Single-Photon Spin-Orbit Coupling for Cluster State Quantum Computation

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Outline

• What is it?
  Overview of photon spin-orbit coupling (SOC)

• How does it work?
  Physics of SOC

• What is it good for?
  Applications of SOC for cluster state quantum computing

• Conclusions
  Progress toward observing and controlling the effect
What is it?
Overview of photon SOC
Overview of photon SOC

• Ingredients:
  – Inhomogeneous medium
  – Cylindrical symmetry
  – Quasi-paraxial photon:
    \( \kappa \ll |k| \), where \( \kappa \equiv \sqrt{k_x^2 + k_y^2} \)
  – Quasi-monochromatic

• Simplest example:
  step-index optical fiber
  – \( n_{\text{core}} > n_{\text{cladding}} \)
  – Infinite cladding radius
Overview of photon SOC

- **Ingredients:**
  - Inhomogeneous medium
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    \[ \kappa \ll |\mathbf{k}|, \text{ where } \kappa \equiv \sqrt{k_x^2 + k_y^2} \]
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- **Simplest example:**
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  - \( n_{\text{core}} > n_{\text{cladding}} \)
  - Infinite cladding radius
The spin-orbit coupling effect

- Field in fiber proportional to \( E(r,t) \propto e^{i(\beta z - \alpha)} \)

- Define the fiber V parameter:
  \[
  V \equiv \frac{\omega a}{c} \sqrt{n_{\text{core}}^2 - n_{\text{cladding}}^2}
  \]

- Assume the far-from-cutoff limit: \( V >> 1 \)

- Solve Maxwell equations to obtain fiber eigenmodes
The spin-orbit coupling effect

- Field in fiber proportional to $E(r,t) \propto e^{i(\beta z - \alpha)}$

- $V \equiv \frac{\omega a}{c} \sqrt{n_{\text{core}}^2 - n_{\text{cladding}}^2}$

- The effect:
  - Photon eigenmodes have well defined values of both spin $\sigma$ and orbital angular momentum $m_\ell$ along z axis
  - Modal propagation constants $\beta$ split according to

$$\beta \approx k \left\{ 1 - \frac{f_1}{V^2} + \frac{f_2}{V^3} - \frac{f_3}{V^4} (\sigma \cdot m_\ell) \right\} \quad \text{when} \quad n_{\text{core}}^2 \approx n_{\text{cladding}}^2$$

(the “$f$” functions depend on simple fiber parameters)
Observable consequence of SOC

• If two fiber photon eigenmodes with different values of the quantity $\mathbf{\sigma} \cdot m_\ell$ propagate in superposition, the relative phase between them will vary as the photon field propagates, due to their velocity mismatch.

• This phase shift gives rise to a $\mathbf{\sigma}$-dependent rotational effect. For example:

$$
\sigma \cdot m_\ell = -2\hbar \quad \sigma \cdot m_\ell = +2\hbar
$$

The superposition mode field rotates clockwise or counterclockwise, as determined by the sign of $\mathbf{\sigma}$.

• This is a spin-controlled spatial-mode Hadamard gate
Spin-controlled Hadamard gate

\[ E \propto \cos \left( m_\ell \phi \right) \rightarrow \cos \left[ m_\ell \left( \phi + \sigma \Delta \beta_z \right) \right] \]

\[ \sigma \cdot m_\ell = -\hbar \quad \sigma \cdot m_\ell = +\hbar \]

\[ |0\rangle \rightarrow 45^\circ \rightarrow |0\rangle + |1\rangle \]

Flipping the photon spin (circular polarization) also flips the direction of rotation of the superposition spatial mode.

\[ \sigma \cdot m_\ell = -2\hbar \quad \sigma \cdot m_\ell = +2\hbar \]

\[ |0\rangle \rightarrow 22.5^\circ \rightarrow |0\rangle + |1\rangle \]
How does it work?
Physics of spin-orbit coupling
How does it work?

