Period Doubling Route to Chaos in a RLD Circuit

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Using a function generator to drive a circuit consisting of a resistor, inductor, and diode, the period doubling route to chaos was observed. Through varying the frequency of the function generator at a constant voltage, both chaotic and periodic regimes were found. By determining the frequencies at which period bifurcations occurred and averaging the results of several trials, the Feigenbaum constant δ was determined to be 4.6 ± 0.3, which is within 1.5% of the known value.

I. INTRODUCTION

A chaotic system is one that evolves deterministically, is non-linear, and exhibits both aperiodic behavior and a high sensitivity to initial conditions. These sorts of systems are commonplace and can be found in systems ranging from shell growth to weather patterns to the orbits of gravitating astronomical bodies. Although all chaotic systems are non-linear, since linear systems do not possess enough degrees of freedom, nonlinearity is not enough in itself for a system to be chaotic. Fundamental to chaos is the sensitivity to initial conditions, also known as the butterfly effect, a term coined by Edward Lorenz, an early pioneer of what would become Chaos Theory. Although the long-term behavior of chaotic systems is impossible to determine, chaotic systems are not random themselves and are strictly deterministic. Lorenz characterized this behavior in his definition of a chaotic system, a system wherein "the present determines the future, but the approximate present does not approximately determine the future" [1].

The principle of causality states that "every event has a cause," and lies as the cornerstone of natural science. Throughout the development of physics from Johannes Kepler's three laws of planetary motion to Newton's differential calculus, the notion that the universe's behavior could be precisely predicted (given sufficiently accurate data) remained central. Phenomena for which it is impossible to predict the behavior, such as Brownian motion, were attributed to insufficient information either about initial conditions, the laws governing the systems' evolution, or both. Doubt that the future may be inherently unpredictable did not arise in the scientific community until extreme sensitivity to initial conditions was discovered by Henri Poincaré. When studying the n-body problem wherein the famous two-body problem (where behavior is determined by gravitational attraction) is extended to an arbitrary number of bodies, Poincaré found that the set of possible orbits or phase space of the system varied such that an arbitrarily small change in initial conditions could have an immense impact on the behavior of the system, making predictions of long term behavior inaccurate. Unlike linear systems wherein a small change in initial conditions creates a small change in resultant behavior on the same order of magnitude of the initial change, nonlinear systems are capable of exhibiting changes in resultant behavior many orders of magnitude greater than the small change in initial conditions. Although far from a formal theory of chaos, and despite going ignored for many years, this discovery of sensitivity to initial conditions in nonlinear systems was the first major step in the development of chaos theory [2].

Sensitivity to initial conditions was rediscovered by Lorenz in 1963 while he was attempting to simulate weather patterns. He found that changing the rounding in his program from occurring after three digits to six digits resulted in highly divergent results. Through the use of iterative processes such as those used in computing, these tiny differences are amplified, resulting in long term behavior that can vary drastically. This amplification is what is famously referred to as the butterfly effect, a name that comes from Lorenz's talk in 1972 titled, "Predictability: does the flap of a butterfly's wing in Brazil set off a tornado in Texas?" Lorenz went on to discover that sometimes there also exist attractors in the phase spaces of chaotic systems: sets of chaotic solutions that tend towards certain behaviors [2]. These attractors were further studied and categorized by David Ruelle, who went on to describe them mathematically as geometric behaviors and tendencies in phase space. Furthermore, Ruelle found that these attractors could be fixed points in phase space, where multiple trajectories would tend towards the same long term behavior, limit cycles and toruses, where certain periodic and quasi-periodic cycles are tended towards, or strange attractors, which have fractal structure and dimension, and were the type discovered by Lorenz in his simulation of weather patterns [3].

The contribution that unified individual chaotic phenomena as a greater category of phenomena with certain characteristic behaviors, making chaos theory a universal theory instead of a collection of predictions, was made by Mitchell Jay Feigenbaum in 1975. A major part of this contribution was the scenario of period doubling, which was based on the logistic map of biologist Robert May. This period doubling is a route to chaos, wherein the number of possible behaviors exhibited by the system bifurcates repeatedly, doubling the period of the system, until the behavior of the system is chaotic and without repetition (aperiodic). The development of the bifurcation diagram of the logistic map allowed for a visualization of an important route to chaos, applicable to a wide range of natural phenomena, from population growth to the swinging of a driven pendulum. Feigenbaum's biggest contribution to unifying chaotic phenomena was the discovery of a mathematical constant fundamental to chaotic systems, now known as Feigenbaum's first constant or Feigenbaum's δ. This constant is the ratio of the values where successive bifurcations occur, and holds for any system with a quadratic maxima in its logistic map, which includes the majority of systems that occur in the natural world. Feigenbaum went on to discover that in the set of fractals known as the Mandelbrot set, Feigenbaum's constant is also the ratio of successive diameters of the circles. These developments allowed Feigenbaum to develop new cartographical methods to draw maps computationally, and brought chaos theory up to a new level of applicability and generality [4].

