Angstrom's Method of Determining Thermal Conductivity

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This experiment used Angstrom's method to determine the thermal conductivity of a brass rod. A periodic square heat pulse was applied to one end of a brass rod while leaving the other end at room temperature. Using Fourier analysis on the temperature data collected, the thermal conductivity of the rod was found to be 133 ± 84 *W/(m*°K)* and 163 ± 35 *W/(m*°K)* for the 1st and 3rd harmonic of the heat wave respectively, which compares well to the accepted value of 128 *W/(m*°K).*

INTRODUCTION

Angstrom developed a method of determining the thermal conductivity of a metal rod by applying an alternating heat pulse to one end while leaving the other end at room temperature. Doing this causes a heat wave to propagate down the rod and creates an observable temperature difference between two points on the rod. This also creates a varying phase relationship between the measured temperature recorded at the first and second points.[1] The thermal conductivity of the rod can be determined if the temperature of these two points is measured as a function of time.

Since the temperature changes in this experiment are periodic, the measurements of the power input used to heat the system are not required. Because of this, absolute measurements of the temperature are not required so that only relative changes in magnitude of temperature as a function of time and position must be recorded. The thermistors used in this experiment therefore do not need to be calibrated and only need to respond linearly over changes of a few degrees.[1]

This experiment uses a power source driven by a function generator to produce a heat pulse in a brass rod. Using a computer program written by a previous College of Wooster student to gather data, Fourier analysis is used to determine the thermal conductivity of the brass rod.

THEORY

For this experiment the heat generated by a chamber at one end of the rod will cause a pulse down the length of the rod. Part of the heat traveling down the length of the rod will be transmitted through conduction, part will heat the rod itself, and part of the heat will be lost to the air through radiation. The following theory follows the development given in Advanced Practical Physics by Worsnop and Flint.[2]

Thermal conductivity is defined through the rate of heat energy lost through the surface of some object.

$$\underbrace{\iiint \frac{dQ}{dt}\,dV}_{\text{Created in rod}} - \underbrace{\oiint R(T - T_0)\,ds}_{\text{Lost to radiation}} + \underbrace{\oiint k\,\vec{\ }\,T\,\ ds}_{\text{Heat conducted}} = \underbrace{\iiint s\rho\frac{\partial T}{\partial t}\,dV}_{\text{Change in rod's temperature}} \quad (1)$$

where *T* is the temperature, is the gradient operator, *ds* is an element of surface area, and *k* is the thermal conductivity of the rod. The negative sign in front of the double integral shows that the direction of heat flow is to lower temperatures.

This expression along with the expression for the heat loss through radiation and the amount of heat created in the rod can be added to get a general equation for the change in heat energy *Q* per change in time *t* of an object of density ρ occupying a volume *V* with a specific heat *s* (see equation 1).

Since there is no heat source within the rod, equation 1 simplifies since dQ/dt = 0. Green's first theorem is used to transform the integral over the surface for the conductivity into a volume integral. Since the material is uniform *k=0* in the volume integral because *k* is a constant. Because of this, there will be radiation loss.

For a rod of cross sectional area *A* and perimeter *P*, a wave equation is written for the cylindrical geometry for the bar in this experiment as

$$k\frac{\partial^2 T}{\partial x^2} - s\rho\frac{\partial T}{\partial t} - \frac{PR}{A}(T - T_0) = 0 \qquad (2)$$

where T is the temperature, T_0 is the ambient temperature, and R depends on how emissive the surface of the material is. $T - T_0$ can be simplified and rewritten as τ such that $\tau \equiv T - T_0$.