Physics of spin-orbit coupling

• It is the inhomogeneity of the medium gives rise to the effect—there is no paraxial free-space SOC

• B. Zeldovich (PRA ’91) first mentioned photon SOC:
  – He treats a many-mode fiber with a parabolic index profile
  – Rotation of speckle pattern observed, but not of single modes
  – Kapany and Burke (’72) mention mode rotation, but not in the context of single photons or of spin-orbit coupling.

• We consider individual fiber modes propagating in a step-index few-mode fiber
  – The SOC effect is strongest for the step-index case
  – This is due to the $-\nabla \left[ \mathbf{E} \cdot \nabla (\ln \varepsilon(\mathbf{r})) \right]$ term in the Helmholtz eqn
Photon angular momentum quantum numbers

- Define orbital angular momentum operators and quantum numbers:
  \[
  \hat{L} \equiv -i\hbar (\mathbf{r} \times \nabla)
  \]
  \[
  \hat{L}_z = \hat{L} \cdot \hat{z} = -i\hbar \left( \frac{\partial}{\partial \phi} \right)
  \]
  \[
  \hat{L}_z |\ell\rangle \equiv \left( \hat{L} \cdot \hat{\mathbf{L}} \right) |\ell\rangle = -\hbar^2 \ell (\ell + 1) |\ell\rangle
  \]
  \[
  \hat{L}_z |m_\ell\rangle = \hbar m_\ell |m_\ell\rangle
  \]

- Define “spin-1” operators and quantum numbers:
  \[
  \hat{S} \equiv -i\hbar \epsilon_{ijk} \hat{e}_k
  \]
  \[
  \hat{S}_z = \hat{S} \cdot \hat{e}_3 = -i\hbar \epsilon_{ij3} = \begin{pmatrix} 0 & -i\hbar & 0 \\ +i\hbar & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
  \]
  \[
  \hat{S}_z |\sigma\rangle = \hbar \sigma |\sigma\rangle
  \]
Photon angular momentum quantum numbers

- Define orbital angular momentum operators and quantum numbers:

\[ \hat{L} = -i\hbar (\mathbf{r} \times \nabla) \]

\[ \hat{L}_z = \hat{L} \cdot \hat{z} = -i\hbar \frac{\partial}{\partial \phi} \]

\[ \hat{L}^2 |\ell\rangle \equiv (\hat{L} \cdot \hat{L}) |\ell\rangle = -\hbar^2 \ell (\ell + 1) |\ell\rangle \]

\[ \hat{L}_z |m_\ell\rangle = \hbar m_\ell |m_\ell\rangle \]

- Define “spin-1” operators and quantum numbers:

\[ \hat{S} \equiv -i\hbar \epsilon_{ijk} \hat{e}_k \]

\[ \hat{S}_z - \hat{S} \cdot \hat{S}_3 = -i\hbar \epsilon_{\sigma\sigma'} \begin{pmatrix} 0 & -i\hbar & 0 \\ +i\hbar & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ \hat{S}_z |\sigma\rangle = \hbar \sigma |\sigma\rangle \]

The above are NOT generally good photon quantum numbers!
Photon angular momentum quantum numbers

\[
\hat{L}^2 |\ell\rangle = -\hbar^2 \ell (\ell - 1) |\ell\rangle \\
\hat{L}_z |m_\ell\rangle = \hbar m_\ell |m_\ell\rangle \\
\hat{S}_z |\sigma\rangle = \hbar \sigma |\sigma\rangle
\]

The above are NOT generally good photon quantum numbers!

