Internal and External Behavior of a Simulated Bead Pile

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May 5, 2000

This study deals with a computer model of a three-dimension pile of identical beads. The model seeks to emulate the distributions of avalanches off of an experimental pile of spherical glass beads shown to display characteristics of Self-Organized Criticality. The model can produce information about the internal activity on the "pile" to compare with the external avalanche data. The distribution of external avalanches in the computer model follows a power law of exponent b = -1.7 ± 0.2 which corresponds to a three dimensional experimental pile that has external avalanches following a power law with an exponent of b = –1.6 0± 0.01.[3] The distribution of internal avalanches does not appear to follow a power law, but instead hovers around a single value before tapering off.

INTRODUCTION

Bak, Tang and Wiesenfeld proposed Self-Organized Criticality (SOC) as a means to understand certain complex dynamical systems that develop to a critical state and then remain there.[8] This end state is considered critical because no direct causality exists between an event in the system and the magnitude of its effect. Self-Organized Critical systems are characterized by a power-law distribution of the sizes of events. Many different natural phenomena have been identified as potential candidates for SOC. Among these are fault lines, economic trends, biological extinction, and granular materials.

The classic example of SOC is a pile of sand where grains are added to the pile one at a time. As grains are added, the pile will eventually come to its critical angle of repose. At this point, a new grain may cause a large avalanche or have no effect at all. The avalanching will keep the pile's angle of repose around the critical point.[7]

Models of piles of granular materials tend to focus on either the external avalanches falling off of the pile, or the internal activity on the pile. The majority of studies focusing on the internal activities of granular piles have not been three-dimensional. An early experimental "two dimensional" model was produced by Frette et al.[6] at the University of Oslo, Norway in 1996. The Oslo model consisted of a pile of rice grains between two glass plates just wide enough to hold a single grain. Grains were added at one end of the pile. Photographs were taken of the pile at regular intervals to count the number of grains that changed positions in the pile between each picture. The number of grains that shifted was called the avalanche size. The grains of rice that were the closest to spherical had a exponent of 2.4 ± 0.2 for the power-law distribution of avalanche sizes.

Two separate computer models of the Oslo rice pile were studied by Bengrine et al.[4] in 1999 and Amaral et al. [5] in 1996. Bengrine et al. [4] found the exponent for the power-law distribution of avalanches measured in grains to be 1.53. Amaral et al. [5] looked at the distribution of dissipated potential energy and found the same exponent of 1.53. Both of these models consisted of a line of sites where each site was able to hold "grains." Grains were then added to the first site one at a time. The exact rules dictating the transfer of beads differed; however, both models used local rules relating the heights of neighboring squares along with random elements.

The model used in this study is also controlled by local rules relating the height of neighboring sites. However, here a three-dimension pile of identical beads is modeled as opposed to a two-dimensional pile of rice. A square grid where grains are added to the center square replaces the line of sites used in previous models. The model seeks to emulate the distributions of external avalanches off of an experimental pile of spherical glass beads that has been shown to display characteristics of SOC.[3] The number of beads that change sites between grain additions is recorded for analysis as well which is not possible with the experimental model.

Theory

Avalanches of all sizes cause the number of beads on the pile to fluctuate within a range of values. A graph of the number of beads on the pile versus time will show steady periods of pile building scattered with small avalanches interrupted by large avalanches. If the system displays Self-Organized Criticality, the frequency of avalanches of a given size versus that size will follow a power law. The number of avalanches of a given size N(s) varies as the avalanche size s raised to the power b.

$$N(s)= N_o s^b \qquad (1)$$

This behavior can be seen by plotting the fractional occurrence of a range of avalanche sizes versus the average avalanche size within the range. The fractional occurrence is defined by $F_0=N(s)/((w_b)(a_t))$, where N(s) is the number of avalanches in the bin, w_b is the number of avalanche sizes in the bin and a_t is the total number of avalanches occurring in the data set. The bin is simply a range of avalanche sizes. The bins used in this analysis are [0,1), [1,2), [2,3), [3,4), [4,6), [6,10), [10,16), [16,25), [25,40), [40,63), [63, 100), [100, 160), [160, 250), and [250, 400).

In order to observe the power law behavior of the system, the fractional occurrence versus average avalanche size is displayed on a log-log plot. The center of the range of avalanche values for each group can be defined by taking 10 raised to the power of the difference of the log base 10 of the first value in the bin and the last value in the bin divided by 2.

