Acoustic Trapping, a Theoretical Analysis and Computational Implementation.

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The parameters of acoustic traps were analysed to determine,the minimum requirements a acoustic trap to successfully contain a particle. This analysis was done computationally with C code, to allow thousands of configurations of the trap to be run and simulated all together. A three dimensional sample space was then analysed with dimensions in frequency of the trapping waves, amplitude thereof, and speed of sound of the medium. The results confirmed the mathematical results that predicted that the average trapping force is proportional to many factors including, the amplitude squared, square of the frequency and inversely proportional to the speed of sound of the medium raised to the fourth power when responding to a displacement from the origin. When responding to a initial velocity the average force is proportional to the amplitude,the frequency and inversely proportional to the speed of sound of the medium squared.

I. INTRODUCTION

An acoustic trap is a device which can use sonic pressure to manipulate and control small objects within a particular region. It is found to be particularly useful in medicine and other fields, where small objects need to be suspended in a particular region for study. It uses a number of transducers in order to generate a standing wave in the direction it wishes to trap the object. This can be achieved with a variety of different designs that format the locations of the key elements differently, but fundamentally use the same principle, of using a standing wave to generate an effective low point in potential energy.

This device is related to another device known as an ion trap, which is used to study individual ions. Even though these devices seem very different, they share many properties in that they use waves in order to keep a particle contained at a particular point. In spite of this though, there are still significant differences between acoustic an ion trap. Most significantly, ion traps, mostly, do not use waves in space, but rather time [4]. Acoustic traps use standing waves, in contrast, which oscillate in both space and time [1]. They use standing waves, and trap particles inside the node of a particular standing wave. Additionally, they can also reflect the sound waves of the transducer off a curved surface to provide a trapping effect both in both the direction along the travelling direction of the wave, and also perpendicular to that [1] [2]. Alternatively traps can use a fluid flowing in one direction, with a sound wave travelling perpendicular to it in order to establish a standing wave, which can also have objects inserted into it through the flow. An example is shown in Fig. 1.

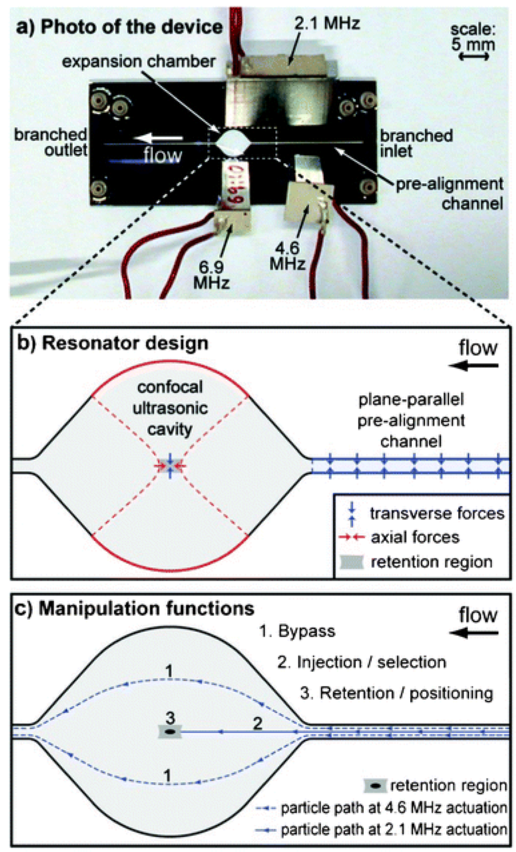

FIG. 1: Example of a flow based acoustic trap. This is an image from [1].

The most simple model for an acoustic trap has a simple standing wave, which a particle gets trapped in the node thereof. This design is demonstrated in two dimensions of Fig. 2. This design is the most basic, and provides a basis from which some of the other designs are evaluated. The core of this model is a standing wave generated by two transducers on each side of the center of the trap. This basic idea forms the base for most of my models, though the other designs rely on similar principles.

