Musical Mapping of Chaotic Attractors

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(Dated: May 7, 2009)

The representation of physical phenomenon is overwhelmingly visual. Although this has many uses, and serves as an sufficient if not near perfect means of representation, it only acknowledges one of our senses. Musical representation offers a refreshing alternative. Time varying aural effects are equally well if not better suited for the representation of some physical systems. Chaotic attractors provide an interesting point of comparison. In this experiment, Mathematica's "Play" function was used to generate music of time-varying frequency, amplitude, and origin using the governing equations of the Lorenz, Rossler, and Chua attractors. Despite conservative use of aural effects, not only were different attractors easily identifiable aurally, changes in an attractor's parameters could also be idenified. The musical samples generated provide an experience of these attractors unlike any visual representation.

I. INTRODUCTION

A. Algorithmic Composition

From a musical standpoint, mathematics has a long history of application as a means to generate or reveal the essential compositional structure of a musical composition. While Pythagoras' explanation of the overtone series is fundamental, more modern examples such as Bach's numerology or the use of the Fibonacci sequence in the music of Debussy and Bartok are also significant. Needless to say, musicians have not overlooked advances in science and computer technology. These advances have produced more complex and sophisticated mathematical methods for musical composition and analysis including the integration of probability, stochastic theory, and 1/f noise. The sensitive nature of chaotic trajectories offers a means for computer-generated variability and in a sense, creativity.

The study of chaos and chaotic attractors has been made possible in large part by advances in computer technology. Although mathematics and the sciences have benefited incredibly from computer based computation, composers have also made use of these advances. Algorithmic composition is a form of composition in which a computer is allowed to generate musical content from a set of rules defined by a composer. When algorithmic composition is used to musically represent physical phenomenon, the composer chooses how to represent the equations governing the physical phenomenon, but then allows the computer to generate the musical results of those choices.

B. Uses of Musical Representation in Physics

Graphical simulations of physical phenomenon are overwhelmingly visual. While graphs and figures are no doubt sensible and easy to comprehend, they appeal to only one of our senses. An alternative means of representation is musical. The use of aural perception in teaching and learning physics is undeveloped as of yet. However, aural perception is an exciting way to experience physics. Despite limitations in the speed of sample generation, musical representation carries a higher dimensionality and cognitive economy.

Although higher dimensionality is a phenomenon typically equated with theoretical particle physics, this is not its only form. Multi-dimensional systems are also present in thermal and quantum physics. For example, the interdependence of temperature, pressure, volume, internal energy, and entropy can be represented concurrently as a five dimensional plot. However, even a simple system such as projectile motion can be represented multidimensionally using the position, velocity, and acceleration vectors of the x-y plane. For the present experiment, the x, y, and z components of the chaotic attractors were represented musically.

C. Chaos and Chaotic Attractors

Chaos defines the behavior of certain dynamical systems whose time-evolution is highly sensitive to initial conditions. The famous butterfly analogy postulates that a simple change such as the flapping of a butterfly wing could generate a catastrophic change in the weather. Because of this sensitivity, chaotic motion seems random. However, chaotic systems are deterministic in that their future states are completely determined by their present states[1].

A chaotic attractor is a dynamical system in which initial conditions eventually converge toward a particular set in their trajectories. Having reached this set, slight perturbations will not affect its motion. For example, in the well known Lorenz attractor, once the trajectory has entered the wings, slight perturbations will not affect its continued trajectory along the wings.

II. THEORY

A. The Musical Representation of Physical Phenomenon

Changes in frequency, amplitude, timbre, duration, and origin of a sound over time are aural effects that can be used to represent dimensions. When used in an appropriate manner, this higher dimensionality can enable a higher order experience than a visual representation. Not only can more dimensions be represented, but also the joining of multiple aural effects can add depth to the experience of a smaller number of dimensions. Defining cognitive economy as the amount of information that can be coherently experienced through time, musical representation has a higher cognitive economy.

