The Chaotic Dynamics of a Bouncing Ball

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A period doubling route to chaos was observed in a bouncing ball bearing by changing the amplitude and holding the frequency constant. It was found that Feigenbaum's delta was δ_F=1.48±0.17 which does not agree with the accepted value of 4.669620161.

I INTRODUCTION AND BACKGROUND

The field of chaos began back in the early 1960's when Edward Lorenz, a meteorologist, was investigating[1] long term weather forecasting, using an old Royal McBee computer to simulate convection rolls in the atmosphere. His model was as simple as possible, using a trio of coupled differential equations. Despite the simplicity of the three differential equations, they contain two nonlinear terms within them. In the winter of 1961 Lorenz discovered the far reaching effects of these nonlinear terms, when he took an important shortcut[2] while examining a run in further detail. Instead of starting the sequence from the beginning again, he used output from the middle of a previous printout as his initial conditions. What he found was dramatic, for the first few simulated days, the two curves were close together, but after those initial days, they diverged into radically different weather patterns. What Lorenz discovered was sensitivity to initial conditions or what is also known as the butterfly effect. This simply states that a slight change in initial conditions can have drastic effects later on.

In the early 1970's, Robert May, a biologist who was originally a theoretical physicist, was looking at variations in populations using the difference equation,[2] $x_{n+1} = \lambda x_n(1-x_n)$, where x_{n+1} is the population for the following year , x_n is the current years population and λ is the nonlinear parameter. This equation works like a feedback loop. For a given λ, an initial population, x_o, is evaluated and yields x_1. Then x_1 is fed back into the equation giving x_2 and so on. May found that for $\lambda < 1$, the population would die or converge to zero after many iterations, but for $1 < \lambda < 3$, the population after several iterations, would settle to a fixed nonzero number. As the nonlinear parameter passed through 3 it would branch, or period double, and after several years the population would settle down to two different numbers, low one year, high the next. As λ was further increased, the system will continue to period double on a route to chaos.

Soon Mitchell Feigenbaum was thinking about the quadratic map.[2] This was a simple way of looking at the logistic map that May had been using for populations. See figure 1 on the next page for a picture of quadratic map.

A parabola is drawn, with a maximum height equal to the nonlinear parameter, λ. The line y=x is also drawn. The initial seed, x_o, is selected and a vertical line is drawn from x_o to the parabola. To use x_o, as the next seed, a horizontal line is drawn from the parabola to the line y=x. From the line y=x, another vertical line is drawn to the parabola, and the process is repeated indefinitely. This will yield the same results that May found, but is a nicer way to observe this. As Feigenbaum continued to study the quadratic map, he began to look at the geometry of the bifurcation diagram.

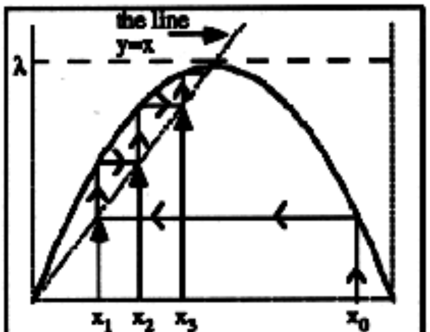

Fig. 1 This a picture[3] of the quadratic map. It starts with an initial seed, X_0, and a line is drawn up to the parabola. From the parabola, a line is drawn to the line y=x, which is the dashed line. This process is repeated indefinitely. The parameter λ, is just the distance that the parabola is from the x-axis.

What he found was a relationship between the ratio of $\lambda_{n+1}-\lambda_n$ and $\lambda_n-\lambda_{n-1}$, in which the system goes from period n to period 2n. This relationship is,

$$\delta_F \equiv \lim_{n\to\infty} \delta_n = 4.66920161...,$$

where $\delta_n = \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n}$.

The following experiment was originally designed[4] in 1985 by N.B. Tufillaro and A. M. Albano from the Department of Physics at Bryn Mawr College. They designed the experiment to be suitable as a simple undergraduate experiment to explore some of the basic properties of nonlinear dynamics, including period doubling, exploring the quadratic map and Feigenbaum's δ. Later this experiment was reproduced[5] by Stephen R. Levengood, a Physics major at Fort Lewis College. Levengood used a similar experimental setup to that of Tufillaro et. al. with the exception that he added a second oscilloscope to display the piezoelectric output vs. the output of the function generator. Levengood also tried the experiment with a glass ball. Levengood's results did not agree with those originally found by Tufilaro et. al. This experiment has also be reproduced by former students here at the College of Wooster.