- However, in the aforementioned paraxial and far-from-cutoff limits, \(m_\ell\) and \(\sigma\) are good quantum numbers for the photon fields in a step-index fiber:

\[
\hat{L}_z \mathbf{E}(\mathbf{r}, t) \to -i\hbar \frac{\partial}{\partial \phi} \mathbf{E}(\mathbf{r}, t) = \hbar m_\ell \mathbf{E}(\mathbf{r}, t)
\]

\[
\hat{S}_z \mathbf{E}(\mathbf{r}, t) \to \begin{pmatrix}
0 & -i\hbar & 0 \\
+i\hbar & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \mathbf{E}(\mathbf{r}, t) = \hbar \sigma \mathbf{E}(\mathbf{r}, t)
\]
Fiber photon eigenmodes

- Fiber photon eigenmodes can be labeled as follows:

\[ E \propto J_{m_\ell} (kr) e^{im_\ell \phi} e^{i(\beta z - \omega t)} e_\sigma \leftrightarrow |n, m_\ell, \omega, \sigma\rangle \]

\[
\begin{align*}
  n &= 1, 2, 3, \ldots \\
  m_\ell &= 0, \pm \hbar, \pm 2\hbar, \ldots \\
  \omega &\in \mathbb{R} \\
  \sigma &= \pm \hbar \\
  e_\sigma &\equiv \hat{x} + i\sigma \hat{y}
\end{align*}
\]
Fiber photon eigenmodes can be labeled as follows:

\[ E \propto J_{m_\ell} (kr) e^{im_\ell \phi} e^{i(\beta_z - \omega t)} e_\sigma \leftrightarrow |n, m_\ell, \omega, \sigma\rangle \]

\[ n = 1, 2, 3, \ldots \quad \omega \in \mathbb{R} \]

\[ m_\ell = 0, \pm \hbar, \pm 2\hbar, \ldots \quad \sigma = \pm \hbar \]
Fiber photon eigenmodes

- Fiber photon eigenmodes can be labeled as follows:

$$E \propto J_{m_\ell}(\kappa r) e^{im_\ell \phi} e^{i(\beta z - \omega t)} e^\sigma \leftrightarrow |n, m_\ell, \omega, \sigma\rangle$$

$$n = 1, 2, 3, \ldots \quad \omega \in \mathbb{R}$$

$$m_\ell = 0, \pm \hbar, \pm 2\hbar, \ldots \quad \sigma = \pm \hbar$$

- Propagation constant

$$\beta \equiv \sqrt{k^2 - \kappa^2} \approx k \left(1 - \frac{1}{2} \left(\frac{\kappa}{k}\right)^2\right)$$

- Transverse wavenumber

$$\kappa(n, m_\ell, \omega, \sigma)$$
Fiber photon eigenmodes

- The roles of $m_\ell$ and $\sigma$ are now explicit:

$$
E \propto J_{m_\ell}(kr) e^{i m_\ell \phi} e^{i (\beta z - \omega t)} e_\sigma \leftrightarrow |n, m_\ell, \omega, \sigma\rangle
$$

$$
-i\hbar \frac{\partial}{\partial \phi} E(r, t) = \hbar m_\ell E(r, t)
$$

$$
\begin{pmatrix}
0 & -i\hbar & 0 \\
+i\hbar & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} E(r, t) \begin{pmatrix} 1 \\
i\sigma \\
0
\end{pmatrix} = \hbar \sigma E(r, t) \begin{pmatrix} 1 \\
i\sigma \\
0
\end{pmatrix}
$$

$$
\beta \approx k \left\{ 1 - \frac{f_1}{V^2} + \frac{f_2}{V^3} - \frac{f_3}{V^4} (\sigma \cdot m_\ell) \right\}
$$
Eigenmode pictures

\[ E \propto J_{m_\ell} (kr) e^{im_\ell \phi} e^{i(\beta_z - \omega t)} e_\sigma \leftrightarrow |n, m_\ell, \omega, \sigma\rangle \]
\[ |n, m_\ell, \omega, \sigma\rangle \]

\[ \omega \text{ is fixed} \]

\[ V \equiv \frac{\omega a}{c} \sqrt{n_{\text{core}}^2 - n_{\text{cladding}}^2} \]
\[ |n, m_\ell, \omega, \sigma \rangle \]

\( \omega \) is fixed
\( n = 1 \)
\( |m_\ell| > 1 \)

\[ V \equiv \frac{\omega a}{c} \sqrt{n_{\text{core}}^2 - n_{\text{cladding}}^2} \]