B. Chaotic Attractors

Chaotic attractors are non-linear systems that are attracted to particular orbits in space-time. Although there are a variety of kinds of attractors, only the Lorenz, Rossler, and Chua attractors were explored in this experiment. They are similar in that they are defined by three differential equations with three time-dependent variables. The flow of an attractor through time can be described using a group of equations distinct to each attractor. Slight differences in the equations with respect to their constants or starting points can vary the motion of the attractor significantly over time. Although the projection of an orbit and the shape of the attractors are generally preserved, motion within an orbit can vary between periodic and chaotic. The general shape of an attractor can be stretched, contracted or twisted in 3-space, but it will always have a defining shape. Because the chosen attractors are described by three equations with three time dependent spatial variables, it is very easy to generate plots using a standard 3-dimensional parametric plot. However, they are also apt for musical representation. An attractor's extreme sensitivity to initial conditions, dynamical nature, and attracting orbits allows for the generation of an infinite amount of distinct yet structurally similar musical material. Although the musical flow of a single attractor may become bland with time, as with the visual representation, different attractors will sound different.

The Lorenz attractor is a historically significant attractor. Working in the dawn of the computer age, with a computer capable of 60 multiplications per second and total memory of 16 kB, Lorenz set to work modeling changes in the atmosphere. Due to the Poincare-Bendixson Theorem, a chaotic solution could not be found with fewer than 3 differential equations. Although the system initially was implemented with 12 equations, after approximately 20 years Lorenz had simplified it to the three equation model[2]. It can be expressed using the three ordinary differential equations,

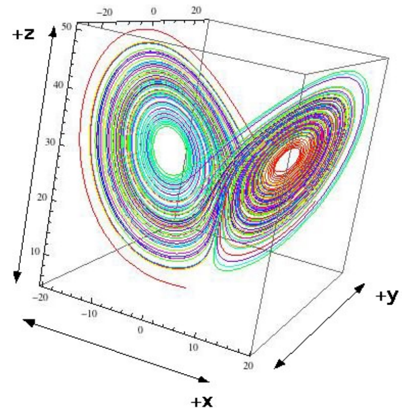

FIG. 1: A parametric plot of the Lorenz attractor generated by Mathematica for σ=10, b=8/3 and r=30. As shown here, the z-axis is always positive and the x and y axis vary between two attracting equilibrium.

$$\dot{x} = -\sigma x + \sigma y, \tag{1}$$
$$\dot{y} = -xz + rx - y, \tag{2}$$
$$\dot{z} = xy - bz, \tag{3}$$

where the Prandtl number σ, the Reynolds number r, and b are all parameters of the system. The dot on top of the x, y, and z refers to the next iterate. This produces a highly idealized motion of a fluid in which the warm fluid rises and the cold fluid sinks. The result is the famous butterfly, which can be viewed in Figure 1.

Lorenz found that for fixed $\sigma = 10$ and $b = 8/3$, the system behaves chaotically for values of r greater than $r \approx 24.74$.

The Lorenz attractor is a very popular attractor and comes with a great amount of interesting phenomenon. However, it is not the only chaotic attractor determined by differential equations. In 1976, the German scientist Otto Rossler found a simpler set of differential equations which produces another chaotic attractor. The equations are,

$$\dot{x} = -y - z, \tag{4}$$
$$\dot{y} = x + ay, \tag{5}$$
$$\dot{z} = b + (x - c)z, \tag{6}$$

where a, b, and c are parameters characterized by the system. The simplicity of these equations comes from the fact that two are linear in nature. Only the $\dot{z}$ component carries any non-linearity. Although initially they were designed to behave similarly to the Lorenz attractor but to be easier to analyze, these theoretical equations were later found to be useful in modeling chemical reactions[3]. The Rossler attractor is strikingly different than the Lorenz attractor and can be viewed in Figure 2.

