Motion of a leaky pendulum

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The period of a physical pendulum whose mass and effective length varied with time has been studied and the results are compared to a non-leaky physical pendulum. The period of the system was determined from the data collected through a potentiometer attached to the pivotal point of the pendulum and then interfaced with a computer running Science Workshop software. The results agreed well with the qualitative theoretical assumptions of the motion of the pendulum.

I. INTRODUCTION

A pendulum is a rigid body of any shape pivoted about a fixed horizontal axis about which it is free to oscillate under the influence of gravity. In contrast to the simple pendulum whose ideal point mass is suspended from a weightless cord, a pendulum made from a real physical body is called a physical pendulum.

For sufficiently small amplitudes the period of the motion of the pendulum is virtually independent of its amplitude. For such cases, it primarily depends on the geometry and the mass of the pendulum, and on the local value of g, the acceleration of gravity. Pendulums have therefore been used as very reliable control elements in clocks, or inversely as instruments to measure g.[1]

If, on the other hand, the mass and the effective length of the pendulum is varied with time, its motion will be different than that of a simple physical pendulum. As the mass and the effective length of the pendulum change with time, its period will also change.

In the following experiment, the motion of a physical pendulum whose mass and effective length vary with time was investigated, and the results were compared to the theoretical calculations. This lab was designed to introduce the undergraduate student to one of the fundamental problems of mechanics in an inexpensive fashion. The analysis of the data was easy enough even for an introductory student, and the apparatus can be used for simple classroom demonstrations.

II. THEORY

The pendulum used in the experiment was modeled as a fluid column of length h measured from the bottom end of an aluminum tube, which was attached to a meter stick as can be seen in Fig.1. The period of a physical pendulum is given by

$$T_{empty} = 2\pi\sqrt{I_1 / m_1 g L_1} \qquad \text{(eq.1)}$$

as derived in Appendix A. Here, L_1 is the distance from the axis of rotation of the pendulum to the center of mass of the empty pendulum, I_1 is the moment of inertia of the empty pendulum, m_1 is the mass of the empty pendulum, and g is the acceleration due to gravity. The mass of the pendulum can be calculated by adding the mass of the empty tube and the mass of the meter stick. Figure 1 shows the pendulum and the distances used in the calculations.

The composite system with the tube filled with fluid can be described by calculating the center of mass y and its moment of inertia *I* as *h* varies from full tube (h=L) to empty tube (h=0). The center of mass of the composite system is given by

$$z = L - y + d \qquad \text{(eq.2)}$$

where y is the center of mass from the bottom of the stick and can be calculated from the equation

$$y = \frac{(h/2)m_{sand}(h/L)}{m_{tube} + m_{ms} + (h/L)m_{sand}} + \frac{(m_{tube}L/2) + \left(\frac{L+x}{2}\right)m_{ms}}{m_{tube} + m_{ms} + (h/L)m_{sand}} \quad \text{(eq.3)}$$

Here, m_{sand} is the mass of the fluid when the tube is full, L is the length of the aluminum tube, and x is the distance from the top of the meter stick to the top of the aluminum tube. The expressions m_{tube} and m_{ms} are the masses of the empty aluminum tube and of the meter stick respectively, whereas the expression $m_{sand}(h/L)$ is the mass of the sand when the height of the column is h. The center of mass of the sand column is located at (h/2). The derivation of y can be found in Appendix B.

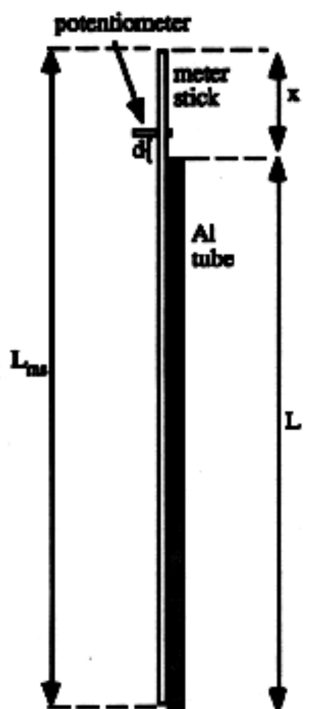

Fig.1. The leaky pendulum and its dimensions.

The moment of inertia of the composite system is given by

$$I = I_1 + I_2, \quad \text{(eq.3)}$$

where I_1 is the moment of inertia of the empty pendulum and I_2 is the moment of inertia of the sand column.

