Autocorrelation of Filtered White Noise

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A computer equipped with a data acquisition board samples the output from a white noise generator. When the white noise is passed through a low-pass filter, the autocorrelation function of the output is shown to decrease exponentially with a decay constant equal to the time constant of the low-pass filter. This experiment presents data for a range of time constants and gives an analysis for the corresponding autocorrelation functions.

INTRODUCTION

This lab serves to validate the experimental process laid out by J. Passmore and associates in their paper "Autocorrelation of electrical noise: An undergraduate experiment"[1] in which filtered white noise is analyzed. A question is raised about the nature of the output of white noise which has been passed through a low-pass filter of resistance R and capacitance C. The noise is demonstrated to show correlation for time intervals below RC when passed through such filters. The autocorrelation function is developed using Fourier transform methods. This lab looks at the autocorrelation functions for white noise and four low-pass filters.

The autocorrelation function is a method for determining how an input signal varies with time. An incoming signal which is periodic should exhibit perfect correlation at time intervals equal to multiples of its period. This means that the signal at such an interval is in the same state as the initial input signal. The overall effect of the autocorrelation function is to demonstrate how well a signal correlates with itself over a period of time. The calculation of this function involves Fourier transforms. By transforming an incoming signal into its frequency-space representation, performing basic functions on this form of the signal, and changing the altered signal into a time-space representation again, we quickly calculate the autocorrelation function of the signal.

The method of autocorrelation presented in this is largely software based. In contrast Stevens R. Miller[2] develops a method in which an autocorrelator can be built out of analog components to compute the autocorrelation function directly.

The autocorrelation has a variety of applications ranging from scattering experiments to using photon-counting and time-correlation that allows one to better understand light statistics.[3]

THEORY

White noise is defined as noise in which all frequencies are contained within the noise in equal amplitudes. This is equivalent to stating that the modulus squared of the Fourier Transform of the input noise voltage is independent of frequency over a long sampling time.[1] When the noise is passed through a low pass filter of resistance R and capacitance C, a correlation among the low frequency noise elements becomes apparent within the span of the time constant.

The time-varying input noise voltage, $V_N(t)$, can be described in linear circuitry terms by a series of impulses. If a circuit's response to an impulse can be determined, then the response to a time varying voltage is described by a sum of impulses of appropriate magnitudes accompanied by the appropriate delay response. The output from the circuit at output time t is

$$V_o(t) = \int_{-\infty}^{\infty} V_N(\tau)h(\tau - t)d\tau, \qquad (1)$$

where $h(\tau)$ is the impulse response of the circuit at input time τ. This means that the response of the circuit to the input signal is the convolution of the impulse response function of the circuit with the input signal.

The convolution theorem[4] states that the Fourier transform of a convolution of two functions is equivalent to the product of the Fourier transforms of the two functions being convoluted (2):

[illegible] (2)

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The autocorrelation function, Z(t), of the output voltage is defined[1] as

$$Z(t) = \int_{-\infty}^{\infty} \tilde{V}_o(\tau)V_o(\tau + t)d\tau. \qquad (3)$$

The autocorrelation function seeks to measure the degree to which the output changes as a function of time. The correlation theorem[1] allows the Fourier transform of the autocorrelation function to be written as the modulus squared of the Fourier transform of the output noise voltage:

$$\tilde{Z}(\omega) = \tilde{V}_o^*(\omega)\tilde{V}_o(\omega) = \left|\tilde{V}_o(\omega)\right|^2. \qquad (4)$$

The inverse Fourier transform of this function simply needs to be taken in order to obtain the autocorrelation function.

Books on linear circuit analysis show that the impulse response function for a low-pass filter is a decreasing exponential with a time constant of RC[1]. This leads to the Fourier transform pair of

$$h(t) = h(0)\exp\left(-\frac{t}{RC}\right) \qquad (5)$$

and

$$\tilde{h}(\omega) = \frac{h(0)RC}{1 + i\omega RC}, \qquad (6)$$

where h(0) is the output voltage at t=0 caused by an initial impulse.

Combining equation (2) and equation (4) yields the relation

$$\tilde{Z}(\omega) = \left|\tilde{V}_N(\omega)\right|^2 \frac{[h(0)RC]^2}{1+(\omega RC)^2} = \frac{K}{1+(\omega RC)^2}, \qquad (7)$$

where $K = \left|\tilde{V}_N(\omega)\right|^2 [h(0)RC]^2$. Calculating the inverse Fourier transform of (7) gives

$$Z(t) = K' \exp\left(-\frac{t}{RC}\right), \qquad (8)$$

where K'=K/(2RC). This shows that the autocorrelation function is an exponentially decreasing function with a time constant equal to RC, as is expected for a low-pass filter.

Normally, except for at t=0, the autocorrelation function of white noise is zero everywhere. This is because the noise voltages change rapidly enough within short time intervals to not allow any correlation to develop. This is primarily due to the high frequency elements within the white noise. When the noise is passed through a low-pass filter, the high frequency elements are eliminated and the output voltage varies more slowly. With only lower frequency components of the noise passing through the output, a correlation over short time spans (below RC) arises.

