at the stable fixed equilibrium point θ_+.[3-4] θ_+ will continue to decrease as ω decreases until both balls are resting at $\theta=0$ when $\omega=0$.

The largest uncertainty in this experiment arises from the experimental setup and the data collection. The hoop is not perfectly aligned with either of the axes about which it rotates, so small oscillations are induced in the hoop as it rotates. These oscillations cause the position of the balls in their respective equilibrium positions to oscillate by approximately $\pm1°$.

Another uncertainty is that the position of the balls can oscillate drastically in the time necessary to compute ω. Thirty frames of video are necessary to compute data for 1 second in iMovie, and in those thirty frames the ball can move as much as $20°$ (especially if the ball is just breaking away from an unstable fixed point). This produced large uncertainty values in the position of the balls at the critical values of ω. However, when the balls reached their stable equilibrium points with the hoop rotating at ω_{max}, the oscillations about the fixed points was greatly decreased ($\pm$approx $3°$).

CONCLUSION

It can be concluded that this mechanical model is valid for investigation of basic nonlinear phenomenon (stable and unstable fixed points) and the phase transition. The data collected using the apparatus can be used to construct a bifurcation diagram which shows the behavior of the system as the parameter ω increases from 0 to ω_{max}. This bifurcation diagram will follow the form of a pitchfork bifurcation. It can be concluded that the data obtained were not very precise. However, the bifurcation diagrams constructed followed the same general form of a pitchfork bifurcation, and allowed for the observation of a stable fixed point becoming unstable, two new stable fixed points forming, and

then these stable fixed points becoming unstable again.

[1] R. V. Mancuso. Am. J. Phys. **68** (3), March 2000, p. 271-7.

[2] S. H. Strogatz, <u>Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering.</u> (Addison-Wesley Publishing, Reading, 1994).

[3] G. Fletcher, "A Mechanical Analog of First- and Second-Order Phase Transitions." Am. J. Phys. **65**, 1 (January 1997).

[4] Etienne Guyon, "Second-order Phase Transitions: Models and Analogies." Am. J. Phys. **43**, 10 (October 1975).