In equation 3, I_1 is basically the sum of the moment of inertia of the meter stick and the moment of inertia of the empty tube, and it can be calculated using the parallel axis theorem.[3] If m_{ms} is the mass of the meter stick, and L_{ms} is the length of the meter stick (and also the pendulum), then the moment of inertia of the meter stick is given by

$$I_{ms} = \frac{1}{12}m_{ms}L_{ms}^2 + m_{ms}\left(\frac{L_{ms}}{2} - x + d\right)^2 \quad \text{(eq.4)}$$

where d is the distance from the axis of rotation to the top end of the aluminum tube.

The moment of inertia of the aluminum tube can be calculated in a similar manner with the equation

$$I_{tube} = \frac{1}{12}m_{tube}L^2 + m_{tube}d_{tube}^2 \quad \text{(eq.5)}$$

where m_{tube} and L are the mass and length of the empty aluminum tube respectively, and d_{tube} is the distance from the center of mass of the tube to the pivot point of the composite system.

I_2, on the other hand can be calculated from the following equation

$$I_2 = \frac{1}{12}m_{sand}\frac{h^3}{L} + \frac{m_{sand}}{L}\left(\frac{h}{2}\right)\left[(L_{tube} + d) - \frac{h}{2}\right]^2 \quad \text{(eq.6)}$$

whose derivation can be found in Appendix C.
The period of the composite system is then given by

$$T = 2\pi\sqrt{\frac{I_1 + I_2}{(m_1 + (h/L)m_{sand})gz}} \qquad \text{(eq.7)}$$

where h is the sand column height measured from the bottom of the tube.

If equation 7 is examined more closely, it can be seen that the argument inside the square root increases as the value of h decreases, thus increasing the value of period T. The drop in the value of h also causes the value of y to decrease as it can be seen in equation 2. One can therefore argue that the period of the system should initially increase due to the fact that the effective length of the pendulum increases as the fluid drains out. The increase in the effective length of the pendulum is caused by the dropping center of mass of the system. Eventually, when the fluid mass becomes too small to maintain this trend, the center of mass moves upward, causing the period to approach the empty tube value.[1]

III. EQUIPMENT

In order to conduct the experiment the following equipment was used:
-leaky pendulum: The leaky pendulum used in the experiment is custom built from a meter stick, an aluminum tube, and a rubber stop with a hole drilled though it. As it can be seen from Fig.1. in the theory section, the aluminum tube was duct taped to the meter stick at a point with a distance x=11 centimeter from the top of the meter stick. The lower end of the aluminum tube was clogged with the rubber stop, whose mass was measured with a Mettler AJ100 electronic balance. A one-turn potentiometer linked to the data acquisition components of the system was then attached to the meter stick 10 centimeters from the top part of the meter stick, and its shaft served as the pivot point for the pendulum to oscillate. The combined mass of the empty pendulum was 327.7 gr. ± 0.05 gr.

Figure 2 shows the mass and the dimensions of the individual parts of the leaky pendulum system.

m (m.s)	141.0 ± 0.05 g
m (tube)	174.6 ± 0.05 g
m (rubber stop)	12.0508 ± 0.00005 g
m (pendulum)	327.7 ± 0.05 g
L	0.910 m
L (m.s)	1.000 m
d	0.010 m
x	0.110 m

Fig.2. The mass and dimension of the individual parts of the leaky pendulum.

-PASCO CI-6560 Signal Interface II: This piece of equipment was used to interface the potentiometer and the leaky pendulum to a Macintosh Quadra 700 computer running Science Workshop data acquisition software. It has four digital channels and three analog channels of which only two analog channels were used.
-PASCO CI-6552 Power Amplifier II: The amplifier was used to provide the potentiometer with a constant DC voltage of amplitude 10 volts to act as a reference voltage. It was arbitrarily connected to the analog channel C on the CI-6560 signal interface because resolution was not a problem due to the one-turn potentiometer.
-PASCO CI-6503 Voltage Sensor: The voltage sensor was designed to be used with the PASCO computer interface system and was able to measure voltages from -10 V to +10 V from the potentiometer. It was connected to the signal interface through analog channel B.
-KS 14381 500 Ω potentiometer: Connected to the pendulum, and acting as its pivot point, this one-turn potentiometer produced electrical signals proportional to the angular displacement of the pendulum. These signals were then recorded by the Macintosh Quadra 700 computer. The circuit diagram of the potentiometer can be seen in Figure 3 which also specifies the colors of poles of the potentiometer. Here the S+ pole acts