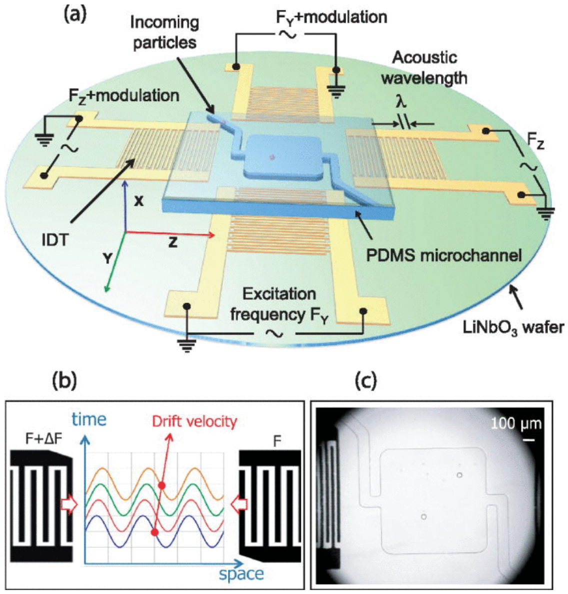

FIG. 2: A Simple two dimensional trap. This is an image from [5].

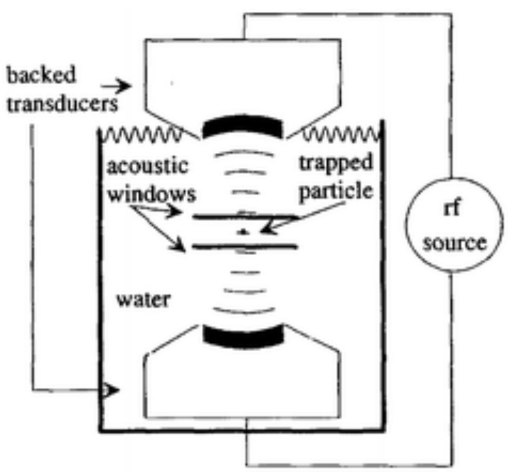

FIG. 3: A trap using two curved transducers. This is an image from [1].

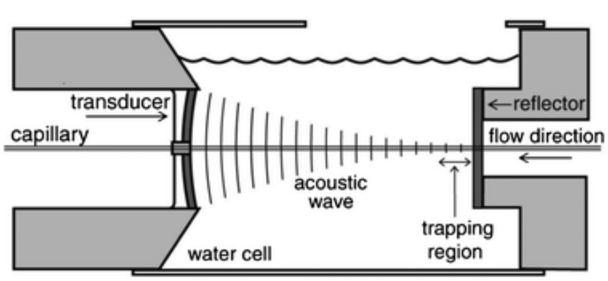

FIG. 4: A trap using a single curved transducer, and flow controls. This is an image from [1].

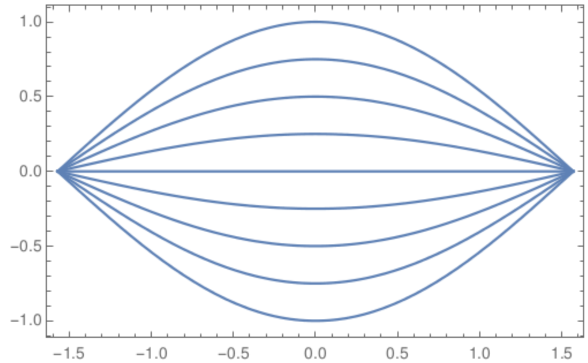

FIG. 5: An oscillating cosine function. The function represents the deviation from normal pressure due to the sound wave.

II. THEORY

A. One dimensional simple model

Consider a particle confined to one dimension of space in a fluid. If two transducers are placed on either side of the particle, emitting the same frequency, a standing wave forms between them. If near the center of the two transducers, near where the particle sits, the two sides emit sound waves that have approximately the same amplitude A, angular frequency ω, and speed of sound c. The pressure of the fluid relative to background pressure is

$$p(x,t) = A(\cos(\omega x/c + \omega t) + \cos(\omega x/c - \omega t)). \qquad (1)$$

Note that while the amplitude of the wave decreases as a function of distance from the transducer, but everything that takes place, takes place within a narrow region, where the amplitude is approximately constant. In addition, ω/c is in use rather than the standard variable k for the wave number, since the frequency, and speed of sound of the medium are what the experimenter has control over, to some extent. This also reduces the number of variables in use. In some cases, where plane waves in a confined space are applied, the amplitude does not significantly decrease across the distance, as in Fig. 2. Equation 1 then simplifies to

$$p(x,t) = 2A\cos(\omega x/c)\cos(\omega t). \qquad (2)$$

Now we must consider how much force is applied to a particle if exposed to this pressure gradient. If this particle has some width and area exposed to the x axis where the pressure varies, then a difference in pressure between the two sides of a particle will mean that a force is acting on the particle. If there is a spherical particle then the force applied by the air pressure on one side is

$$F_{side}(x,t) = -\iint_S p(x+\delta x,t)\vec{n}dA, \qquad (3)$$

where R is the radius of the particle, and $\vec{n}$ is the vector normal to the surface particle. More explicitly,

$$F_{side}(x,t) = -\int_0^{2\pi}\int_0^{\pi/2} p(x+R\cos(\theta),t)R^2\cos(\theta)d\theta d\phi, \quad (4)$$