The Experimental Model

The reference for comparison with the computer model is an experimental system. The experimental system consists of a pile of 3mm diameter glass beads at its critical angle of repose, resting on a circular base on a Mettler Balance. Beads are dropped onto the center of the pile one at a time. Avalanches occur when beads fall off of the base and are no longer measured by the balance. The mass reading on the balance is allowed to become stable before the next bead is dropped onto the pile. Each bead drop, and the resulting avalanches until a stable mass reading is recorded is considered to be one event. The size of an avalanche is determined by dividing the difference in the mass of the pile between consecutive events by the mass of one bead, 0.035 g.

Base diameters from five to eleven inches were used in the experimental study, and no correlation between the base size and the behavior of the pile was found.[3] Hanna Wagner[2] worked with the same apparatus substituting a 7 X 7 inch square base for the circular base. She found no difference in the behavior of the system on a square base than that on a circular one.

Computer Model

The computer was constructed using CodeWarrior IDE 4.0 for the Macintosh. The distribution of avalanches was found using Wingz. Other analysis was performed with Microsoft Excel '98 for the Macintosh and Igor Pro.

The "bead pile" consists of a grid of squares. The number of squares in the grid can be altered. Here a 29 squares X 29 squares grid and a 39 squares X 39 squares grid are studied. Each square on the grid can hold "beads". At the start of the simulation, there are no beads on the grid. The number of beads in the center square is incremented by one at the start of each event. The number of beads in the center square is then

Table 1
External Avalanches in the 29 X 29 grid

Average Avalanche Size (beads)	Distribution of Avalanche sizes (beads)	Fractional Occurrence	Error in Fractional Occurrence
1.41	5194	0.522	0.001
2.45	2470	0.248	0.001
3.46	1201	0.121	0.001
4.90	545	0.0274	0.0001
7.75	403	0.0101	0.0001
12.65	122	0.00205	0.00002
20.00	6	0.00007	0.00001
31.62	1	0.000007	0.000007

Table 1: The distribution and fractional occurrence of avalanches off of the pile.

compared to the number of beads in each of its neighbors in a random order. If the center square has more beads than the chosen neighbor, it may give anywhere from zero to one half the difference between their numbers of beads to the neighbor. Afterwards, the receiving square can have the same or less than the number of beads as the giving square.

The number of beads the square gives to its neighbor is determined randomly. There is equal probability for any number of beads from zero to one half the difference in beads held by the two squares to be given. The greater the difference in the number of beads held by each

Table 3
External Avalanches in the 39 X 39 grid

Average Avalanche Size (beads)	Distribution of Avalanche sizes (beads)	Fractional Occurrence	Error in Fractional Occurrence
1.41	4246	0.527	0.001
2.45	2035	0.253	0.001
3.46	920	0.114	0.001
4.90	452	0.0281	0.0001
7.75	310	0.00963	0.00003
12.65	87	0.00180	0.00002
20	1	0.0000138	0.0000138

Table 3: The distribution and fractional occurrence of avalanches off of the pile.

Table 2
All Avalanches in the 29 X 29 grid

Average Avalanche Size (beads)	Distribution of Avalanche sizes (beads)	Fractional Occurrence	Error in Fractional Occurrence
1.41	582	0.0324	0.0001
2.45	1015	0.0566	0.0001
3.46	1082	0.0603	0.0001
4.90	1027	0.0286	0.0001
7.75	1958	0.0273	0.0001
12.65	3187	0.0296	0.0001
20.00	3298	0.0204	0.0001
31.62	2861	0.0106	0.0001
50.20	2005	0.00486	0.00001
79.37	760	0.00114	0.00001
126.49	160	0.000149	0.000001
200.00	10	0.000006	0.000001

Table 2: The distribution and fractional occurrence of avalanches within the pile.