as the pivot point of the pendulum, and as the pendulum oscillates, it moves between the V+ and V- poles of the potentiometer. This motion of the pendulum therefore produces electrical signals that are proportional to the angular displacement of the pendulum.
-Science Workshop 1.0.1 Data Acquisition Software: Written by PASCO Scientific, this program collected data via the interface system that is explained above.
-Igor 1.25 Software: This program was used to analyze the acquired data and to determine the period of the pendulum.

IV. PROCEDURE

The data collection procedure was rather simple after the equipment was set up as shown in Figure 3. In order to collect data, Science Workshop 1.0.1, a data acquisition and analysis software was used. Before starting with the data collection process of the experiment, the potentiometer and the program had to be calibrated accordingly, so that the above mentioned software could be utilized. The calibration was done by displacing the pendulum to its maximum and minimum points and giving a value of +10 volts and -10 volts to the readings taken at these points respectively. Before starting the actual data acquisition part of the experiment with the leaky pendulum, two control data sets were collected, one with the empty tube, one with the tube filled with sand, but not leaking. These data sets were collected in order to check the performance of the setup, and in order to have some control data so that the end results from the leaky pendulum could be compared to a non-leaky one.

After this was done, two more data sets were collected with the sand leaking from the pendulum while the pendulum was oscillating. In order to make the sand leak, the duct tape preventing the flow of the sand was removed from the rubber stop with the hole. Again, the data was collected with Science Workshop software and saved to the hard drive of the computer. This data was later analyzed with another program, Igor 1.25. For the analysis part of the experiment, the period of oscillation of the pendulum was determined by finding the time lapsed between each point of maximum amplitude. The period values for each run or data set were then plotted versus time, and then the resulting graphs were fitted with fourth order polynomials in the form of

$y = K_0 + K_1x + K_2x^2 + K_3x^3$. Finally, these curve fits were compared to the theoretical data.

The theoretical data was obtained by deriving the equation describing the period of the system and then calculating several values for the period as the mass changes. In order to do the calculations Microsoft Excel spreadsheet program was used. By deriving the equation, it was assumed that the change in the mass of the pendulum was linear with time. Using this assumption, the time for the sand to be completely drained was measured with a stopwatch, and an equation relating the sand column height to time was formulated. This equation relating the sand column height to time was determined to be $h=-0.044\,t$. Several values for the column height h were calculated with Microsoft Excel and then these values were used in equation 7 so that theoretical values could be obtained. Figure 4 shows the column height versus time plot which was used in the determination of the equation relating the sand column height to time. Finally, the results were analyzed by Igor 1.25 and plotted for comparison with the experimental results.