II THEORY

The setup for this experiment consists of a Mechanical Vibrator that undergoes simple harmonic motion. In its simplest form, the equation for simple harmonic motion is,

$$\ddot{x} + \omega_o^2 x = 0, \qquad \text{Eqn. 1}$$

where $\ddot{x} = \frac{d^2x}{dt^2}$ and $\omega_o^2 = \frac{k}{m}$. Eqn. 1 can be rewritten as follows,

$$\ddot{x} = -\omega_o^2 x, \qquad \text{Eqn. 2}$$

giving an equation for the acceleration of the Mechanical Vibrator. The acceleration will be a maximum when x is at its maximum displacement, A, defined as the amplitude. The acceleration for the watchglass varies as both the amplitude and the frequency are changed. The acceleration for the ball is constant, g, the acceleration due to gravity. If $\omega_o^2 A < g$ then the net force on the ball at all points in the oscillation will be downward. When $\omega_o^2 A = g$ then the net force on ball at the peak of the oscillation will be zero. It is when $\omega_o^2 A > g$, that there is a net force upward on the ball at the peak of the oscillation and the ball begins to bounce. So substituting in g for $\ddot{x}$ into Eqn. 2 gives,

$$g = +\omega_o^2 A, \qquad \text{Eqn. 3}$$

with $\vec{g}$ opposite to $\vec{A}$, where $\omega_o = 2\pi f$. This implies that,

$$\frac{1}{f^2} = \frac{4\pi^2}{g} A, \qquad \text{Eqn. 4}$$

which gives a plot of $\frac{1}{f^2}$ vs A a slope of $\frac{4\pi^2}{g}$. Equation 3 is the necessary condition that the ball just leaves the watchglass. For an experimental plot of this, refer to figure 2 below.

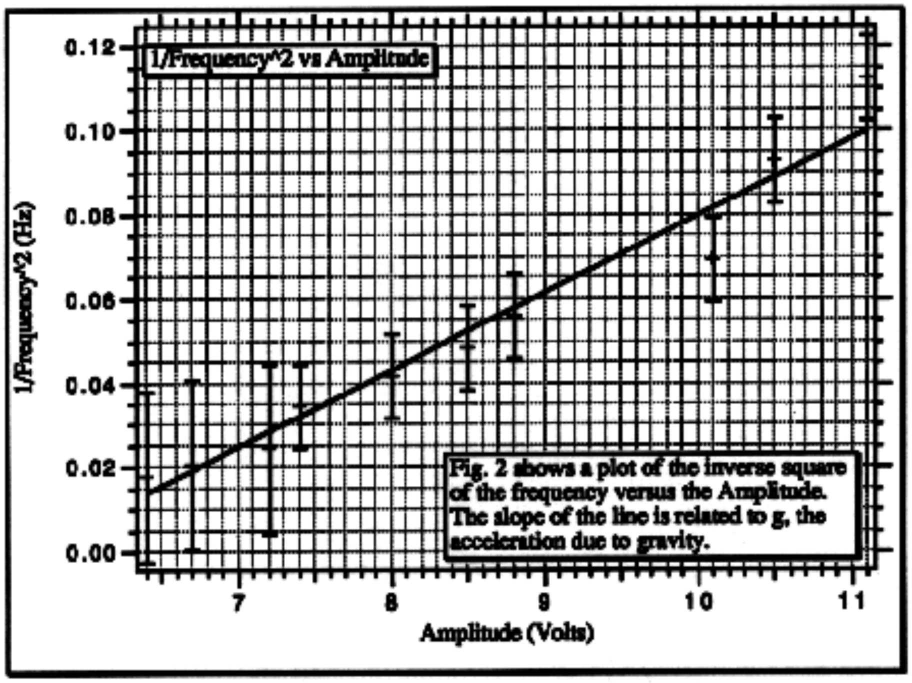

Fig. 2 shows a plot of the inverse square of the frequency versus the Amplitude. The slope of the line is related to g, the acceleration due to gravity.