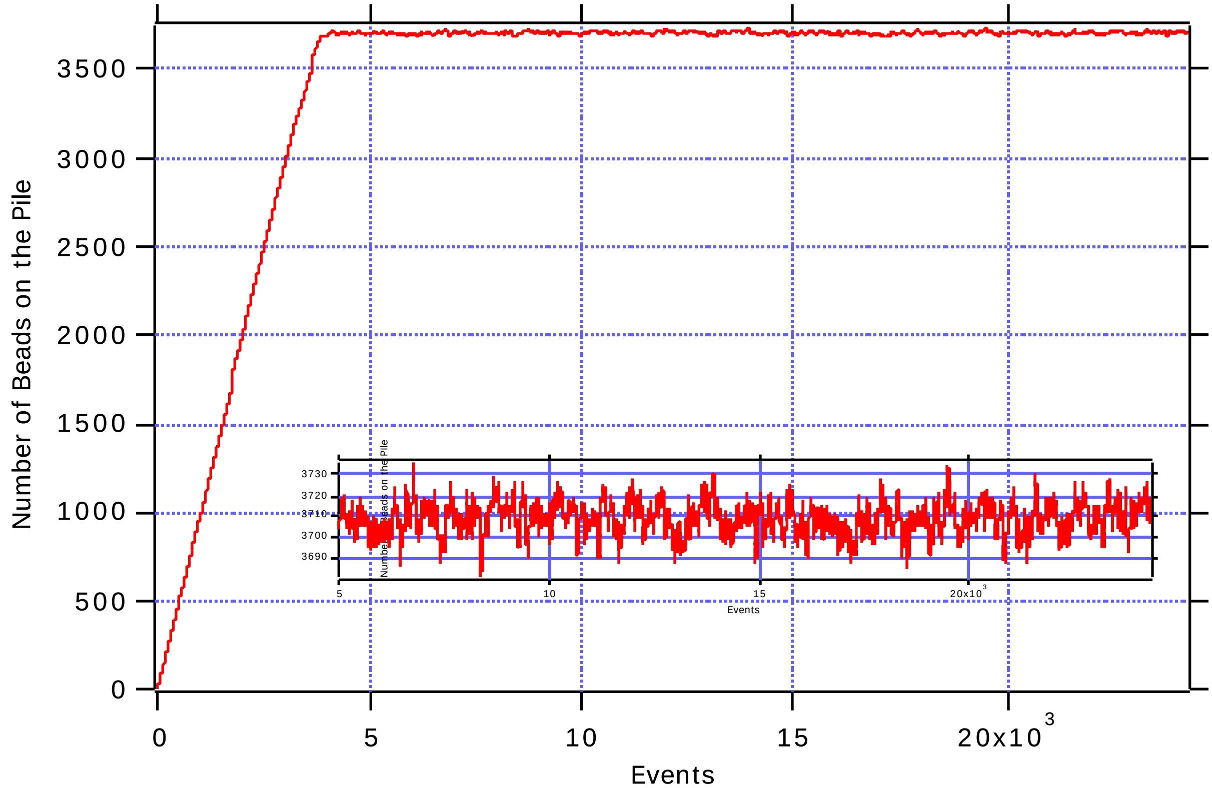
giving square. It will be compared to each of its eight neighbors in a random order as the center square, the greater the chance that the giving square will transfer beads to the receiving square.

After the center square has been compared to each of its eight neighbors (above, below, right, left, and the four corners), one of the neighbors is chosen at random to be the square was compared

Table 4
All Avalanches in the 39 X 39 grid

Average Avalanche Size (beads)	Distribution of Avalanche sizes (beads)	Fractional Occurrence	Error in Fractional Occurrence
1.41	277	0.0186	0.0001
2.45	569	0.0382	0.0001
3.46	715	0.0480	0.0001
4.90	717	0.0241	0.0001
7.75	1313	0.0220	0.0001
12.65	2264	0.0253	0.0001
20.00	2532	0.0189	0.0001
31.62	2701	0.0121	0.0001
50.20	2216	0.00647	0.00001
79.37	1215	0.00221	0.00001
126.49	331	0.000370	0.000001
200.00	41	0.0000306	0.0000007

Table 4: The distribution and fractional occurrence of avalanches within the pile.



Number of Beads on the Pile versus Events for the 29 X 29 grid

Figure 1: The simulated pile grew slowly, reaching its critical angle of repose after approximately 4000 events. It then began to show SOC behavior, as can be seen in the smaller box.

to each of its neighbors. This process is repeated for all eight of the center square's neighbors in a random order. The process is repeated with the next sixteen squares surrounding the center square. Each of these sixteen squares will be selected in a random order to be compared to its neighbors and given the opportunity to give beads to them.

Squares can give beads back to the square they received them from or to squares closer to the center than themselves. Squares on the edges of the grid are treated in the same manner. Imaginary neighbors who always hold zero beads for the sake of comparison with beads on the pile, can be given beads from the squares on the edge of the grid. Beads given to the imaginary neighbors off of the grid are said to fall off of the pile and cannot fall back on. This process continues, looping back to the center square if necessary, until no beads are exchanged between squares.