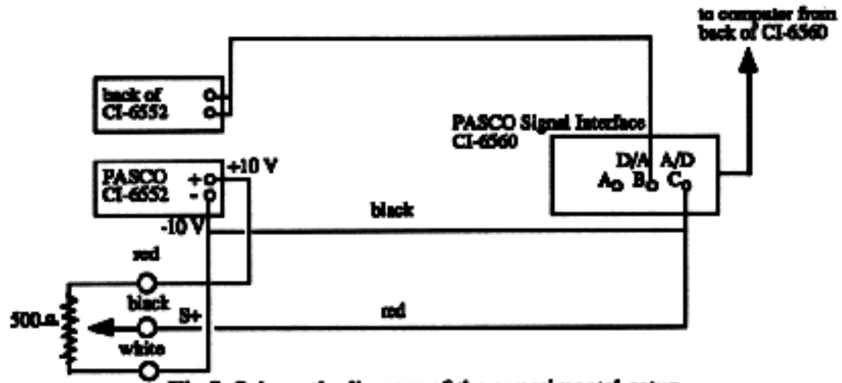

Fig.3. Schematic diagram of the experimental setup

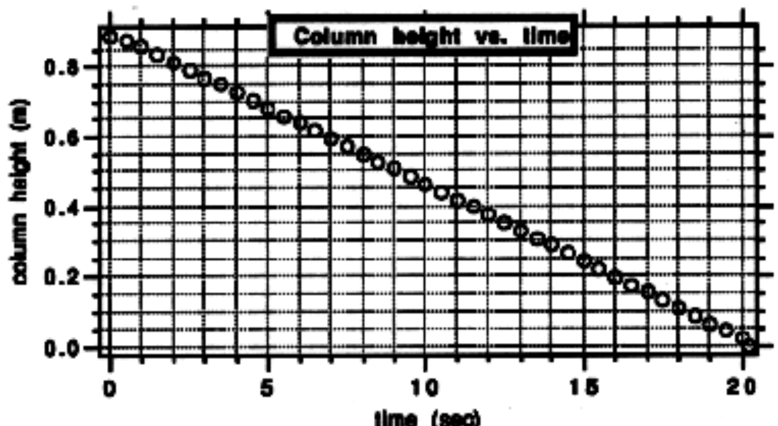

Fig.4. Column height versus time plot. According to this plot the equation relating time and the column height was determined to be $h = -0.044\ t$.

V. DATA ANALYSIS & RESULTS

After the data was analyzed with Igor 1.25 as explained in the procedure section, the following graphs were obtained. Fig.5, which was used in order to calculate the period of the system when the tube was empty, shows the oscillation of the empty pendulum. Figures 6 (a) and (b) show the results from the analysis of experimental data, whereas Fig.7 is the superposition of figures 6 (a) and (b) and the results from the theoretical calculations.

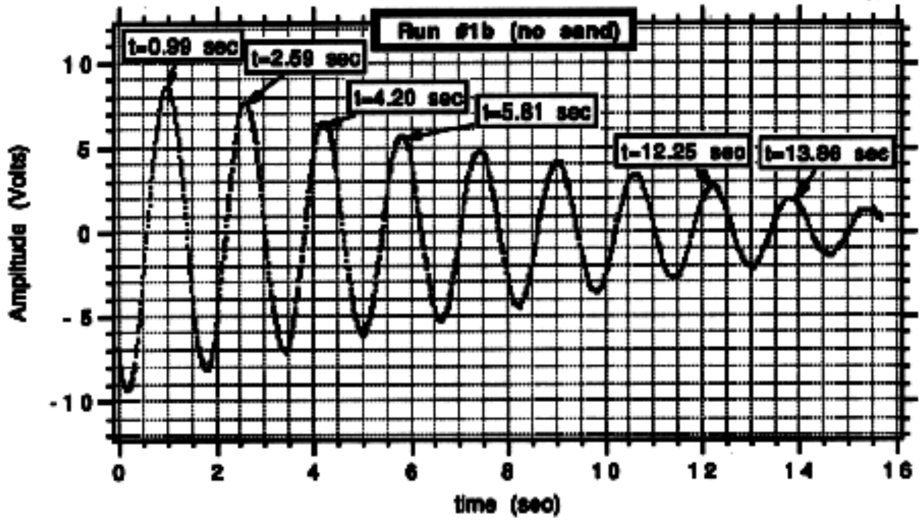

Fig.5. Period determination of the empty system.

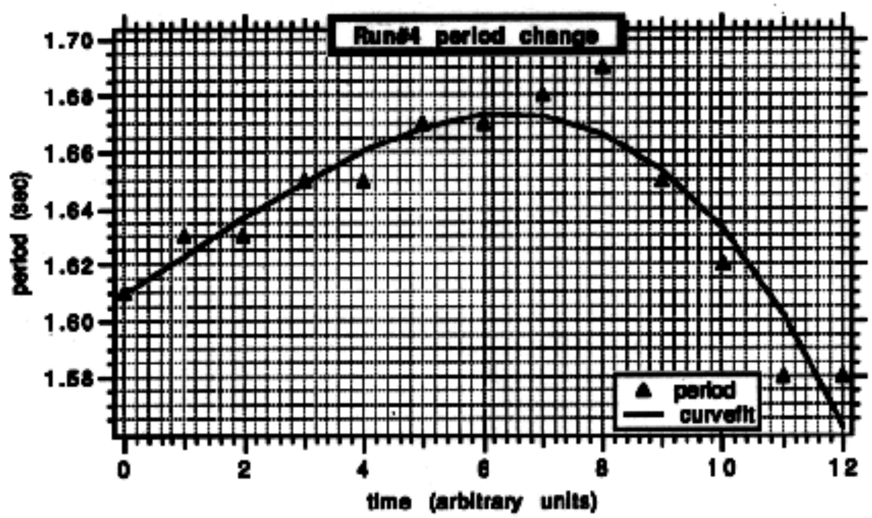

Fig.6 (a). Period change with time for the leaky pendulum, Run#4. The equation used to fit the data is a fourth order polynomial in the form of $y = K_0 + K_1x + K_2x^2 + K_3x^3$, where K_0=1.60992, K_1= 0.0126557, K_2= 0.00067839, and K_3= -0.000172067.