II. THEORY

Nonlinear systems are common in the real world, and a very simple one is the driven RLD circuit, consisting of a resistor, an inductor, and a diode connected to a signal generator. In this system, there are multiple degrees of freedom due to the presence of the diode, which is a non-linear element that controls thue direction in which current can flow. This is because diodes have an asymmetric conductivity due to being composed of two doped semiconductors, one with an excess of negative charges and one with an excess of positively charged holes. The diode prevents current from traveling in one direction as electrons are attracted to the positively charged holes, filling them and creating a depletion zone in the middle, while allowing the current to flow in the other direction as electrons are repelled through the depletion zone, filling holes on the other side. Because of this, the resistance across the diode varies and depends non-linearly based on the positive and negative charge build up. This nonlinear element ensures nonlinear behavior in the circuit [5].

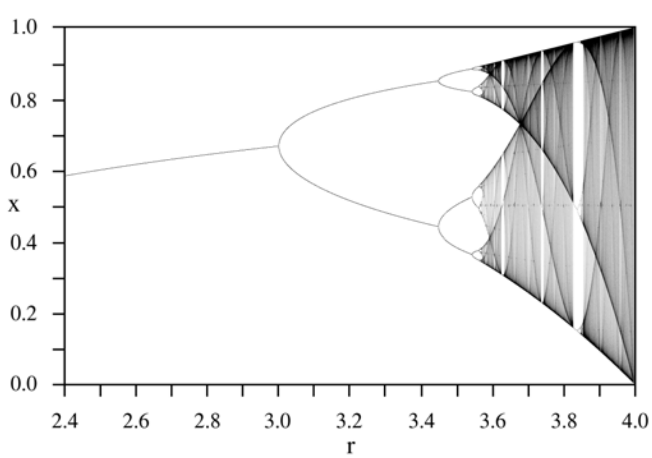

FIG. 1: Bifurcation diagram of the logistics map from the Wikipedia page on Mitchell Feigenbaum [4]. The parameter r is used instead of λ.

Another key element of the circuit is the inductor, which has an effective capacitance and creates a potential the electrons must overcome that increases with amount of the current passing through the inductor. At sufficiently high frequencies, the inductor's capacitance interferes with its behavior, and the inductor becomes self-resonant [7]. This resonance acts to amplify the signal, making the period doubling of the non-linear RLD circuit easier to observe and record.

The logistics map is a simple, nonlinear model of population with linear birth and quadratic death [6]. This means that the maxima is quadratic, and that bifurcations should occur with a ratio equal to Feigenbaum's constant, $\delta = 4.6692\ldots$. If x_n is the normalized population of the n^{th} generation, then the normalized population of the $n+1^{th}$ generation is given by

$$x_{n+1} = \lambda x_n(1 - x_n), \tag{1}$$

where λ is an input parameter that corresponds to a physical input. Varying λ between zero and four yields long-term behavior that can vary widely. By mapping the possible long term behaviors of x as a function of λ, it is possible to create a bifurcation diagram of the logistics map, which illustrates how period doubling occurs at specific values of λ, and can be seen in Fig. 1.

Feigenbaum's delta is defined to be the ratio of distance in λ of successive bifurcations, and is given by

$$\delta = \lim_{b\to\infty} \frac{\lambda_{b+1} - \lambda_b}{\lambda_{b+2} - \lambda_{b+1}}, \tag{2}$$

where λ_b, λ_{b+1}, and λ_{b+2} are the locations of successive bifurcations. Note that δ is a dimensionless quantity and that as long as the units of λ_b, λ_{b+1}, and λ_{b+2} are the same, they will cancel, making the equation extremely versatile in describing a wide range of phenomena. As

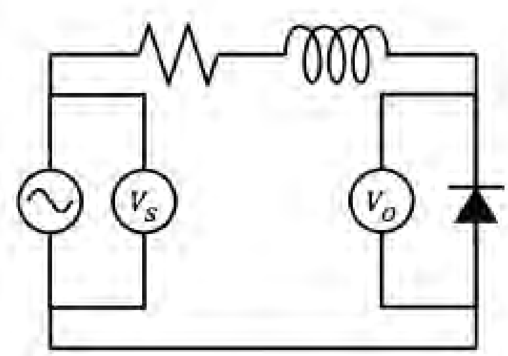

FIG. 2: Diagram for a nonlinear, driven circuit from the Jr. I.S. Manual [6].