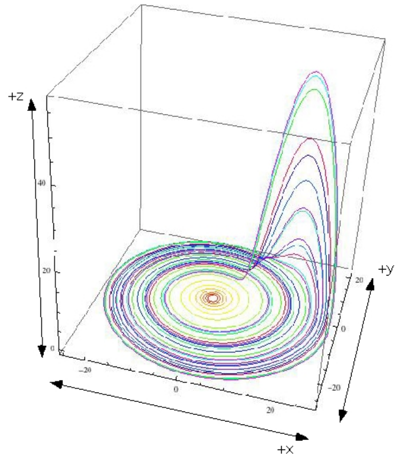

FIG. 2: A parametric plot of the Rossler attractor generated by Mathematica for a=0.1, b=0.1 and c=18. As shown here, the x and y components circle around the origin and the z component is only non-zero in the first quadrant.

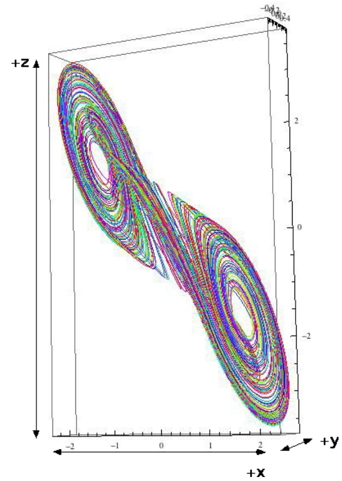

FIG. 3: A parametric plot of the Chua attractor generated by Mathematica for c_1=15.6, c_2=1, c_3=25.58, m_0=-8/7, and m_1=-5/7. For certain values of c_3, it becomes two separate yet similar attractors.

Although the Chua attractor is not as simple mathematically as the Lorenz or Rossler attractors, it is unique in that it doesn't require computer simulations to visualize the chaotic behavior. The Chua attractor can be generated relatively easily in an electronics lab using two capacitors, a resistor, an inductor, and a non-linear diode. After creating the circuit, it can be viewed using a simple oscilloscope[2]. The equations for the Chua attractor are

$$\dot{x} = c_1(y - x - g(x)), \tag{7}$$
$$\dot{y} = c_2(x - y + z), \tag{8}$$
$$\dot{z} = -c_3 y \tag{9}$$

where

$$g(x) = m_1 x + \frac{m_0 - m_1}{2}(|x+1| - |x-1|), \tag{10}$$

and c_1, c_2, c_3, m_0, and m_1 are all parameters that characterize the circuit. The function $g(x)$ is the only non-linearity present in the system. It represents three different current regimes for the non-linear diode. The Chua attractor is different than the Lorenz or Rossler attractor in that for certain parameter values, it creates a disconnected set. Figure 3 represents the connected attractor. The disconnected attractor will be shown later in Figures 12 and 13.

Together, these attractors represent interesting and important chaotic systems in the field of chaos. They are all characterized by three differential equations with at least one non-linearity. They also are attracted to particular orbits which serve to make interesting visual plots. By mapping them musically, equally interesting musical samples can be generated.

C. Musical Formalism

Although there are five aural effects that can be manipulated, only the amplitude, frequency, and origin of the sound were used in this project. With this in mind, only the frequency and spatial component of the sound will need theoretical explanation. The frequency of a wave, measured in Hertz (Hz) is the number of complete cycles of a sine wave in the time of one second. A pitch of our tuning system can therefore be expressed as simply as

$$A \cdot \sin[F \times 2\pi t], \tag{11}$$

where t is the time, A is the amplitude, and F is the frequency. If the frequency were 440 Hz, then the sine wave would have 440 cycles per second. By increasing the amplitude, the peaks of each wave would be higher and a louder wave will result.