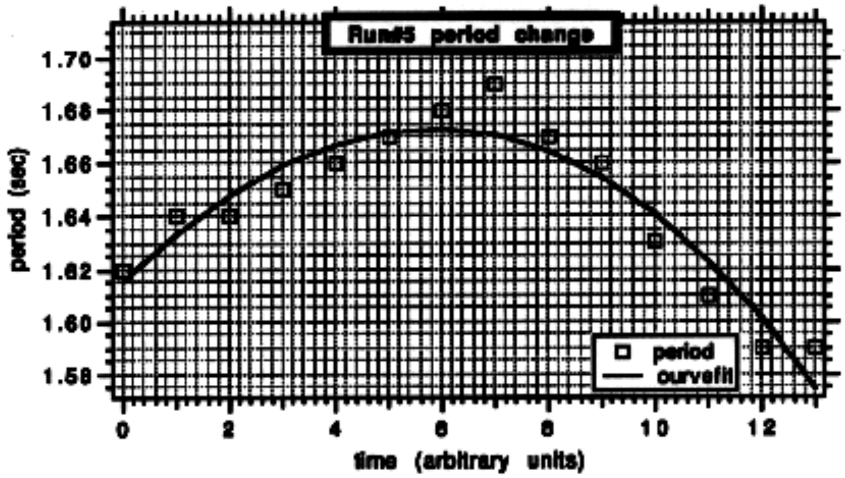

Fig.6 (b). Period change with time for the leaky pendulum, Run#5. The data was fit with a fourth order polynomial in the form of $y = K_0 + K_1x + K_2x^2 + K_3x^3$, and the value of the parameters are K_0=1.61614, K_1= 0.018917, K_2= -0.00144808, and K_3= -1.94861e-05.

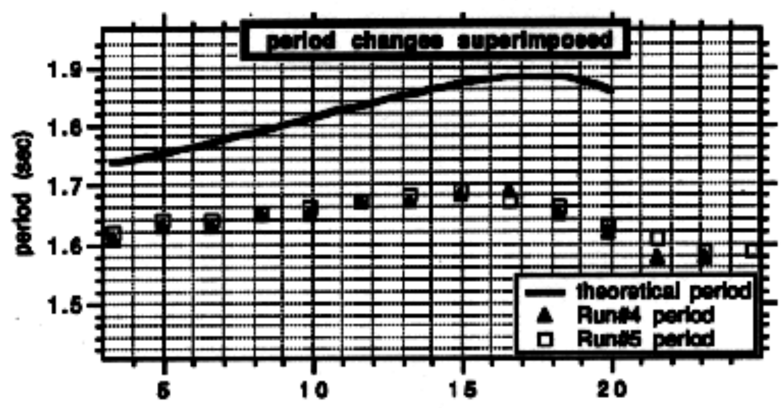

Fig.7. Superposition of data from runs #4 and #5 and the values of the period obtained theoretically. The plot shows the change in period.

In Fig.7, Run#4, Run#5, and the theoretical results are superimposed in order to show that the period of the system changes in a similar fashion for both data sets. Before analyzing Fig.7 and comparing the results of Run#4 and Run#5, it should be noted that the initial conditions for both data sets were not identical. The mass of the pendulum and the height it was released differed for both runs. The results are superimposed so that a qualitative understanding of the behavior of the system could be provided.

Figure 8, on the other hand, shows only the theoretical results calculated with the equation 7 for Run#4.

As we can see from the Fig.8, the period of the pendulum shows an initial increase followed by a decrease, due to the theory explained earlier in the theory section.