From the addition of a bead to the center square until no exchanges of beads take place is called one event. At the beginning of each event, the avalanche size and the external avalanche size are set equal to zero. Each time beads transfer from one square to another, the number of beads moving is added to the size of the avalanche. After the event takes place, the number of beads to be transferred to the imaginary squares off of the grid are added and called the external avalanche. The total number of beads on the grid is the recorded as the number of beads on the pile. This information is sent to an outfile for later analysis. Next, another bead is added to the center square and the process begins again for the next event. Data was collected for two grid sizes, 29 squares X 29 squares and 39 squares X 39 squares. For each grid size, the external avalanche data was compared to the total avalanche data. 24,289 events were recorded for the smaller grid, and over 19,289 events occurred after the system entered a critical state. 27,963 events were recorded for the larger grid and over 15,963 events occurred during the critical state. Only avalanches occurring after the system appeared to reach its critical point were considered in

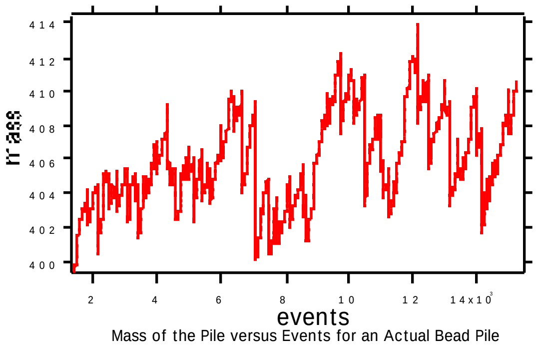


Figure 2: The mass of an experimental bead pile on a circular base with a diameter of 5 inches versus events. There are long periods of pile building interrupted by avalanches of all sizes. [Ref. 3]

calculating the fractional occurrence. The critical point was assumed to occur after the number of beads on the pile reached the point where it began to fluctuate within a range of values (see Figure 1). An error of plus or minus one bead was attributed to the number of avalanches within a given range

The external avalanche data is included in the data for all avalanches along with transfers of beads on the grid. The sum of the fractional occurrences for the external avalanches is approximately one, which is expected. However, the sum of the fractional occurrences for avalanche sizes when considering all avalanches is less than one. This is a result of the larger bin sizes for larger avalanches.

ANALYSIS AND INTERPRETATION

Number of Beads Vs Events

The number of beads versus events plot illustrates Self-Organized Criticality nicely. The 29 X 29 bead pile steadily grew with only a few minor external avalanches until it reached its critical state after around 5,000 events. The pile then began to produce external avalanches of varying sizes, keeping the number of beads between 3,680 and 3,740 beads.

The 39 X 39 bead pile behaved in a similar manor. However, the larger system fluctuates around a larger number of beads at its critical point.

The number of beads versus events graphs are similar to the mass of the pile versus events graph for the experimental bead pile. The experimental pile fluctuates between around 398 g to 414 g. This corresponds to around 11,300 to 11,800 beads where each bead weighs approximately 0.035 g.

The experimental pile holds far more beads than the computer model with the grid sizes used here. As a result, the computer model still lacks the range of avalanche sizes seen in the experimental model. This makes discerning the power-law relationship for the system more difficult.

Power-Law Relationship of External Avalanches

The experimental bead pile follows a power-law with an exponent of –1.60 ± 0.01 as shown in figure 3.[3] The data fits the power law quite well, however the larger avalanche sizes curve below the fit line a bit.

The external avalanche data points for the computer model appear to fit a similar power-law relationship. The straight line in Figure 4 shows the power law fits for the external avalanche data for both base sizes on the simulated bead pile where the squares represent the larger grid and the circles represent the smaller. The power law fit was calculated separately for each data set using equation 1. A slope of –1.7 ± 0.1 was found for the smaller pile and –1.7 ± 0.2 for the larger pile. Since the slopes are the same, the first few

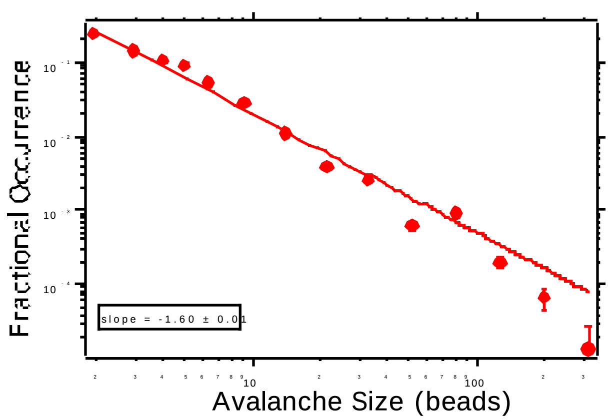


Figure 3: The power-law graph for external avalanches off of an experimental pile with a 5 inch circular base. The line indicated in the figure is a weighted power law fit to all of the data points. [Ref 3]