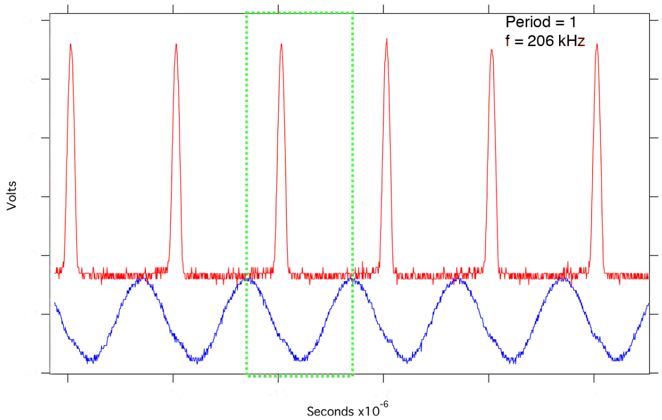

FIG. 3: Plot of voltage versus time for the sine wave generated by the signal generator (blue) compared to the signal from the RLD circuit (red) at a frequency of 206 kHz, illustrating period 1 behavior. A dashed box indicates a single period of the signal from the RLD circuit (green).

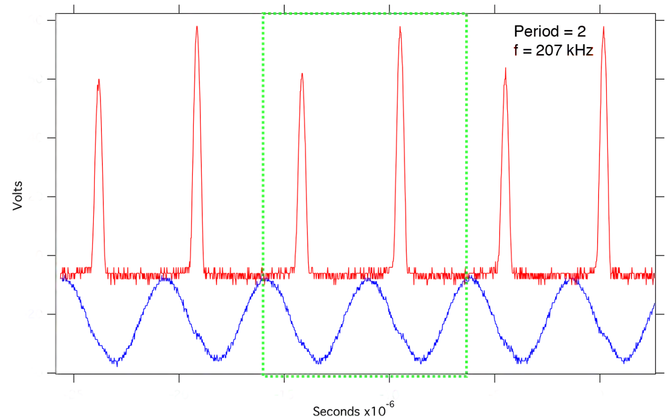

FIG. 4: Plot of voltage versus time for the sine wave generated by the signal generator (blue) compared to the signal from the RLD circuit (red) at a frequency of 207 kHz, illustrating period 2 behavior. A dashed box indicates a single period of the signal from the RLD circuit (green).

the generations of the bifurcations b approach infinity, the ratio approaches δ. Thus, by recording at least three values of λ at which bifurcations occur, it is possible to approximate Feigenbaum's constant δ by taking it to be

$$\delta \approx \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_2}, \tag{3}$$

with the approximation becoming asymptotically more accurate for later bifurcations.

III. PROCEDURE

In order to create a nonlinear, driven circuit, a 100 μH inductor, a diode, and a 3.9 Ω resistor were attached to a grounded signal generator, which can be seen in Fig. 2. In order to view the periodicity of the circuit, this was in turn attached to a oscilloscope. A direct input from the signal generator was also attached to the oscilloscope, so the circuit's behavior could be compared to a consistent signal. By increasing the frequency of the signal generator while holding the voltage constant, the periodicity of the circuit changed, exhibiting period doubling until it was indistinguishable from random noise and was truly chaotic. To determine at what frequencies the period doubling was occurring, the signal from the circuit was observed both as a waveform (versus time) and in phase space (versus the sine wave from the signal generator). The phase space plot is more useful for determining when period doubling occurred as the difference was more distinct visually.