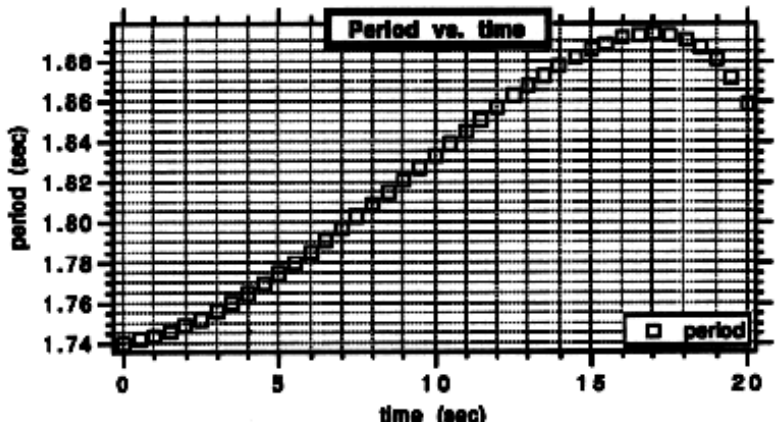

Fig.8. Period change calculated theoretically for the leaky pendulum.

Fig.8. also shows that the values of the period calculated theoretically are considerably larger than that of experimental measurements. This is due to the fact that in the theoretical calculations the frictional energy losses were not taken into consideration. In the real situation, though, the frictional energy loss constitutes an important factor in the behavior of the period. Another cause of the divergence from the experimental data is that the change in the height of the sand column, and therefore the change of its mass with respect to time, could not be determined with good accuracy. In order to approximate dm/dt, it was assumed that the sand leaked in a linear fashion from the aluminum tube. The time required for the sand to be completely drained from the tube was measured three times, and the results were averaged to minimize the discrepancies. From the data obtained in this fashion, an equation relating the column height was derived with the help of the Fig.4.

The period of the empty pendulum, on the other hand, was determined by measuring the time interval between each maximum amplitude point over the whole range of data, and then averaging the results. In order to measure the time intervals between the peaks, Fig.5 was used with Igor 1.25 software. The final calculation gives a period of 1.61 sec for the empty pendulum compared to the calculated theoretical value of 1.85 sec. It should be noted, though, that in the theoretical calculation of the period, the frictional energy loss and the uncertainty in dm/dt were neglected.

The period of the system, as it can be seen from both the experimental results and theoretical calculations, changes with time, since it depends on the mass and the effective length of the pendulum. As the sand in the tube initially drains out, the center of mass drops, causing the effective length of the pendulum to increase and thus increase the period. This happens even though the total mass of the system is ever decreasing. Eventually, as the mass of the sand becomes too small to effect the period, the center of mass of the system moves upward, causing the period to approach empty tube value.

In conclusion, as we see from figures 6 (a) and (b), the period of the system first increases and upon reaching a maximum value, it starts decreasing again. Figure 7 compares the results from two different data sets, and shows that the experiment is reproducible with a fair accuracy, i.e. both data sets give similar qualitative results. Figure 8 shows that the theoretical calculations also yield similar qualitative results for the leaky pendulum. In addition, it should be noted that in actuality the period of the system changes continuously, whereas the experimental measurements of the change in period are averages over one cycle.

Quantitative results for the error analysis could not be obtained for the experimental data due to the fact that the sand column height could not be determined during the experiment. Even though there are no quantitative results for the error in the experiment, some observations concerning them are worth noting. The most accurately measured quantity, besides the constants such as the moment of inertia of the system, mass of the pendulum, etc., in the experiment was the period T of the pendulum. Science Workshop software was able to collect data at 100 Hz, and that provided enough points to determine the period accurately with Igor 1.25. Here, the periods measured were the averages over one cycle, and they could be compared to the instantaneous values from the theoretical calculations, assuming that the theoretical period values are snapshots from ever-changing period values of the system.

One of the biggest problems encountered during the experiment was the slow flow rate of the sand, and the quick damping of the oscillations of the pendulum. Initially, the hole in the rubber stop was too small for the sand to drain completely before the oscillations of the pendulum die out. Even though the size of the hole was increased to ensure data collection, the damping was still too big for the acquisition of a larger data pool, and therefore more useful and accurate information.

The discrepancy of the experimental data from the theoretical calculations can be explained by the above arguments and the fact that the theoretical data was in reality only an approximation of the real system. In order to derive the equation for the theoretical calculation two major assumptions, the linear flow rate of the sand, and the absence of frictional losses, were made. However, the discrepancy in the results for the period of the empty pendulum cannot be explained with the above reasoning since there is no sand leaking which requires an approximation. The reasons for the difference in the experimental and the theoretical results remain to be discovered.