D. Creating a Time Varying Frequency

To generate a time varying frequency, as used in this experiment, a more sophisticated explanation of frequency is needed. The interval of an octave is represented by a doubling of frequency. Considering a frequency of 440 Hz, this means the upper octave has a frequency of 880 Hz, while the lower octave has a frequency of 220 Hz. Although a piano presents each octave as having 12 notes, the notes in the higher octaves have a larger difference in frequency per note than the lower octaves. The frequency difference between two notes on a linear scale (such as the piano) can be understood in terms of the system of "cents". In the system of cents, there

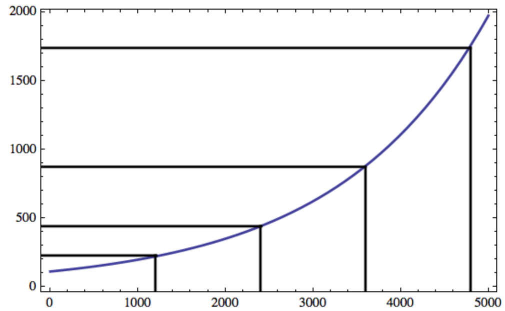

FIG. 4: A graph exhibiting a the tuning system based upon "cents". The black lines represent octave equivalents. The frequency is displayed on the left axis and cents are displayed on the bottom axis. It is generated using Equation refTuning1. Cents provide a means of understanding musical pitch linearly. In this system, each octave has 1200 cents, and each note on the 12 tone scale is 100 cents apart. This system is generalized to work to translate a governing equation of an attractor to our tuning system.

are 1200 cents in an octave, and each successive note in the ascending 12-tone scale has an increase of 100 cents. Knowing the difference in cents between two notes, and the frequency of one, we can calculate frequency of the other using the equation

$$b = a \cdot 2^{\frac{n}{1200}}, \tag{12}$$

where 1200 represents the number of cents in the octave, a represents the known frequency of the first note, b is the calculated frequency, and n represents the difference in cents between the two notes. A graph displaying the tuning system in terms of cents can be seen in Figure 4.

Though equation 12 is of standard use in our tuning system, it can be generalized to accommodate any linear (or non-linear) parametric equation that has a maximum and minimum value. A more general form of the equation can be written

$$b(t) = a \cdot 2^{\frac{P(t)}{O(s)}}, \tag{13}$$

where $P(t)$ is some time varying function, a is the chosen fundamental, $b(t)$ the time varying frequency directly related to the trajectory of $P(t)$, and $O(s)$ is the octave size chosen. By taking the difference of the maximum and minimum of $P(t)$, the range can be calculated and used to generate an octave size. For example, dividing the range by 3 will produce a range in frequency of 3 octaves. By choosing a fundamental frequency a such as 440 Hz, a new function $b(t)$ directly translates the time-varying trajectory of the system to our standard tuning system.

E. The Spatial Perception of Sound

Our perception of the origin of a sound can be understood using simple mathematics. Because of limitations in stereo equipment, only the left-right continuum was employed. When a sound comes from the far left, it makes sence that all sound would come from that speaker; and similarly with the right speaker. However, the space in between requires explanation. The standard representation of these parameters is modeled by two simple equations for the amplitude of the sound leaving the left and right speakers. Orienting the speakers such that positive values of some equation come from the right, and negative values from the left produces

$$L(t) = \frac{(P(t) - 1)^2}{4} \tag{14}$$

$$R(t) = \frac{(P(t) + 1)^2}{4} \tag{15}$$

where $P(t)$ is a chosen parametric equation scaled such that its maximum is 1 and its minimum is -1, $L(t)$ is the amplitude of the sound which comes from the left speaker, and $R(t)$ is the amplitude of the right speaker. A graph of these equations can be viewed in Figure 5. When $P(t)$ is at its maximum, no sound comes from the left, when $P(t)$ is at 0, sound comes half from the left and half from the right.