In this experiment a periodic heat wave is applied to the rod with a frequency of $\omega=2\pi f$, where f is the inverse of the period of the heat wave and of the oscillation of the temperature. Since several combined frequencies are being observed:

$$\tau(x,t) = \sum_{n=-\infty}^{\infty} C_n(x)e^{in\omega t} \qquad (3)$$

Substituting this into equation 2, it is required that $C_n(x)$ must satisfy equation 4

$$\sum\left(K\frac{\partial^2 C_n}{\partial x^2} - HC_n - in\omega C_n\right)e^{in\omega t} = 0 \qquad (4)$$

where $K=k/(s\rho)$, $H=PR/(As\rho)$. Since the function $e^{in\omega t}$ forms a complete orthogonal set, each coefficient in equation 4 must vanish. C_n must then satisfy the differential equation

$$K\frac{\partial^2 C_n}{\partial x^2} - C_n(H + in\omega) = 0 \qquad (5)$$

where K, H, and ω are constants independent of x. Equation 5 lends itself easily to the solution

$$C_n(x) = C_{n0}e^{\pm\sqrt{\lambda_n}x} \qquad (6)$$

where $\lambda_n = [(H+in\omega]/K$. Because λ_n is complex the square root of λ_n is also complex. If $\sqrt{\lambda_n} = \alpha_n + i\beta_n$ and one sets the real and imaginary parts of λ_n equal to each other respectively, then we get equations 7 and 8 below

$$\alpha_n^2 - \beta_n^2 = \frac{H}{K} \qquad (7)$$

and

$$2\alpha_n\beta_n = \frac{n\omega}{K} \qquad (8)$$

In Angstrom's method, α_n and β_n are found experimentally which lets us find the thermal conductivity $k = s\rho K$.

The solutions of the coefficient $C_n(x)$ allows us to find the Fourier series for $\tau(x,t)$ such that

$$\tau(x,t) = \sum C_{n0}e^{-\alpha_n x}e^{i(n\omega t - \beta_n x)} \qquad (9)$$

since β_n is a function of n, but is not linear. The coefficients of C_{n0} determine the form of the heat wave at $x=0$ in equation 3. So that no single form or single wave of the heating function is implied, the solutions may be complex. If the form $C_{n0} = A_{n0}\, e^{i\gamma}$ is chosen where A_{n0} and γ_n are real, then the general case of τ is:

$$\tau(x,t) = \sum A_{n0}e^{-\alpha_n x}\sin(n\omega t - \beta_n x + \gamma_n) \qquad (10)$$

for the temperature variation that can be observed experimentally, where the imaginary part of the complex function has been chosen rather than the real part since the common use of Fourier analysis employs the sine function instead of the cosine function in this form of a real function.

In this experiment the variation in heat with respect to time at two different points along the rod, x_1 and x_2 are being observed. Therefore the amplitude of the *nth* harmonic at x is $A_{n0}e^{-\alpha_n x_1}$ and the phase constant is $\gamma_n - \beta_n x_1$. At x_2 the amplitude for the same harmonic is $A_{n0}e^{-\alpha_n x_2}$ and the phase constant is $\gamma_n - \beta_n x_2$. Because of this the ratio of the amplitudes is

$$r_n = e^{-\alpha_n(x_1 - x_2)} = e^{\alpha_n(x_2 - x_1)} = \frac{A_1}{A_2} \qquad (11)$$

and $$\alpha_n = \frac{\ln(r_n)}{(x_2 - x_1)} \qquad (12)$$

The difference in phase of the two harmonics at the points x_1 and x_2 is $\varphi_n = \beta_n(x_2 - x_1)$ so that $\beta_n = \varphi_n/(x_2 - x_1)$. By determining the components of the Fourier series of the temperature at x_1 and x_2 for the nth harmonic, α_n and β_n can both be determined and we can therefore find k from equation 8:

$$k = s\rho K = \frac{s\rho n\omega}{2\alpha_n \beta_n} = \frac{s\rho n\omega (x_2 - x_1)^2}{2\varphi_n \ln\left(\frac{A_1}{A_2}\right)} \qquad (13)$$

where A_1/A_2 is the ratio of the amplitude of the harmonic n and φ_n is the difference in the phase of harmonic n. Since equation 13 can be used for any harmonic in the system several harmonics can be used for the Fourier analysis.