EXPERIMENT

A National Semiconductor Digital Noise Source chip[5] (MM5437) serves as a white noise generator. Other suggested noise generator devices can be found in Horowitz and Hill.[6] A Macintosh Quadra 650 is equipped with a National Instruments NB-MIO-16 Data Acquisition board to acquire the noise voltages. The data acuisition program is written in LabView 3.1, and all data analysis is carried out with Igor Pro. Electrical schematics and program listings are available upon request.

ANALYSIS AND INTERPRETATION

Each of the data sets represents a sampling of 10,200 voltages sampled at a rate of 6700 samples per second. This equates to a voltage sample time interval of 0.15 ms. For each low-pass filter a single resistor of 3.05 kΩ is used. Four capacitors ranging from 0.19 μF to 2.30 μF are used. The units of the autocorrelation function are arbitrary. Each of the autocorrelation functions have been normalized by dividing the autocorrelation function by the value of the autocorrelation function at t = 0.

Figure 1 presents the results of the first 10 ms of the autocorrelation function for output noise voltage with no filter present. As expected there is an exact correlation only at t=0 for this arrangement. After that the high frequency components of the noise cause the output voltage to change too quickly for any correlation to result. This is demonstrated in the figure by the zero baseline correlation for t > 0.

Figure 2 displays autocorrelation functions for four low-pass filters. The figure lists the values of the fitted time constants. The error in the fits are not included in the figure for the sake of legibility. The errors are included however in Table I below which lists the calculated filter time constants as well as the fit time constants. Figure 2 shows that each set of output noise voltages has a significant correlation for a time interval t < RC. In each case the average voltage is subtracted out before the autocorrelation is calculated.

The resistor and capacitors used in this experiment were measured using a General Radio RLC digital bridge. The error reported in those measurements reflects an approximated value based upon the fluctuating reading for the component. The error in the fitted time constant is computed using parameters taken directly from Igor Pro's fit function. In Igor Pro, each set of data is fit to the function

$$Z(t) = K0 + K1\exp(-K2 * t). \qquad (10)$$

Table II presents the values for the fit constants for each filter along with their associated error.

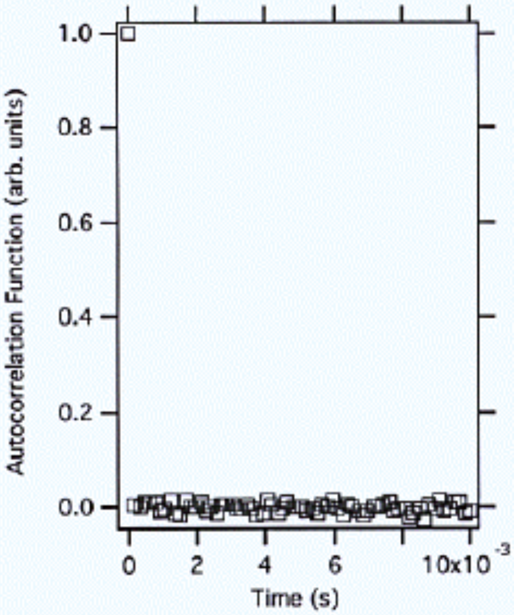

FIG. 1. Autocorrelation function of the output voltage of the noise generator with no low-pass filter present. The autocorrelation function was calculated over 10,200 voltages measured 0.15 ms apart. The plot is normalized by dividing the autocorrelation function Z(t) by the value of the autocorrelation function at t=0.

Table I. Values for the time constants of four low-pass filters are listed here as computed from measured values of R and C along with the results of the individual fits. The value of R for each case is 3.05 ± 0.01 kΩ.

C (μF)	Calculated Time Constant (ms)	Fitted Time Constant (ms)
0.19 ± 0.01	0.56 ± 0.03	0.55 ± 0.01
0.34 ± 0.01	1.04 ± 0.03	1.09 ± 0.02
0.75 ± 0.01	2.30 ± 0.04	2.16 ± 0.01
2.30 ± 0.01	7.01 ± 0.05	9.09 ± 0.07

Table II. Values for the fit constants for each of the low pass filters along with their associated errors. These values are calculated from a least squares exponential fit performed in Igor Pro.

C(μF)	K0(arb. units)	K1(arb. units)	K2 (s)
0.185	-0.003 ± 0.001	1.00 ± 0.01	1817 ± 28
0.34	-0.02 ± 0.003	1.02 ± 0.01	921 ± 18
0.75	-0.01 ± 0.001	1.01 ± 0.003	463 ± 3.0
2.30	-0.19 ± 0.01	1.12 ± 0.01	110 ± 1

The error in the fitted time constant is calculated by averaging the error in tfit of $1/(K2-\delta K2) - 1/K2$ and $1/K2 - 1/(K2+\delta K2)$.