At this point, the ball has passed from the zero state to period 1. For notational convenience, T_n ≡ period n. As the amplitude is increased, the ball goes from T_1 to T_2. At the point the ball begins to bounce, it is in free fall. The acceleration of the ball remains constant, g , but the acceleration of the watchglass changes in time since it is attached to the Mechanical Vibrator which is undergoing simple harmonic motion. Consider a single oscillation of the watchglass as it begins to rise from its lowest position. The acceleration for the ball is always down, but as the watchglass ascends its acceleration is up. Because of this, from the preceding oscillation, the watchglass would achieve its minimum before the ball did, and therefore would collide with each other as the watchglass is rising while the ball is falling down on it. At the point of contact, the ball's velocity will change direction and the ball will go back up. Gravity is still pulling the ball down, and so it would be possible for the ball to strike the watchglass a second time, as the watchglass reaches its maximum height. At this point both the ball and the watchglass have an upward velocity. Both continue to rise, and when the watchglass achieves it's maximum, the ball can strike the watchglass before it begins its decent finishing the oscillation, all ready to begin the process again.

In the bifurcation diagram, it is easy to see Feigenbaum's delta, which will be represented as δ_F. Refer to figure 3 below to see the bifurcation diagram and a geometrical interpretation of δ_F.

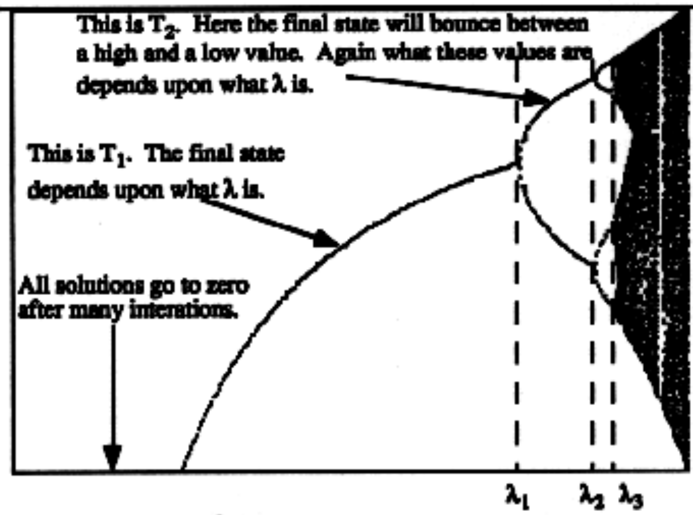

Fig. 3 shows a picture[3] of the bifracation diagram for the logisitic map that Feigenbaum studied. The bottom axis is λ and the left axis is the final state of the system after several iterations. Feigenbaum's δ is just $(\lambda_2-\lambda_1)/(\lambda_3-\lambda_2)$. It is now believed that this number is unversal and is equal to 4.66920161...

This relationship for δ_F is[1],

$$\delta_f = \lim_{n\to\infty} \delta_n = 4.66920161\ldots, \quad \text{Eqn. 5}$$

$$\text{where } \delta_n = \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n}. \qquad \text{Eqn. 6}$$

III APPARATUS

The setup consists of an acoustic Texas Transducer; a Pyrex watchglass; steel ball bearings, one with a diameter of 5±1 mm and a mass of 4.4784±0.0001 g and the other with a diameter of 10±1 mm and a mass of 8.3751±0.0001 g; a Pasco Mechanical Vibrator, Model SF-9324; a Pasco Digital Function Generator-Amplifier; a Keithley 2001 Multimeter; a Hewlett Packard 54600A oscilloscope, and a Stanford Research Systems Low-Noise Preamplifier, Model SR560. The Function Generator provides the sine wave that drives the Mechanical Vibrator. Both the generated function and the signal from the transducer are read on the oscilloscope. The acoustic transducer picks up the sound of the ball bouncing, and this signal is converted into a voltage which can be read by the oscilloscope. The Preamplifier is used to eliminate the 60 cycle noise that occurs, thus allowing the experimenter to see the actual signal from the transducer. The Keithley Multimeter is used to check for the stability of the frequency and to also provide a more accurate measurement of it. For a schematic of the circuitry refer to figure 4 below.