If it is assumed that the pressure in the region is approximately linear, and the force along the $\hat{x}$ is considered, then

$$F_{side}(x,t) = -\int_0^{2\pi}\int_0^{\pi/2}(p(x,t) + \frac{dp}{dx}(x,t)R\cos(\theta)) R^2\cos(\theta)d\theta d\phi \quad (5)$$

This simplifies to

$$F_{side}(x,t) = -p(x,t)\pi R^2 - 2\pi R^3\frac{dp}{dx}\int_0^{\pi/2}\cos^2(\theta)d\theta, \quad (6)$$

or

$$F_{side}(x,t) = -p(x,t)\pi R^2 - \pi^2 R^3\frac{dp}{dx}/2. \quad (7)$$

If we factor in the pressure applied from the other side of the particle then the net force is

$$F_{net}(x,t) = F_{side1} + F_{side2} = -\pi^2 R^3\frac{dp}{dx}. \quad (8)$$

Let $\alpha = \pi^2 R^3$, to simplify further calculations.

Using the equations 2, and 8, the motion of a particle adding a drag term β can be derived, yielding

$$m\ddot{x} = -\beta\dot{x} - 2\alpha A\omega/c\sin(\omega x/c)\cos(\omega t). \quad (9)$$

So far as I can tell, this equation does not have a simple solution that can be applied, either directly through elementary functions, power series, or Fourier transformations. However, numerically integrating the solution did lead to a possible solution, or at least approximation. Notice that this equation of motion is very similar damped driven oscillations. If this equation of motion is integrated for high values of ω, it resembles damped oscillations. Therefore there may be solutions, or approximations in the form of $x(t) = Ae^{-\gamma t}\cos(\psi t + \delta)$, where γ, and ψ are some constants. Note that $k << \omega$ for large ω. This acts as a damped harmonic oscillator, and the motion during individual cycles of the sound wave are ignored. The whole system acts as a mass on a spring. Fig. 15 seems to support this, as the sound wave oscillates 1024 times, in the time it takes the particle to move back and forth 12 times.

B. Basic analysis of the simple model

If the particle is allowed to move freely in this potential, then it may seem that the particle should not receive any significant force in any particular direction. The particle should seem to receive an average of zero force, and be allowed to move approximately in any direction it was set in. This preliminary, however, does not reflect the whole situation when the particle is moving slowly enough near a node in the pressure or anti node in the force that a particle will become trapped there.

Let it be assumed that the particle is moving near $x = 0$ such that $|x| << \lambda = 2\pi c/\omega$. Also let the velocity be $|v| << c$. Lastly assume $\beta << 2\alpha A\omega^2/c^2$ and can be ignored. If it is assumed that the velocity during a single cycle is approximately constant, and the small angle approximation applies to the equation of motion 9, then the average force on the particle during half a cycle is

$$\bar{F} = \omega/\pi\int_{\pi/2\omega}^{3\pi/2\omega} 2\alpha A(\omega/c)^2(x_0 + vt)\cos(\omega t)dt. \quad (10)$$

These particular limits of integration were chosen so that they would align with the pressure gradient always having a negative value. Solving equation 10 gives

$$\bar{F} = \frac{2A\alpha\omega^3}{\pi c^2}(x_0\int_{\pi/2\omega}^{3\pi/2\omega}\cos(\omega t)dt + v\int_{\pi/2\omega}^{3\pi/2\omega} t\cos(\omega t)dt). \quad (11)$$

Finally:

$$\bar{F} = \frac{4A\alpha\omega^2 x_0}{\pi c^2} - \frac{4A\alpha\omega v}{c^2} \quad (12)$$

This relation (equation 12) is the basis for this experiment. It should be noted that the sign difference between the two terms of 12 are unimportant as during the next half cycle the situation is reversed resulting in the opposite result. This does not, however mean that there is zero net force over a whole cycle. If the net force during half a cycle pushes a particle outward, then during the next half cycle it is pushed inward, and since the first half cycle pushes the particle outward, x, and v are greater resulting in a larger push inward.