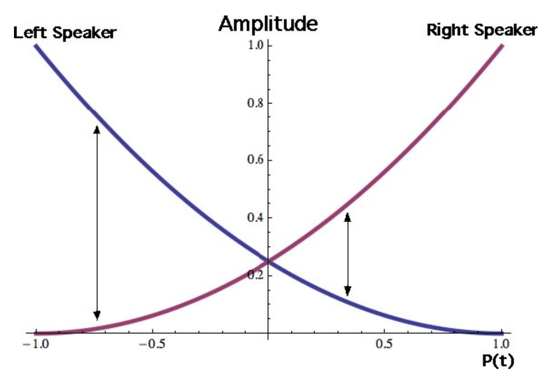

FIG. 5: A graph representing our perception of the origin of a sound. When $P(t)$ has reached its maximum value of 1, all sound comes from the left and none from the right. For $-1 < P(t) < 1$, each speaker has a specific output amplitude.

III. EXPERIMENT AND PROCEDURE

All experimentation involved computer based musical generation. Mathematica 7, specifically the "Play" function, was used to create generate all musical samples. Of the five possible aural dimensions, it was decided to use only use the amplitude, frequency, and spatial dimensions. A completely continuous and time-varying frequency, amplitude, and origin was desired. However, the

Play function could not deal with a continuously varying frequency without strange errors, so a discrete function had to be implemented. The "UnitStep" function was used to sustain a frequency for a discrete time step. The Unit Step function is equal to 0 for $x < 0$ and 1 for $x \geq 0$.

By using this function correctly, the frequency function was re-evaluated and sustained for a 10th of a second every 10th of a second. A figure of the proper use can be viewed in Figure 6.

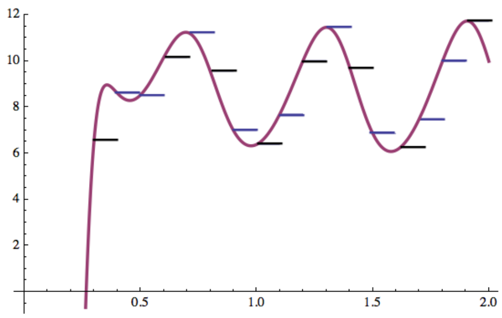

FIG. 6: A graph displaying a continuous function and its translation into a discrete function using the Unit Step function. The bottom axis represents time and the left axis represents the d For the experiment, the "Step" was chosen to be 0.1s. Although the continuous function was preferable, a continuously varying frequency produced aliasing errors.

The code for its proper use is complicated, but can be written as

$$f(t) = \sum_{i=0}^{tStop/tStep} z[i \cdot tStep] \times U[t - (i \cdot tStep)] \times U[((i+1) \cdot tStep) - t],$$

where $F(t)$ is the linear frequency component, z represents the differential equation of the attractor chosen to represent frequency, the function U is the unit step function, t is time, $tStep = 0.1s$, and $tStop$ specifies the stop time. Although this is complicated, the two Unit Step functions are only both non-zero when t satisfies

$$i \cdot tStep < t < (i+1) \cdot tStep. \tag{16}$$

Although each note had a finite duration, because the duration was not varying, the continuous nature of the function was somewhat preserved. Although the Play function generated blips between each 0.1 second sample, these blips were somewhat aesthetically pleasing. In addition to reinforcing a perception of rhythm, the percussive sound that was generated distracts the ear from the pure sound wave which is an aurally jarring quality alone.

After creating a discrete function for the frequency, it was translated into a frequency function which corresponded to the western tuning system as described in Equation 13. The octave size was decided to be a third of the range of the $\dot{x}$, $\dot{y}$, or $\dot{z}$ equation for the attractor that was chosen to represent frequency. The exact pairings for these equations and frequency can be seen in Table I. The fundamental was chosen to be 440 Hz. A graph of the z component of the Lorenz attractor and its translation into frequency can be compared in Figure 7.