IV. RESULTS & ANALYSIS

The period doubling route to chaos was observed for the driven RLD circuit, through increasing the frequency of the signal generator from 100 kHz to 700 kHz while holding the voltage constant at 14 V. This resulted in observable bifurcations through the period doubling four times before the system became chaotic. The highest frequency found with period 1 in the first period-doubling regime was 206 kHz, and a comparison of the signal from the signal generator to that of the RLD circuit can be seen in Fig. 3. Increasing the frequency by a single kHz to 207 kHz drastically changed the behavior of the system, making it suddenly period 2, as can be seen in Fig. 4. This extreme sensitivity to small changes in initial

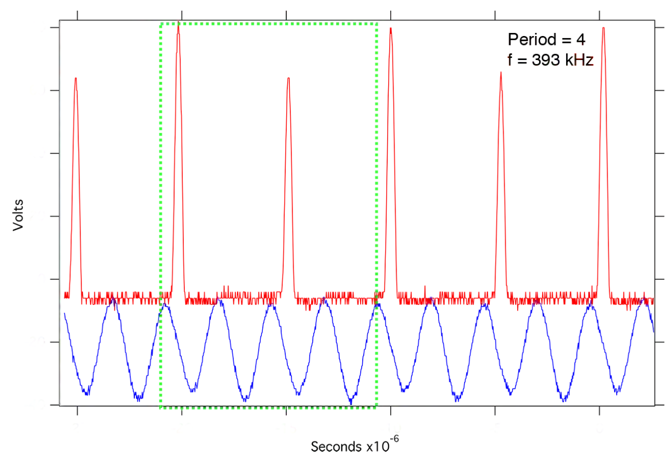

FIG. 5: Plot of voltage versus time for the sine wave generated by the signal generator (blue) compared to the signal from the RLD circuit (red) at a frequency of 393 kHz, illustrating period 4 behavior. A dashed box indicates a single period of the signal from the RLD circuit (green).

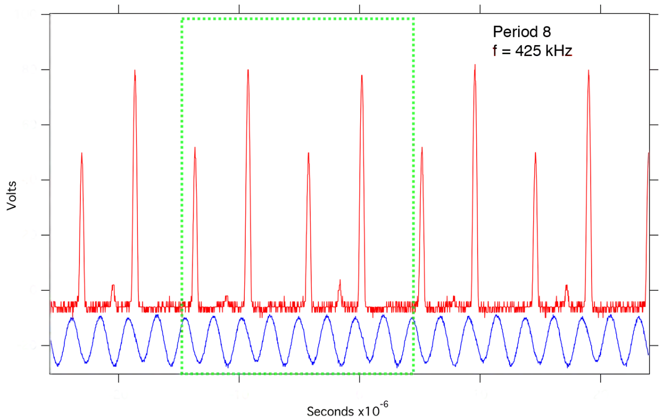

FIG. 6: Plot of voltage versus time for the sine wave generated by the signal generator (blue) compared to the signal from the RLD circuit (red) at a frequency of 425 kHz, illustrating period 8 behavior. A dashed box indicates a single period of the signal from the RLD circuit (green).

conditions is characteristic of chaotic systems. The next bifurcation occurred at a frequency of 393 kHz, resulting in period 4 behavior which can be seen in Fig. 5. This behavior bifurcated again at 425 kHz to period 8 behavior, which can be seen in Fig. 6. The final visible bifurcation occurred at a frequency of 431 kHz, resulting in period 16 behavior which can be seen in Fig. 7. Subsequent bifurcations yielded behavior indiscernible from chaos, as can be seen in Fig. 8 for a frequency of 472 kHz. In this plot, there is no visible repeating pattern to the behavior of the RLD circuit, indicating that its behavior is truly chaotic.

Note that at all frequencies there are visible inconsistencies in the amplitude of the sine wave generated by the function generator, which may have been a source of error. Due to the period increasing by multiples of two, there existed regions where the period of the circuit was twice or four times that of the sine wave but none where it was three times that of the sine wave. Further increasing the frequency through the chaotic region eventually resulted in a non-chaotic region, wherein the period doubling behavior eventually started anew.

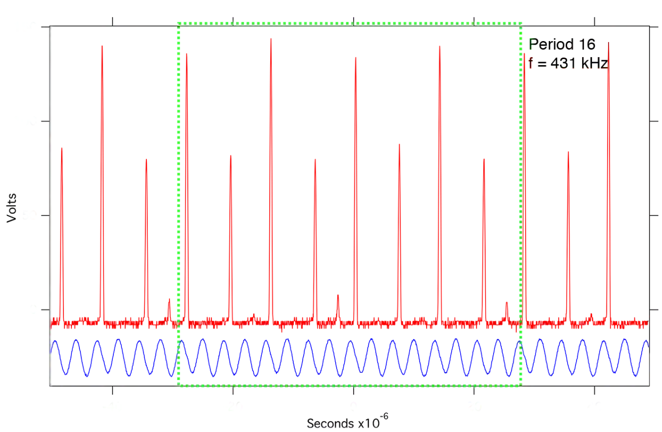

FIG. 7: Plot of voltage versus time for the sine wave generated by the signal generator (blue) compared to the signal from the RLD circuit (red) at a frequency of 431 kHz, illustrating period 16 behavior. A dashed box indicates a single period of the signal from the RLD circuit (green).