\( \sigma = \mp 1 \)
\( m_\ell = \pm 2, \pm 3, \pm 4 \)
- Eigenmodes have cylindrical symmetry

\[ |n, m_\ell, \omega, \sigma\rangle \]

\( \omega \) is fixed

\( n = 1 \)

\( |m_\ell| > 1 \)

\[ V \equiv \frac{\omega a}{c} \sqrt{n_{\text{core}}^2 - n_{\text{cladding}}^2} \]
- Eigenmodes have cylindrical symmetry
- Grouped into families of $|m_\ell|$
• Eigenmodes have cylindrical symmetry
• Grouped into families of $|m_\ell|$ 
• Paired according to $\sigma$

\[ |n, m_\ell, \omega, \sigma \rangle \]

$\omega$ is fixed

$n = 1$

$|m_\ell| > 1$

\[ V \equiv \frac{\omega a}{c} \sqrt{n_{\text{core}}^2 - n_{\text{cladding}}^2} \]
• Eigenmodes have cylindrical symmetry
• Grouped into families of $|m_\ell|$ 
• Paired according to $\sigma$
• Each mode has well defined spin and orbital angular momentum
• The modes of each family have **identical field distributions**
• Parallel vs. anti-parallel momenta: $\sigma \cdot m_\ell = \pm 1$
• Propagation constant $\sigma \cdot m_\ell$ dependent; mode pairs have a small degeneracy lifting in propagation constant

$$|n, m_\ell, \omega, \sigma\rangle$$

$\omega$ is fixed 
$n = 1$ 
$|m_\ell| > 1$

$$V \equiv \frac{\omega a}{c} \sqrt{n_{\text{core}}^2 - n_{\text{cladding}}^2}$$
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$$| n, m_\ell, \omega, \sigma \rangle$$

$\omega$ is fixed

$n = 1$

$|m_\ell| > 1$

$$V \equiv \frac{\omega a}{c} \sqrt{n_{\text{core}}^2 - n_{\text{cladding}}^2}$$
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- Parallel vs. anti-parallel momenta: $\sigma \cdot m_\ell = \pm 1$
- Propagation constant $\sigma \cdot m_\ell$ dependent; mode pairs have a small degeneracy lifting in propagation constant
- **Superposition modes exhibit spin-controlled rotation**

$|n, m_\ell, \omega, \sigma\rangle$

$\omega$ is fixed
$n = 1$
$|m_\ell| > 1$

$V \equiv \frac{\omega a}{c} \sqrt{n_{\text{core}}^2 - n_{\text{cladding}}^2}$
\[ |n, m_{\ell}, \omega, \sigma\rangle \]

\( \omega \) is fixed

\( n = 1 \)

\( |m_{\ell}| > 1 \)

\[ V \equiv \frac{\omega a}{c} \sqrt{n_{\text{core}}^2 - n_{\text{cladding}}^2} \]
We focus next on the subset of modes where $|m_\ell| = 2$.
How does it work?
Propagation depends on absolute $m_j$

$$E \propto J_{m_\ell} (\kappa r) e^{im_\ell \phi} e^{i(\beta z - \omega t)} e_\sigma \leftrightarrow |n, m_\ell, \omega, \sigma\rangle$$

$$|1, +2, \omega, -1\rangle \quad |1, -2, \omega, -1\rangle$$

$\sigma = -1$
$m_\ell = +2$

$\sigma \cdot m_\ell = -2$  (anti-parallel)
$|m_j| = 3$  (faster)

$\sigma = -1$
$m_\ell = -2$

$\sigma \cdot m_\ell = +2$  (parallel)
$|m_j| = 1$  (slower)

$$|1, -2, \omega, +1\rangle \quad |1, +2, \omega, +1\rangle$$

$\sigma = +1$
$m_\ell = -2$

$\sigma \cdot m_\ell = -2$  (anti-parallel)
$|m_j| = 1$  (faster)

$\sigma = +1$
$m_\ell = +2$

$\sigma \cdot m_\ell = +2$  (parallel)
$|m_j| = 3$  (slower)

$$\hat{J}_z \equiv \hat{L}_z + \hat{S}_z \quad m_j \equiv m_\ell + \sigma \quad \hat{J}_z |n, m_\ell, \omega, \sigma\rangle = \hbar m_j |n, m_\ell, \omega, \sigma\rangle$$
How does it work?