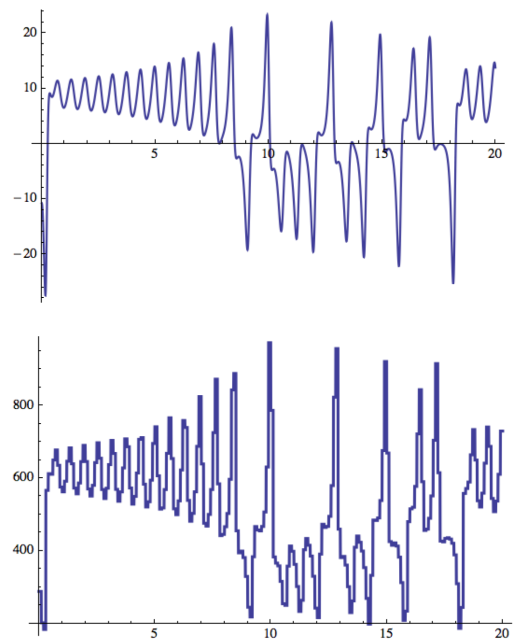

FIG. 7: A comparison of the x component of the Lorentz attractor before and after translation into frequency. The bottom function is characterized by its discrete nature and larger changes in frequency for higher pitches. Although not clearly visible in this picture, the total range of frequencies for a 100 second sample is three octaves.

Several musical examples were generated for each attractor, but the musical pairing of the differential equations did not change for each attractor. The pairing of each equation with musical dimensions can be seen in Table I.

TABLE I: Pairing of Specific Equations with Musical Dimensions.

Attractor	x[t]	y[t]	z[t]
Lorenz	Spatial	Frequency	Amplitude
Rossler	Amplitude	Spatial	Frequency
Chua	Spatial	Amplitude	Frequency

The pairing of the differential equations to musical dimensions was chosen to maximize musical expression. This meant that the equation of the attractor which was most defining of the attractor in general, such as the $\dot{z}$ equation for the Rossler attractor, was mapped to frequency. After choosing the frequency equation, the

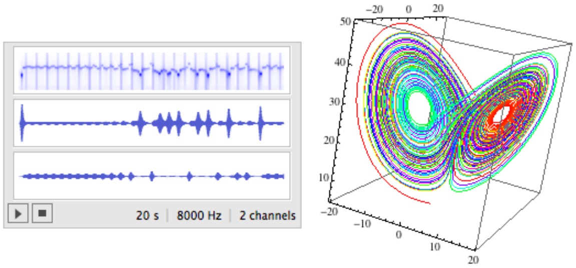

FIG. 8: The output display of a the Lorenz attractor using the Play function and corresponding 3-d plot with parameters σ=10, b=8/3, and r=30.

equation that could benefit from a spatial representation was chosen. Whatever equation remained, typically one which was not very expressive, such as the $\dot{y}$ equation of the Chua attractor, was paired with the amplitude.

Although these equations could be mapped to different musical dimensions, because the focus of the experiment was the chaotic attractors, the content of experimentation was primarily the effects of the change in initial parameters. Specifically the c parameter in the Rossler attractor, and c_3 parameter in the Chua attractor were varied and the musical results analyzed. The result of these changes were musical samples that were alike in structure but noticeably different in musical identity.

IV. RESULTS AND ANALYSIS

For each example presented in this report, there is a corresponding music file.

A. The Lorenz Attractor

The output display of a the Lorenz attractor using the Play function and corresponding 3-d plot with parameters σ=10, b=8/3, and r=30 is represented in Figure 8. The upper portion of the display shows the frequency, while the second and third displays show the amplitudes from each speaker. The middle display represents the amplitude of the left speaker and the amplitude of the right speaker is the lower display. The middle display has a higher amplitude because for this portion of time, z component has a proportionally high magnitude when its value for x is neqative. A darker blue in the upper display implies a louder note.

Although there is much more that can be studied with the Lorenz attractor, it was thought that the other attractors might give more musically significant results. The Rossler and Chua attractor have very clear points of periodic and chaotic orbits.