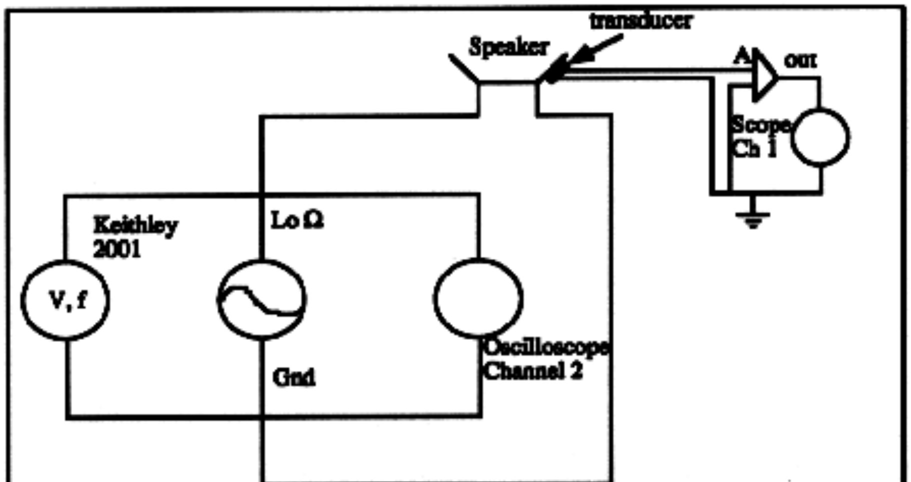

Fig. 4 shows the electrical schematic for the experiment, where A is the input of the Preamplifer, Lo Ω is output of the function generator, and the oscilloscope is a digital storage scope on which the data is read. The Keithley 2001 can measure both AC voltage and the frequency f.

The following settings were used. For channel 1 of the oscilloscope, the settings were: 50 mV per division, position of 100 mV, 100 ms per division, and trigger of 0 V. For channel 2: 5 V per division, position of -10 V, 100 ms per division and trigger of 0 V. The display mode was normal, the trigger mode was normal, and the oscilloscope measured the frequency and the peak-to-peak amplitude of channel 2, the signal from the transducer. The following are the settings for the preamplifier: filter cutoffs at 1 kHz with a high pass at 12 dB per octave, coupling at ACS, source A, gain mode at low noise and the gain set at 5. The Keithley Multimeter needs to be set to "low frequency RMS", AC voltage, so that it can read the low frequencies with stability and accuracy.

IV PROCEDURE

The experiment originally used a piece of piezoelectric material instead of a transducer to detect when the ball striked the watchglass. The piezoelectric material was abandoned because there were problems getting a clear signal and thus made it difficult to detect even period one. A description of the setup and procedure using the piezoelectric is explained by Tufillaro et. al.[4]

After all the connections were made as described in the Apparatus section, all parts of the setup are turned on and the ball is allowed to bounce. Two signals should be clearly seen on the oscilloscope, one that is the sine function driving the Mechanical Vibrator, and the other from the transducer of the ball hitting the watchglass. One should be able to see a clear sinusoidal wave on the oscilloscope. It is best to adjust the amplitude and frequency of the

Mechanical Vibrator so that the ball is in period one. This can be done easily by just listening to and watching the ball bounce until one hears the ball hit once for every cycle of the watchglass. This is a simple process and is a convenient way to check that the signal coming through is clear. It should be observed that for each oscillation of the watchglass, there is one distinct spike in the signal from the acoustic transducer.

The data could have been collected in one of two ways: changing the frequency and holding the amplitude constant, or holding the frequency constant and changing the amplitude. Arbitrarily, it was decided to do the latter, change the amplitude while holding the frequency constant. All data sets were taken using a frequency of 5 Hz. A low frequency was chosen because it is within the range of the apparatus, and it seemed to give the best results, after trying several frequencies. The watchglass is then set into simple harmonic motion using a small amplitude initially. At this point the ball should be resting on top of the watchglass and not bouncing yet. The amplitude is then slowing increasing until the ball goes from resting to just beginning to bounce. Because of the weak signal, it will be hard to see on the oscilloscope, but can clearly be heard. This value of A (or λ) is easier to find by listening for the ball bouncing than to see it on the oscilloscope. The frequency and the amplitude of the transition is recorded. Then the amplitude is further increased until the ball passes from T_1 to T_2 and the values for the frequency and the amplitude are again recorded. This process is repeated for as many period doublings as can be accurately observed.