Now it must be considered, how much stronger the push in the opposite direction is going to be. The average force translates into a amount of impulse. This linearly increases the velocity ($\bar{F}/m = \Delta v$), but the displacement is increased a different factor. If the force on the particle is approximated as constant over the period then $\Delta x = \bar{F}(\Delta t)^2/(2m)$. This is not strictly true, but it provides a simple enough approximation to explain the rest of the experiment. Now, notice that for the x direction, the average force over a complete cycle will be proportional to $A(\omega^2/c^2)\Delta x(\Delta t)^2 \propto A^2(\omega^4/c^4)(\Delta t)^2 x$. But notice that $\Delta t = \pi/\omega$, which means that $\bar{F}_{\text{second half}} \propto A^2\omega^2 x/c^4$. Now consider the difference between the force out and the force in.

$$\bar{F}_{\text{full cycle}} \propto A\omega^2(x + \Delta x)/c^2 + A\omega(v + \Delta v)/c^2 - A\omega^2 x/c^2 - A\omega v/c^2 \quad (13)$$

Putting that all together:

$$\bar{F}_{\text{full cycle}} = \eta A^2\omega^2 x/c^4 + \zeta A\omega v/c^2, \tag{14}$$

where η and ζ are positive real numbers.

C. More Dimensions

Expanding this model into more dimensions is as simple as arranging for standing waves in each of the other dimensions. If the standing waves in two dimensions have the equation:

$$p(x, y, t) = 2A(\cos(\omega x/c)\cos(\omega t) + \cos(\omega y/c)\cos(\omega t)), \tag{15}$$

then it is possible to trap a particle within a two dimensional confined space. It is important to note that the two dimensions act independently of one another, and thus, in spite of the fact that this is very similar to an ion trap in many ways, there is a substantial difference. The electric field lines of the Coulomb force need to have a net flux of zero through the middle of the ion trap, but compressible fluids have no such requirement, though incompressible fluids do. This experiment concentrates mainly on compressible fluids, however the same method that makes ion traps possible, also makes acoustic traps in incompressible fluids possible, which is to put the standing waves in the two directions 90° out of phase to one another so that there in never any net flux in or out of the center of the trap. In both cases, adding additional dimensions is irrelevant, and the predictions for the proportionality of the system stay about the same.

D. Curved transducers, and Multidimensional Control with One Transducer Set

I have previously claimed that the same mathematics applies to multiple directions even with the vast variety of models that exist. With the case of the design in Fig. 2, that is obvious. It is less so with the curved transducers. If we consider a pair of point sound sources spaced the same distance from the center of a trap, we get the following pressure pattern:

$$r_1 = \sqrt{(x+a)^2 + y^2 + z^2}, \tag{16}$$

$$r_2 = \sqrt{(x-a)^2 + y^2 + z^2}, \tag{17}$$

$$p(x, y, z, t) = \sum_{i=1}^{2} A(r_i)\cos(\omega r_i/c \pm \omega t). \tag{18}$$

At the center of the trap $A(r_1) = A(r_2)$ and are approximately constant. It is clear that along the x axis the same standing wave as described in equation 2 results. If the wave is analysed along the y or z axes, similar results occur:

$$p(0, y, 0, t) = 2A\cos(\omega\sqrt{a^2 + y^2}/c)\cos(\omega t). \tag{19}$$

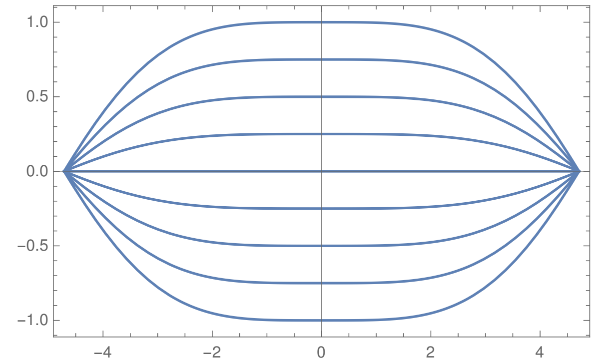

FIG. 6: Example of a potential perpendicular to the direction of the main standing wave, in arbitrary units.

This is a distorted version of the original potential in equation 2. Note that this function becomes more linear as y becomes larger, and even though it may not have the same containment ability when $|y| < a$ it still functions as a trap. This is demonstrated in Fig. 6.

III. PROCEDURE

In order to evaluate the approximations that allow it to be assumed that the trap will work, it can be modelled in a few different manners. Ideally one would build an actual acoustic trap, but there are limits to what I could build. Unfortunately, a functioning acoustic trap is beyond my materials constraints. However, I was able to develop a couple of computer models that allow for the simulation of the system.