B. The Rossler Attractor

Two outputs of the Rossler attractor are represented in Figures 9 and 10. The trajectory is very simple compared with the Lorenz and Chua attractors, and this result is more simple musically as well. A clear cyclic pattern is easily recognizable in both samples. This is because the trajectory oscillates almost periodically between the maximum and minimum values for $x[t]$ and $y[t]$. Although frequency is still the dominant musical dimension, this periodicity makes the Rossler is easiest to identify musically.

The output display of the Rossler attractor with a=0.1, b=0.1 and c=6 and the equivalent purely visual representation is displayed in Figures 9. This set of parameters produces a period 2 attractor. This is evident by the fact that the first and third humps in the upper display are equal in size and darkness. Furthermore, they are followed by sections of no darkness in which the amplitude is 0 (no sound). The amplitude goes to zero because the exact minimum point in the x[t] component reaches its minimum every second revolution.

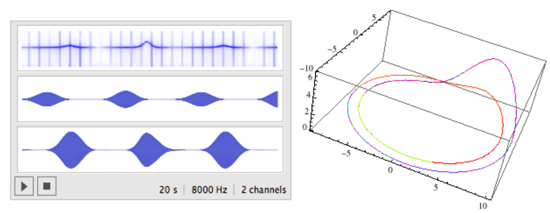

FIG. 9: The output display of the Rossler attractor with a=0.1, b=0.1 and c=6 and the equivalent purely visual representation.

The output display of the Rossler attractor with a=0.1, b=0.1, and c=9 using both the Play function and a 3-D plot is shown in 10. Although the z-range is almost identical to the periodic attractor in 9, this example is distinct aurally because of its chaotic trajectory.

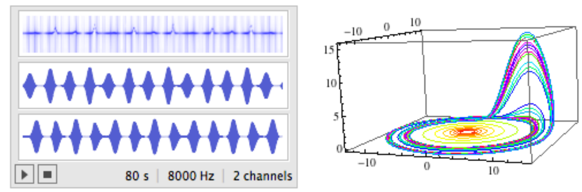

FIG. 10: The output display of the Rossler attractor with a=0.1, b=0.1, and c=9 using both the Play function and a 3-D plot.

C. The Chua Attractor

The Chua attractor is disconnected for particular parameter values. For the disconnected sets, two different frequency functions corresponding to orbits were employed. The output for a connected attractor looks like Figure 11, while the output for the disconnected attractor is represented in Figure 12 and 13.

The output display of the Chua attractor with c_1=15.6, c_2=1, c_3=25.58, m_0=-8/7, and m_1=-5/7 using the play function is shown in Figure 11. The 3-D plot is the same as displayed in Figure 3. Although the Chua attractor has two attracting equilibrium like the Lorentz attractor, the values for the frequency overlap between the two attracting equilibrium. The aural result is that the change in frequency between the two basins is not as distinct, and the fluctuations in frequency are not a quick as in the Lorenz attractor.

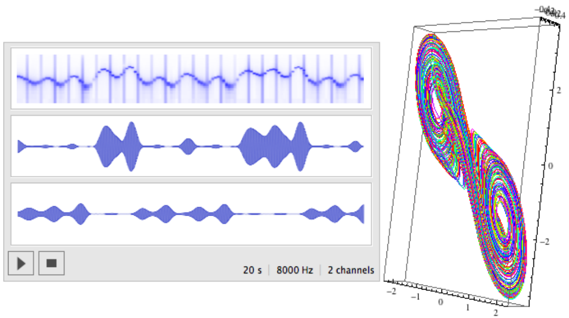

FIG. 11: The output display of the Chua attractor with c_1=15.6, c_2=1, c_3=25.58, m_0=-8/7, and m_1=-5/7 using the play function.