Propagation depends on absolute $m_j$

$$E \propto J_{m_\ell} (kr) e^{im_\ell \phi} e^{i(\beta z - \omega t)} e_\sigma \leftrightarrow |n, m_\ell, \omega, \sigma\rangle$$

$$|1, +2, \omega, -1\rangle + |1, -2, \omega, -1\rangle \quad |1, -2, \omega, +1\rangle + |1, +2, \omega, +1\rangle$$

Counter-clockwise

Clockwise

$$\hat{J}_z \equiv \hat{L}_z + \hat{S}_z \quad m_j \equiv m_\ell + \sigma \quad \hat{J}_z |n, m_\ell, \omega, \sigma\rangle = \hbar m_j |n, m_\ell, \omega, \sigma\rangle$$
How does it work?

Propagation depends on absolute $m_j$.

$$E \propto J_{m_\ell} (kr) e^{im_\ell \phi} e^{i(\beta z - \omega t)} e_\sigma \leftrightarrow |n, m_\ell, \omega, \sigma\rangle$$

$|1, +2, \omega, -1\rangle$      $|1, -2, \omega, -1\rangle$  $|1, -2, \omega, +1\rangle$  $|1, +2, \omega, +1\rangle$

$\sigma = -1$  $\sigma = -1$  $\sigma = +1$  $\sigma = +1$

$m_\ell = +2$  $m_\ell = -2$  $m_\ell = -2$  $m_\ell = +2$

$\sigma \cdot m_\ell = -2$  $\sigma \cdot m_\ell = +2$  $\sigma \cdot m_\ell = -2$  $\sigma \cdot m_\ell = +2$

(anti-parallel)  (parallel)  (anti-parallel)  (parallel)

$|m_j| = 1$  $|m_j| = 3$  $|m_j| = 1$  $|m_j| = 3$

(faster)  (slower)  (faster)  (slower)

$$\hat{J}_z \equiv \hat{L}_z + \hat{S}_z \quad m_j \equiv m_\ell + \sigma \quad \hat{J}_z |n, m_\ell, \omega, \sigma\rangle = \hbar m_j |n, m_\ell, \omega, \sigma\rangle$$
What is it good for?
Applications of SOC: cluster states
What is it good for?

Applications of SOC: Hermite-Gauss (HG) spatial-mode-entangled cluster states

\[ |0\rangle = \begin{cases} \text{We denote the Bell state} \\ |00\rangle + |11\rangle \text{ by } |\circ \circ \rangle + |\bullet \bullet \rangle \end{cases} \]

\[ |1\rangle = \begin{cases} \text{And the GHZ state} \\ |00\rangle + |11\rangle \text{ by } |\circ \circ \circ \circ \rangle + |\bullet \bullet \bullet \rangle \end{cases} \]

\[ |0\rangle = \begin{cases} \text{The following is applicable to} \\ \text{either of the choices to the left:} \\ \text{we focus on first order modes} \end{cases} \]

\[ |1\rangle = \begin{cases} \end{cases} \]
What is it good for?

Applications of SOC: Hermite-Gauss (HG) spatial-mode-entangled cluster states

We denote the Bell state \( |00\rangle + |11\rangle \) by \( \bullet\bullet\bullet\bullet + \bullet\bullet\bullet\bullet \)

And the GHZ state \( |00\rangle + |11\rangle \) by \( \bullet\bullet\bullet\bullet\bullet\bullet + \bullet\bullet\bullet\bullet\bullet\bullet \)

The following is applicable to either of the choices to the left: we focus on first order modes
What is it good for?
Applications of SOC: Hermite-Gauss (HG) spatial-mode-entangled cluster states

\[ |0\rangle = \begin{array}{c}
\end{array} \]

\[ |1\rangle = \begin{array}{c}
\end{array} \]