This computer model used the C programming language for its efficiency and its simplicity. The C code numerically integrated the equation of motion Eq. 9 for many values of A, ω, and c. The values chosen are largely arbitrary, but they do have some basis on reality. The values for A started at 1024, and increased by a multiple of $\sqrt{2}$ each step. Sixteen values of A were sampled. These values absorb the constant α, an inverse factor of m, and a factor of two. The values for ω start at 256, and again increase by $\sqrt{2}$ each cycle. Again, sixteen values were sampled. these values extend from very low frequencies in the audible range, to ones that actual acoustic traps use. Finally, c starts at 86, and again increments by two every two cycles, or $\sqrt{2}$ every cycle. Eight cycles of these were sampled, and the values were chosen to center the sampling around 344, the actual speed of sound in air. Later in the analysis phase, using *Igor Pro,* the results are compared for proportionally to the predictions made in equation 12. To do so the program cycled through a range of values for A, ω, and c. Each frame of the numerical integration the force on the particle is evaluated then the values are averaged to assess the trapping force of the trap. This evaluation was done twice. Once with the initial conditions such that there is a small initial displacement, and no initial velocity, and the other such that there was no initial displacement, and a small initial velocity. This is done because the dependence on frequency changes depending on if the system is reacting to an initial velocity or displacement. Note that applying this to more dimensions is not helpful, since the forces in each dimension act entirely independently of one another. Moreover, the basic mathematics that applies to the acoustic trap in this case, also applies to the other designs. Once the average force over $2^{20} = 1\ 048\ 576$ cycles was evaluated, the *Igor Pro* analysis could start.

Powers of two were selected throughout this process, so that the floating point format could calculate them more accurately. This works the same way as powers of ten in a decimal system (just as multiplying by 1000 is easy in decimal, multiplying by $2^{10} = 1024$ is easy in binary). Floating point is a common format to represent large and small numbers that effectively works like scientific notation in base two. This is especially important in the time step.

Firstly the data was imported from a comma separated file into "Igor Pro." All of the data for which the particle escaped the well were not considered in the evaluation. Next one dimensional cross sections of the data was taken through the center of each axis, to compare the results on a logarithmic plot. This allowed for evaluation of a power relation between each independent variable, and the average force applied. Lastly the data for each of the runs was compared with $A^2\omega^2 x/c^4$ or $A\omega v/c^2$ respectively. The quotient of the actual result and these values were then averaged, and a standard deviation evaluated, to get values for η and ζ.

IV. RESULTS

As the data are three dimensional, and including the force values four dimensional, displaying the data can be accomplished by taking two dimensional cross sections. The cross sections are shown in the following three sets of two graphs, with data from the displacement start simulation with square data points, and circular for the velocity start simulation. Each of the sets represents, dependence on amplitude, dependence on frequency, and dependence on speed of sound.

When the amplitude is compared against the trapping force the power law that relates them is:

$$\bar{F} \propto A^{k_A}. \tag{20}$$

The results for the displacement based simulation are shown in Fig. 7. The power k_A was found to be 2.08 ± 0.03, where the simplified mathematics predicts that it should be exactly two. For the velocity based simulation $k_A = 1.06 \pm 0.02$, where exactly one was predicted. This is shown in Fig. 8.

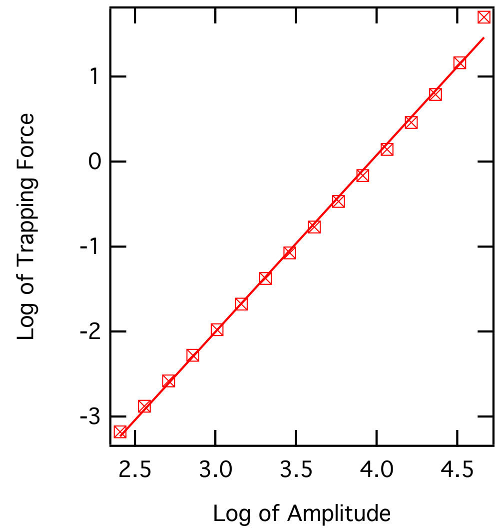

FIG. 7: The logarithm of the amplitude versus the logarithm of the trapping force, with starting displacement.