The output display of the disconnected Chua attractor with $c_3 = 50$ using both the Play function and a 3-D plot is shown in 12. The Chua attractor is not connected, and each part is displaying a period 1 trajectory. The Play output shows very interesting results. The two frequencies are not exactly aligned, and don't exactly parallel to each other in space. This is shown by their difference in peak frequencies. The lower attractor has a peak frequency later than the upper attractor. Furthermore, the lower trajectory is slightly more negative in the y component. This can be seen by the period of no darkness in its motion of ascending frequency. The likely reason for this is the start parameters that were chosen. The two sets clearly have periods of equal length. Choosing proper initial values could result in two frequencies moving in parallel motion through time.

The output display of the disconnected Chua attractor with $c_3 = 33$ using both the Play function and a 3-D plot is shown in 13 . For this value of c_3, the trajectory is chaotic for both attracting sets. This plot is clearly different from Figure 12. Besides for not being periodic, it is also apparent that the peak frequency of the lower attractor is happening before the upper attractor.

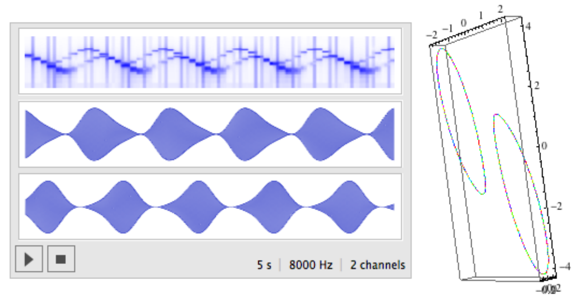

FIG. 12: The output display of the disconnected Chua attractor with $c_3 = 50$ using both the Play function and a 3-D plot.

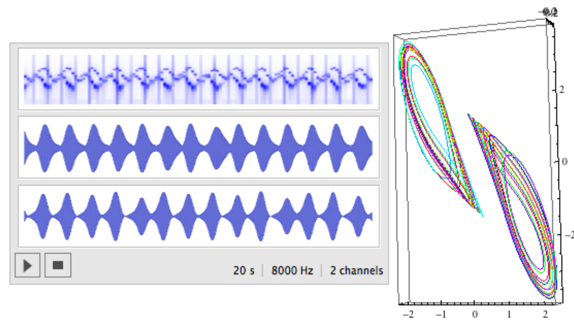

FIG. 13: The output display of the disconnected Chua attractor with $c_3 = 33$ using both the Play function and a 3-D plot.

An infinite number of musical samples can be generated by varying the initial parameters. However, the majority of the sound will sound the same. The largest difference in sound for each attractor will be its periodic versus its chaotic orbits. However, this does not discredit the musical representation. Visual representations can create an infinite number of images of the attractor, but only the periodic versus chaotic orbits will be easy to differentiate.

V. CONCLUSION

The musical mapping of chaotic attractors allows for an experience of chaos that is unparalleled visually. Though the beauty of the attractor can be seen in is distinctive 3-dimensional shape, musical representation provides distinctive representations as it flows through time. None of the Lorenz, Rossler, or Chua attractors sounded alike despite their similarity in timbre and the conservative use of aural effects. Furthermore, periodic and chaotic orbits were easily distinguishable for specific attractors.

VI. ACKNOWLEDGEMENTS

I'd like to thank Dr. Lindner for suggesting the topic to me, providing key insight, and providing substantial support in the early stages. I'd like to thank Dr. Brown of the Music department for helping me to understand musical dimensionality as well as introducing key concepts in music technology. Also, Dr. Lehman for advising and suggesting a discrete frequency function when the continuous function was providing disheartening results. Finally, Frank King for his assistance in Mathematica and his idea to use the Unit Step function.

[1] "Chaos Theory," *Wikipedia, The Free Encyclopedia*, (Accessed on May 24, 2009).

[2] K. Alligood, T. Sauer, and J. Yorke, *Chaos: An Introduction to Dynamical Systems.* (Springer-Verlag, New York, 1997).

[3] "Rossler Attractor." *Wikipedia, The Free Encyclopedia*, (Accessed on May 24, 2009).