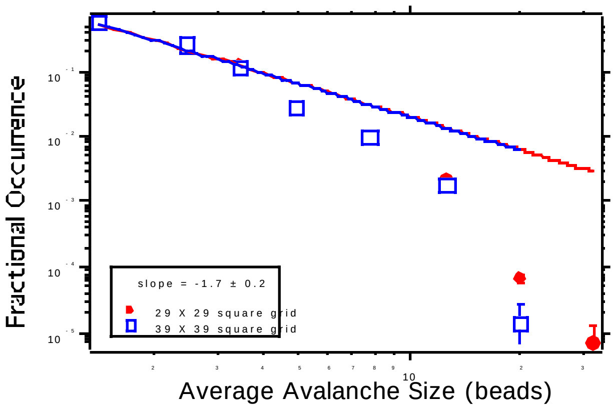


Figure 4: The power-law graph for external avalanches off of the simulated pile for two different grid sizes. The slope indicated in the figure is a power law fit that matches both sets of data within the error indicated.

points and the two line fits fall on top of each other. The exponent for this system is then b = -1. 7 ± 0.2. The similarity in the distribution of avalanches for the two grid sizes is expected since Self-Organized Criticality is size independent.

Very few large external avalanches occurred during the data runs, adding uncertainty to our data. It has been observed that with longer data runs, the distribution of larger avalanches for the experimental bead pile will come closer to a powerlaw.[3] These larger avalanches are so rare, that it takes a long time before enough of them occur to determine their frequency accurately. Here we see that the distributions for the larger pile drop off faster than that of the smaller pile. This can be explained in part by the fact that more events were analyzed for the smaller simulated pile than the larger simulated pile.

Considering the Internal Avalanches

The total avalanche distribution is the number of exchanges of “beads” during the event (when a “bead” is added to the center square until there is no more activity on the pile). Again, there is strikingly similar behavior for the two different grid sizes.

All avalanches less than ten “beads” seem to occur with nearly the same frequency for both grid sizes. This effect does make sense.

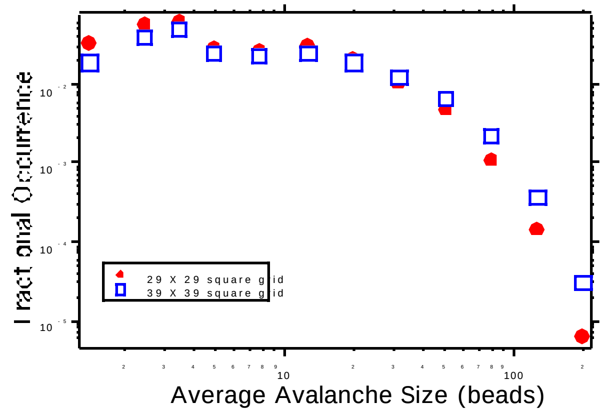


Figure 5: The fractional occurrence for all avalanches in the simulated pile for two different grid sizes.

Once a large number of beads are on the grid, it will be rare for only a few to shift position after a bead is added to the center square. Since the program continues to calculate the size of the avalanche until the pile stabilizes, it is also reasonable to conclude that it takes around the same amount of shifting before the pile settles each time.

The computer models constructed by Bengrine et al.[4] and Amaral et al.[5] show similar effects for avalanches of relatively few beads or low changes in potential energy respectively. The Oslo pile presented by Frette et al. [6] shows this effect as well. Frette et al. [6] and Bengrine et al. [4] define an avalanche in the same manner as my model: the total exchange of grains within an event. Amaral et al. [5] record the total change in potential energy between events. All three groups report that after the flat beginning, the distribution becomes a power law. It is possible that given more time or a larger grid size, the distribution of avalanches for my model would also fit a power law distribution.

Another interesting aspect of the internal behavior of the pile can be seen by observing the number of beads given from a square to its neighbor during each individual transfer. Once the pile reaches its critical state, almost all transfers were of exactly one bead. Overloading the pile at the start of the run would cause transfers of greater numbers of beads, however once the pile stabilized squares seemed to exchange a single bead at a time. This may indicate that the pile became flat at this point.

CONCLUSION

The external avalanche exponent of 1.7 ± 0.2 found with the computer model fits fairly well with the exponent of 1.60 ± 0.01 found for the experimental pile. To better confirm the exponent for the computer model, longer runs or runs with larger grids will be helpful. This will allow higher occurrences of all avalanches and a wider distribution of avalanche sizes.

Focusing on the internal or external activity on the pile changed the distribution of avalanche sizes dramatically, however the behavior of the two different size piles is quite similar. This is consistent with the size independence of Self-Organized Criticality. It would also be helpful to look at the number of beads in each individual square after each event to see if the grid is becoming a pile with a peak in the center.

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