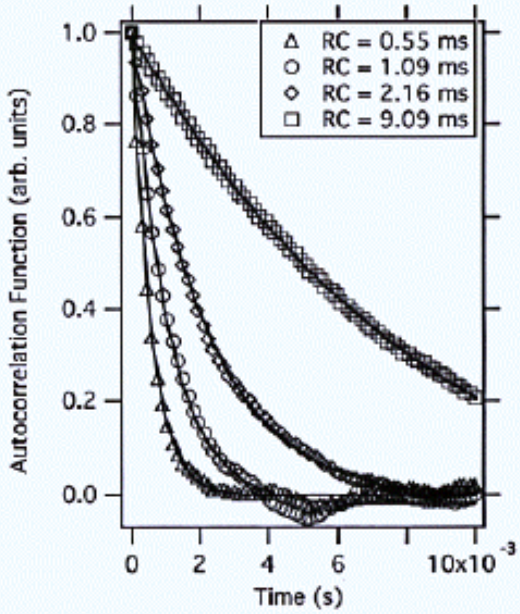

FIG. 2. Autocorrelation function of the output voltage of the noise generator with various low-pass filters present. The autocorrelation functions were calculated over 10,200 voltages measured 0.15 ms apart. The plot is normalized by dividing the autocorrelation function Z(t) by the value of the autocorrelation function at t=0. The lines represent least squares fits made over a 10 ms interval of the autocorrelation function

DISCUSSION

Figure 1 is straightforward in demonstrating that unfiltered white noise is uncorrelated over any length of time. The graph is a confirmation of this fact. Figure 2 serves to demonstrate more interesting phenomena. One of the first items noticed when inspecting the graph is that the autocorrelation functions appear to oscillate sinusoidally as they approach an asymptotic limit. This would seem to indicate that there could be some residual structure present in the signal which the autocorrelation function is incorporating. It is possible that this could be a 60 Hz signal which is riding along with the noise. One method of testing this case would be to create an adding circuit which would add an amplified 60 Hz signal to the noise and pass the combined signal through the filter and see what the autocorrelation function looks like.

By inspection of Table II and Figure 2, it seems that each of the autocorrelation functions approaches a slightly negative asymptote. The noise originally rests on some DC voltage. When the autocorrelation function is calculated with the DC offset present, the normalized autocorrelation functions approach a large non-zero limit. In order to have the function limit approach zero for long time intervals, the average output voltage is subtracted from the data set. J. Passmore suggests that a bias is introduced because the average voltage is subtracted from a finite set of voltages and this leads to a negative asymptote.[1] He shows that as more sample points are taken, the effects of the negative bias are reduced so that a sampling of 10,200 voltages are enough to ensure that the effects of the negative bias are negligible.

Up to this point very little has been said as to how well the calculated time constants compare with the fitted time constants. The first three time constants in Table I agree within 6% of each other's values, and the first two are within tolerable error of each other. The data for these filters fit the theory remarkably well. The fourth filter fitted time constant though is off from the expected calculated time constant by 30%. The first three filters are fit over a period of 10 ms (68 points) and yield the aforementioned degree of consistency.

A fit over the same time interval for the fourth filter produces a value which is in disagreement with the expected value. When the fourth filter is fit over a time interval roughly twice as long (150 points), a fitted time constant of 7.46 ± 0.19 ms results. This is much closer to the expected 7.01 ± 0.05 ms calculated time constant. This could have resulted because the autocorrelation function fluctuates slightly (possibly due to a resident frequency structure), so enough data points were not averaged over in order to gain an accurate representation of the exponential decay. However if the fourth filter fit is averaged over 200 points, the time constant drops further to a value of approximately 6 ms. This could be explained by a sinusoidal variation being present and the fit was covering a region at which this variation was reaching a minimum. If this is the case, the solution to the problem is to simply take the fit over an even longer time interval to see if the fitted time constant approaches its expected calculated value. If there is a sinusoidal influence, it may suffice to average over one full period of the sinusoidal variation in order to obtain the expected value for the time constant.

The data in Figure 2 represents only one calculated autocorrelation function which has been normalized. The data presented by Passmore calculated on sets of 10,200 points. It is possible that the sinusoidal fluctuations might average out as more autocorrelation functions for the same filter are calculated. If the Figure 2 in Passmore's article is examined,[1] a small sinusoidal variation in the autocorrelation functions can be detected. These variations would fit with the hypothesis that the variational effects decrease with an average of normalized autocorrelations. The variations would also fit the explanation that there was some resident structure in Passmore's circuit as well. An excellent continuation of this experiment would be to delve into determining the source of these sinusoidal variations.

CONCLUSION

The data from the four filters indicates that the autocorrelation function of white noise passed through a low-pass filter decreases exponentially with a decay constant equal to the low-pass filter's time constant. The data also clearly demonstrates that a strong correlation exists in the noise output for noise passed through a low-pass filter when the correlation time interval is less than the filter's time constant.

A number of extensions to this experiment immediately leap to mind. Observations could be made on circuits ranging from bandpass filters to cascaded low-pass filters. One could expand this approach to attempt to pull a weak simple signal from a noisy source. This experiment serves as a nice stepping stone into the realm of signal processing. It touches on key concepts of Fourier transforms, noise, correlation, convolution, and linear circuits.

REFERENCES

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