Our states have two key properties:
1.) Hadamard gate equivalent to a rotation
2.) They have opposing 1-D parities

\[ \frac{1}{\sqrt{2}} \begin{pmatrix}
HG_{10} & HG_{01}
\end{pmatrix} = \begin{pmatrix}
HG_{10}
\end{pmatrix} = \begin{pmatrix}
HG_{01}
\end{pmatrix} \]

Sums/differences make rotations

\[ \frac{1}{\sqrt{2}} \begin{pmatrix}
HG_{10} & HG_{01}
\end{pmatrix} = \begin{pmatrix}
HG_{01}
\end{pmatrix} = \begin{pmatrix}
HG_{10}
\end{pmatrix} \]

Odd/even upon y axis reflection

\[ E(x) \]

\[ x \]

\[ E(x) \]

\[ x \]
What is it good for?
The simplest example: type-I fusion (Brown et.al., 2005)

- Initial resource: two Bell states
- Input one photon from each Bell state
- The remaining two photons propagate freely
- One of the input photons is measured, the other exits the gate as output
- The result is an entangled three-qubit cluster state: $|\Psi\rangle = |\Psi\rangle + |\Phi\rangle \rightarrow |\psi\rangle = |\psi\rangle \pm |\Phi\rangle$
What is it good for?

Fusion gate elements

Mode Detector

Mode Rotator (Hadamard)

Mode Sorter
What is it good for?
fusion gate elements

Use transverse spatial modes as qubits

Odd/even upon y axis reflection

Sums/differences make rotations

$\frac{1}{\sqrt{2}} \left( \begin{array}{c} H_{G10} \\ + \\ H_{G01} \end{array} \right)$

$\frac{1}{\sqrt{2}} \left( \begin{array}{c} H_{G10} \\ - \\ H_{G01} \end{array} \right)$

$|0\rangle \rightarrow |0\rangle + |1\rangle$

Step-index fiber

Mode Rotator ($\text{Hadamard}$)

Mode Sorter

Mode Detector

$E(x)$

$E(x)$

$A$ port

$B$ port

$H_{G01}$

$H_{G10}$

$\text{BS}$

$E'$ port

$E$ port
What is it good for?
fusion gate elements

• The Mach-Zehnder style interferometer below is not stable, but we can overcome this by constructing a monolithic prism-like device
• The reflections and interference occur at internal interfaces in the “prism”

\[ |0\rangle \rightarrow |0\rangle + |1\rangle \]

Mode Rotator (Hadamard)

Step-index fiber

Mode Sorter

Mode Detector
What is it good for?
All fiber-based fusion gate elements

fiber compressor, relative HG-mode phase shifter
neutral phase shifter

Mode Sorter

|0⟩ → |0⟩ + |1⟩

Mode Rotator (Hadamard)

Step-index fiber

Mode Detector
Outlook

Single-photon spin-orbit coupling

• No clear advantages to using photonic spatial modes over the standard polarization encoding for cluster states

• But there are physical differences!
  – Polarization “parity” is local, spatial “parity” is global
  – Spatial modes must be sorted via interferometry (not a PBS)

• Could this make spatial modes more robust in some situations?

• Interferometry could be in principle provide a less lossy way to sort qubits than polarizing beam splitters

• Fibers restrict the number of degrees of freedom for losses

• All-fiber devices are practically convenient to work with

• Encode both the photon’s spin and spatial degrees of freedom?
Conclusions

Single-photon spin-orbit coupling

• What is it?
  \[ \beta \text{ splitting due to degeneracy lifting in } \sigma \cdot m_\ell : a \text{ larger } |m_j| = |m_\ell + \sigma| \text{ gives rise to a slower propagation speed} \]

• How does it work?
  \[ \sigma \cdot m_\ell - \text{split modes have a varying relative phase} \]

• What is it good for?
  SOC produces a mode rotation effect that can be used to implement a spin-controlled Hadamard gate for multiple HG-entangled spatial modes.

• How will we observe it? Experiments are underway…