In order to increase the accuracy of experimental results, I would recommend to decrease the friction of the pendulum at the pivot point. This can be done by using a potentiometer with lesser friction. A system with less frictional losses would enable a larger data collection with sand leaking at a slower rate. Another setup for a similar experiment is described by Mires and Peter, where they utilized a pendulum made of glass supported on sapphire blades to reduce the friction.[2]

APPENDIX A: DERIVATION OF EQUATION 1

A pendulum is a device that executes periodic motion by oscillating about a horizontal axis. As shown in Figure 9, the center of mass of the pendulum is a distance L_1 below the axis of rotation. As the pendulum oscillates, its center of mass moves back and forth along a circular arc with a radius L_1. The position of the center of mass along this arc can be specified either by the variable x, the arc length measured from the equilibrium position, or by θ, the angle between the vertical direction and the line connecting the axis and the center of mass. These two quantities are related by $x = L_1\theta$. Since the pendulum rotates about a fixed axis, its motion obeys $\sum\tau = I\alpha$, where the only torque is caused by the weight of the pendulum. As it can be seen from Fig.9, the lever arm of the weight is $L_1 \sin\theta$, and therefore the torque τ is given by

$$\tau = -mgL_1 \sin\theta$$
$$= -mgL_1 \sin\left(\frac{x}{L_1}\right) \qquad \text{(eq.A1)}$$

where the minus sign denotes a restoring torque. Using $\sum\tau = I\alpha$, it can be shown that

$$I\alpha = -mgL_1 \sin\left(\frac{x}{L_1}\right) \qquad \text{(eq.A2)}$$

Since the tangential acceleration a_t of the center of mass is equal $L_1\alpha$, we have

$$a_t = L_1\alpha = -\frac{mgL_1^2}{I}\sin\left(\frac{x}{L_1}\right) \qquad \text{(eq.A3)}$$

For small angles $\sin(x/L_1) \approx x/L_1$, and therefore the acceleration of the center of mass is

$$a_t = -\left(\frac{mgL_1}{I}\right)x \qquad \text{(eq.A4)}$$

But this equation is just the simple harmonic motion equation with

$$\omega^2 = \frac{mgL_1}{I} \qquad \text{(eq.A5)}$$

Thus the period T of a pendulum for small angles is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgL_1}} \qquad \text{(eq.A6).}$$

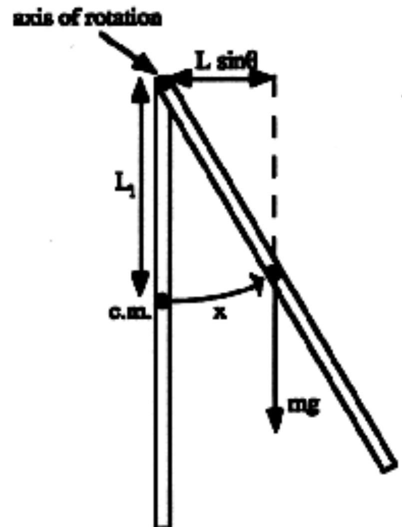

Fig.9. The physical pendulum

APPENDIX B: CALCULATION OF THE CENTER OF MASS OF THE PENDULUM

Since the center of mass of a composite system is given by

$$x = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \qquad \text{(eq.B1)[4]}$$

the center of mass of the pendulum can be calculated by finding the centers of mass of the individual components of the system. The first component of the pendulum is the aluminum tube which has a length of L. Since the aluminum tube has constant density, and a symmetrical shape, its center of mass is given by

$$c.m._{tube} = \frac{L}{2} \qquad \text{(eq.B2)}$$

The center of mass of the meter stick can be calculated in a similar fashion, keeping in mind that the length of the meter stick is $L_{ms} = (L + x)$, where x is measured to be 11 centimeters. Thus,

$$c.m_{ms} = \frac{L_{ms}}{2} = \frac{L + x}{2} \qquad \text{(eq.B3)}$$

Since the height of the sand column is h, the center of mass of the sand column is

$$c.m_{sand} = \frac{h}{2} \qquad \text{(eq.B4)}$$

Substituting equations B2, B3, and B4 into the equation B1 gives

$$y = \frac{(h/2)m_{sand}(h/L)}{m_{tube} + m_{ms} + (h/L)m_{sand}} + \frac{(m_{tube}L/2) + \left(\frac{L+x}{2}\right)m_{ms}}{m_{tube} + m_{ms} + (h/L)m_{sand}} \qquad \text{(eq.B5)}$$

which is the center of mass of the pendulum from the bottom of the meter stick. Here the expression $m_{sand}(h/L)$ denotes the mass of the sand when the height of the column is h.