EXPERIMENT

For this experiment, heat was applied to a brass rod in the form of a square wave pulse. The heater which produces this wave, is powered by a Kepco BOP 100 power supply that applies a voltage to run the heater. This power supply is connected to a Tektronix TN503 function generator that is set to produce a square pulse at approximately 10^{-3} Hz with an amplitude of 10 volts. The function generator causes the power supply to switch on and off periodically every 500 seconds.

The rod was covered with foam tubing and bubble wrap to keep heat loss from conduction and convection to a minimum. The two thermistors, which are placed at x_1 and x_2 which are 15.1 ± 0.02cm apart, are set into the rod and sealed in place with epoxy. The thermistors are connected in series with a known reference resistor of 15 kΩ and a 1.5 volt dry cell battery is used as a constant source of voltage. The reference resistor allows for the calculation of the current that flows through the thermistors and allows for the calculation of the temperature through a Hewlett Packard 3421A data acquisition unit.

Data was collected with a computer program written in LabVIEW 4.1. The program reads the voltages from the data acquisition unit and changes these voltages to temperatures and then plots the data on the computer so that the data from both thermistors can be seen.[3]

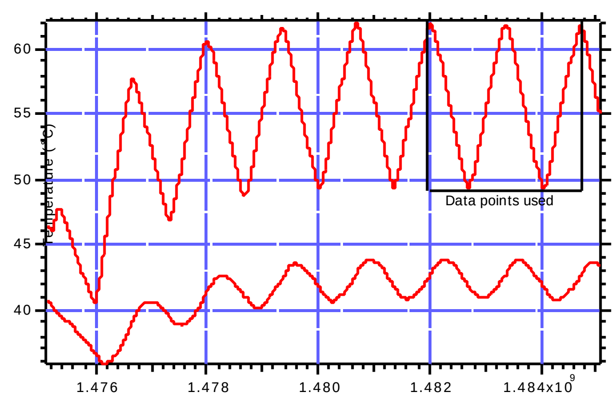
Two sets of data were taken for this experiment. The first set of data only contained about 600 data points. This was too few to use for an accurate analysis using a Fourier transformation in Igor Pro. However, it was possible to use this data to find the driving frequency needed for the next data set to give 256 data points per wavelength, since the Fast Fourier analysis algorithm used by Igor Pro requires 2^n points. The driving frequency set to 0.74×10^{-3} Hz and a data set of approximately 1900 data points was recorded. Much of the beginning data was eliminated so that only data taken after the heating of the system had reached equilibrium was used for analysis. Graphing these data points allows the data to be narrowed down until only 512 points remained, enough to use Fourier analysis, provided an integral number of wavelengths. Matching the end of the wave with its beginning was important to getting useful results since having this integral number of wavelengths keeps the Fourier analysis from generating too much error from compensating for a change in amplitude while making calculations.

The Fourier transformation function in Igor Pro was used to find the imaginary and real parts for the set of temperature data. With this, the magnitude of the heat wave and the phase difference between the heat waves was found and recorded for the near and far thermistors.

ANALYSIS AND INTERPERETATION

The second set of data was collected with a set frequency f of 0.74×10^{-3} Hz and showed the near and far thermistors temperatures in figure 1. The collected data takes about 3 full wavelengths to reach equilibrium and a regular wave pattern.

FIG.1. Data collected which shows the temperature as a function of time of the near thermistor (top) and the far thermistor (bottom).



Looking at both sets of data together makes it possible see the phase difference of the two waves. Because data taken for our analysis before the system reached an equilibrium position is unusable we must eliminate data points in the graph before equilibrium is reached. Only 512 points are kept since it is a value of 2^n that is useable for Fourier analysis. The selected data points used for the analysis can be seen in figure 2.