V DATA

A total of four data runs were done. In all runs, it was found that the stability of the frequency was consistent, but was of a different value than was set on the function generator. All values for the frequency were read and recorded from the Keithley. The first two runs used a ball of mass 4.4784±0.0001 g and the last two runs used a ball of mass 8.3751±0.0001 g. The data can be found in Table 1 below.

Run #	Transition	Freq (Hz)	Amp (V)
1	0 to period 1	4.8±0.2	8.8±0.1
1	T1 to T2	4.8±0.1	10.5±0.1
1	T2 to T4	4.89±0.01	11.2±0.1
1	T4 to T8	4.90±0.01	11.7±0.2
1			
1	Delta-->	1.40±0.76	
2	0 to period 1	4.87±0.02	8.5±0.1
2	T1 to T2	4.88±0.02	10.8±0.1
2	T2 to T4	4.90±0.01	11.72±0.01
2	T4 to T8	4.90±0.01	12.34±0.01
2			
2	Delta-->	1.48±0.17	
3	0 to period 1	4.90±0.02	8.2±0.1
3	T1 to T2	4.88±0.02	10.8±0.1
3	T2 to T4	4.88±0.2	11.5±0.1
3	T4 to T8	4.89±0.1	12.2±0.1
3			
3	Delta-->	1.00±0.35	
4	0 to period 1	4.85±0.02	8.2±0.1
4	T1 to T2	4.85±0.03	10.8±0.1
4	T2 to T4	4.89±0.01	11.5±0.1
4	T4 to T8	Uncertain	Uncertain

Table 1. Shows the data for all four runs. The first two data runs used a ball of mass 4.4784±0.0001 g and the last two used a ball of mass 8.3751±0.0001 g. The last three columns are transistion, frequency and amplitude respectively.

VI ANALYSIS

The best experimental value of δ_F=1.48±0.37 does not agree at all with the excepted value of $\delta_F = 4.66920161$. The experimental value is off by more than a factor of three. One reason for this large error is that finding T_8 is very difficult, but without it, an estimate of δ_F can not be made. Another difficulty is that many times there were packets of spikes, and it was hard to know whether to count the whole packet as one or to count the individual spikes. Because of this it is hard to determine precisely when the system goes from say T_2 to T_4. Also because there was a fluctuation in the frequency, it would then affect the value of λ for the transistions. This helps to account for the drastically large errors.

The propagated error in δ_F starts with Eqn. 6,

$$\delta_F = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_2}. \qquad \text{Eqn. 7}$$

From Eqn. 7, the uncertainty in δ_F is,

$$d\delta = \left[\frac{1}{(\lambda_3 - \lambda_2)^2} d\lambda_1^2 + \frac{(\lambda_3 - \lambda_1)^2}{(\lambda_3 - \lambda_2)^4} d\lambda_2^2 + \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_3 - \lambda_2)^4} d\lambda_3^2 \right]^{1/2}, \qquad \text{Eqn. 8}$$

and then substituting the appropriate data for set #3 gives $d\delta = 0.17$ for the third data set.

VII CONCLUSIONS

The experiment was partially successful in that data was collected, but not so because of the large approximations and difficulties in finding T_8. The best experimental value of $\delta_F = 1.48 \pm 0.17$ was nowhere close to the accepted value. However, a clear T_1, T_2 and T_4 were observed. The experiment seems to work much better with the transducer than the piezoelectric, since the transducer gives a cleaner, clear signal.

Things that can be done to improve this experiment would include having the Mechanical Vibrator suspended instead of resting in the counter, using a generator capable of working with even smaller frequencies, finding a better way to get rid of the 60 cycle noise without the overkill of using a preamplifier, or even using an oscilloscope that could measure smaller time intervals to spread the signal out across the screen, allowing the experimenter to more easily see T_8 or higher periods.

[1] Robert C. Hilborn, Chaos and Nonlinear Dynamics:An Introduction For Scientists And Engineers, (Oxford University press, New York, 1994).
[2] James Gleick, Chaos:Making a New Science, (Penguin, New York, 1988).
[3] John Lindner, "Feigenbaum 1.11", THINK Pascal (The College of Wooster, unpublished).
[4] N.B Tufillaro and A.M. Albano, Am. J. Phys. 54, 939-944 (1986).
[5] Stephen R. Lovengood, J. of Undergrad. Research in Physics, 18, 29-32.