Therefore the center of mass from the axis of rotation is

$$z = L - y + d \qquad \text{(eq.B6)}$$

Figure 10 in Appendix C shows the distances used in the calculation of the center of mass.

APPENDIX C: DERIVATION OF I_2

The parallel axis theorem states that if you know the rotational inertia of a body about any axis that passes through its center of mass, you can find its rotational inertia about any other axis parallel to this.[3] Therefore, the moment of inertia I_2 of the sand column can be calculated using the parallel axis theorem, which is

$$I = I_{cm} + Ms^2 \qquad \text{(eq.C1)}$$

where M is the mass of the body and s is the perpendicular distance between the two parallel axes, whereas the expression I_{cm} is the rotational inertia of the body about a parallel axis through its center of mass. I_{cm} can be calculated by the following integral:

$$I_{cm} = \int r^2 dm \qquad \text{(eq.C2)}$$

where dm can be written as

$$dm = \left(\frac{m_{sand}}{L}\right)dr \qquad \text{(eq.C3)}$$

Substituting eq.C3 into eq.C2, and setting the limits of the integral, we get

$$I_{cm} = \int_{-h/2}^{h/2} \left(\frac{m_{sand}}{L}\right) r^2 dr \qquad \text{(eq.C4)}$$

which can be solved to obtain

$$I_{cm} = \frac{1}{12} m_{sand} \frac{h^3}{L} \qquad \text{(eq.C5).}$$

If $L + d$ is the distance from the axis of rotation of the pendulum to the bottom of the aluminum tube, then s is

$$s = (L + d) - \frac{h}{2} \qquad \text{(eq.C6)}$$

Since the mass of the sand column depends on the height of the column, M can be expressed as

$$M = \frac{m_{sand}}{L}\left(\frac{h}{2}\right) \qquad \text{(eq.C7)}$$

Putting equations C1, C6, and C7, the expression for the moment of inertia I_2 of the sand column becomes

$$I_2 = \frac{1}{12} m_{sand} \frac{h^3}{L} \qquad \text{(eq.C8)}$$

$$+\frac{m_{sand}}{L}\left(\frac{h}{2}\right)\left[(L+d)-\frac{h}{2}\right]^2$$

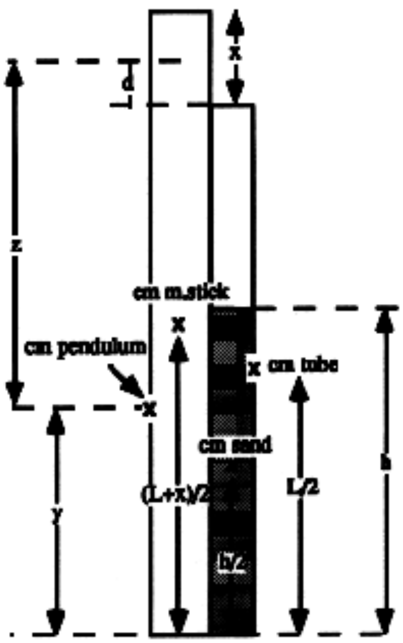

Fig.10. The leaky pendulum and the distances used in the development of the theory.

REFERENCES

[1] J. M. Keller, "Pendulum", McGraw-Hill Encyclopedia of Physics, 2nd ed. edited by Sybil P. Parker (McGraw-Hill, Inc., New York, 1991), p.998-1000.

[2] R. M. Mires and R. D. Peters, "Motion of a leaky pendulum", Am. J. Phys. **62**, 137-139 (1994).

[3] R. Resnick and D. Halliday, Fundamentals of Physics (Wiley, New York, 1988).

[4] A. L. Stanford and J. M. Tanner, Physics for Students of Science and Engineering (Academic Press, Inc., Orlando, 1985).