Using Eq. 3 and the measured frequencies at which bifurcations occurred, Feigenbaum's constant was calculated to be $\delta = 4.6 \pm 0.3$. Because the period sixteen bifurcation was observed and the frequency at which it occurred was recorded, it was possible to compare two approximations of Feigenbaum's constant to see if they were approaching the known value as b_n increased, as is suggested by Eq. 2. From the first through third bifurcation using Eq. 3, Feigenbaum's constant was calculated to be $\delta = 7.6$, and from the second through fourth bifurcation using the same equation, Feigenbaum's constant was calculated to be $\delta = 4.9$. This second value was substantially closer to the known value of $4.6692\ldots$ and indicates that it is possible that the ratio of bifurcations approaches Feigenbaum's constant asymptotically as $b_n \to \infty$, but significantly more bifurcations would need to be recorded to more substantially support this theoretical prediction.

V. CONCLUSIONS

The transition from normal periodic behavior to chaos through period doubling was observed in a driven RLD circuit by varying the parameter of frequency of the forcing function generator. By increasing the frequency with a constant voltage and recording where the period doublings or bifurcations occurred, Feigenbaum's constant was found to be $\delta = 4.6 \pm 0.3$, which is within 1.5% of the known value of $4.6692\ldots$. The theoretically predicted approaching of the actual value of Feigenbaum's constant was observed for two calculations of δ, but more bifurcations would need to have been observed before

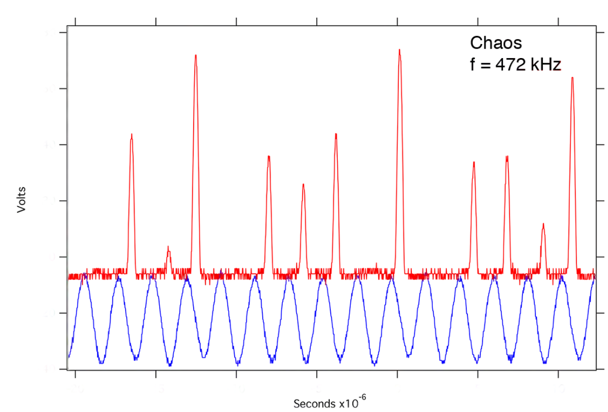

FIG. 8: Plot of voltage versus time for the sine wave generated by the signal generator (blue) compared to the signal from the RLD circuit (red) at a frequency of 472 kHz, illustrating chaotic behavior.

the system transitioned to chaos to show that the approaching of 4.6692... was truly asymptotic. Nonetheless, this result supports the theoretical predictions of Eq. 2, along with the experimentally determined value for δ of 4.6 ± 0.3. In future iterations of this experiment, an oscilloscope should be used with a larger display and higher resolution, as this may allow for observing bifurcations beyond the fifth, allowing for better verification of the asymptotic approaching of the approximation of δ towards the true value Feigenbaum's constant. Additionally, it would be smart to implement a simulation of the RLD circuit in the analysis of chaos, as it would allow for quick production of bifurcation diagrams specific to the experimental setup, and would have been a nice qualitative means of analyzing the period doubling route to chaos. It would also be wise to determine the source of the slight variations in the amplitude of the sine wave produced by the signal generator, which might lay in the connections between the signal generator and the oscilloscope or might be internal to the signal generator itself. If the function generator was producing oscillations of varying amplitude, this would definitely throw off results, and would be testable by running the function generator without the circuit and measuring the output voltage, looking for inconsistencies in the sine wave produced. If this is the case, then despite calculating Feigenbaum's constant with high accuracy, it is undoubtable that the behavior observed was changed by this variation in what should be a uniform sine wave. After all, the characteristic feature of chaotic systems is high-sensitivity to initial conditions, and unaccounted variation in the forcing frequency or amplitude (voltage) would change long-term behavior in a manner that may be impossible to reproduce in later experiments.

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