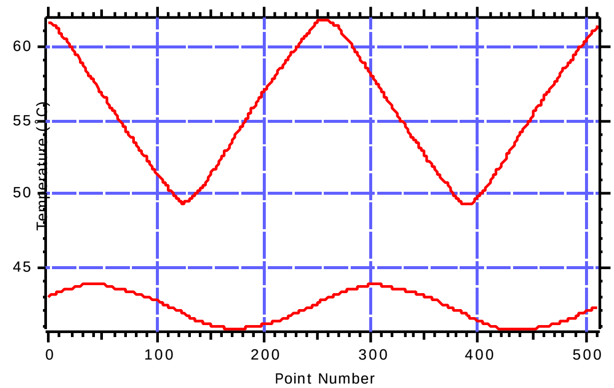


FIG.2. Graph of points used for Fourier analysis for the near thermistor (top) and the far thermistor (bottom).

These points were selected so that an integral number of wavelengths are presented so that any analysis done is more accurate since the wave can be approximated as being continuous and wrapping back on itself.

Igor Pro was used to do Fourier analysis on the data for the near and far thermistors, which gave real and imaginary components for the wave amplitudes as a function of frequency. The magnitude and phase of the near and far thermistors was found using the real and imaginary parts calculated through Fourier analysis. After finding these values the signs of the phases were checked to make sure the values were in the correct quadrant.

Only the data for the first and third harmonic is able to be used to calculate the value of the thermal conductivity. This is due to the use of the square heat wave applied. Because of this only the odd harmonics of the wave are significant for analysis. Only these two harmonics are usable since they are the only ones that have corresponding magnitudes large enough for a match in the peaks of the harmonics to be seen.

Using equation 13 and knowing the values of ω, *s*, ρ, A_1/A_2, $x_1 - x_2$, and φ for the 1st and 3rd harmonics respectively (see Table 1 below) the value of *k* is calculated.

	1st Harmonic	**3rd Harmonic**
ω	47.57*10^{-4} *Hz*	47.57*10^{-4} *Hz*
s	368 *J/(kg*°C)*	368 *J/(kg*°C)*
ρ	8500 kg/m^3	8500 kg/m^3
A_1/A_2	0.36	6.21
$x_1 - x_2$	0.151 *m*	0.151 *m*
φ	1.25 *rads*	1.71 *rads*
k	133 *W/(m*°K)*	163 *W/(m*°K)*

TABLE.1. Values used in equation 13 to calculate the thermal conductivity of the brass rod for the 1st and 3rd harmonics where ω is the driving frequency of the wave and is found from the wave period, *s* is the specific heat of brass, ρ is the density of brass, A_1/A_2 is the ratio of the amplitudes, $x_1 - x_2$ is the distance between the thermistors, and φ is the phase difference calculated between thermistors. Values for *s* and ρ come from the book Tables of Physical and Chemical Constants (ref 4).

CONCLUSIONS

The thermal conductivity of a metal rod can be measured by applying a heat pulse to one end of the rod while leaving the other end at room temperature. Measuring the temperature at two points on the rod as a function of time and using Fourier analysis, we are able to calculate the thermal conductivity *k* by using Angstrom's method. Using equation 13 from above we are able to calculate the thermal conductivity of the brass rod to be 133 ± 91 *W/(m*°K)* and 163 ± 41 *W/(m*°K)* for the 1st and 3rd harmonic of the heat wave respectively. Comparing these values to the accepted value of 128 *W/(m*°K)*[5] we find a percent difference of 4% and 27% for the 1st and 3rd harmonic respectively. The value for *k* calculated from the 3rd harmonic is much higher than the accepted value, but is still within experimental error. The value calculated from the 1st harmonic is much closer to the accepted value.

[1] J.E. Parrott and A.D. Stuckes, *Thermal Conductivity of Solids* (Pion Limited, London, 1975).

[2] B. L. Worsnop and H. T. Flint, *Advanced practical physics for students 9th ed.* (Methuen, London, 1962).

[3] A. Flewelling, *Experimental set up section from lab report*, 1995. Lab reference material.

[4] G.W.C. Kaye and T.H. Laby, *Tables of Physical and Chemical Constants: and some Mathematical Functions 14th ed.* (Longman Inc., New York, 1973).