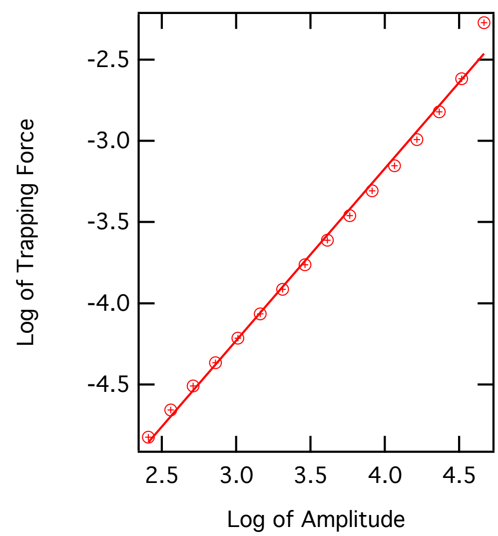

FIG. 8: The amplitude, and the logarithm of the trapping force, with starting velocity.

For the angular frequency comparison, we use k_ω for the exponent. The results for the displacement based experiment are displayed in Fig. 9., $k_\omega = 2.017 \pm 0.008$ with an expected value of exactly two. In the other case, $k_\omega = 1.04 \pm 0.01$ with an expected value of exactly one. Results displayed in Fig. 10.

In the case of speed of sound comparison, both results again are near, but have slightly larger magnitude

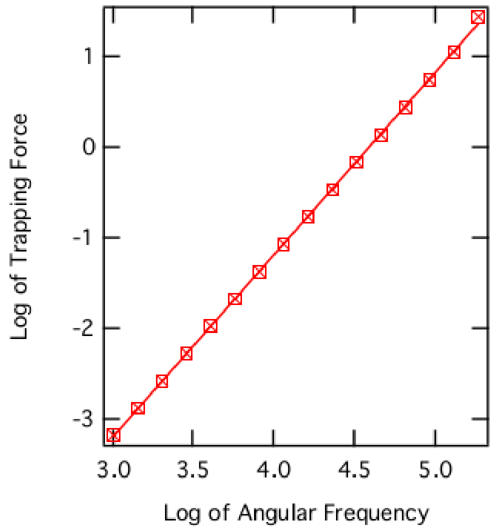

FIG. 9: The logarithm of the angular frequency versus the logarithm of the trapping force, with starting displacement.

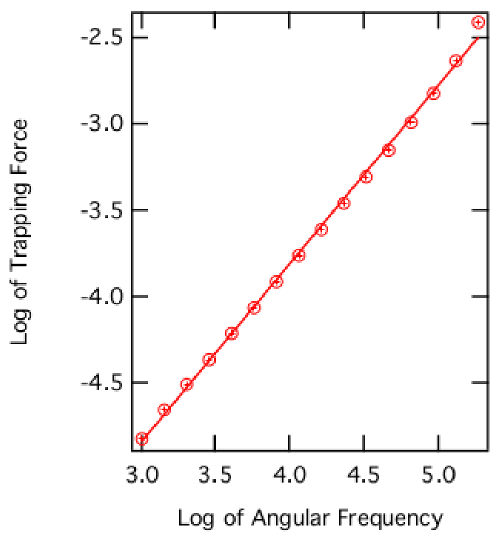

FIG. 10: The logarithm of the angular frequency, and the logarithm of the trapping force, with starting velocity.

than the predicted values. For the displacement based experiment, $k_c = -4.11 \pm 0.04$, with an expected value of exactly negative four. Results displayed in Fig. 11. In the velocity based experiment, $k_c = -2.07 \pm 0.03$, with an expected value of exactly negative two. Results displayed in Fig. 12.

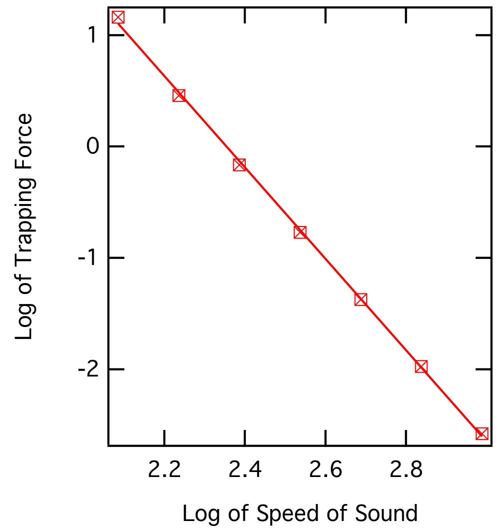

FIG. 11: The logarithm of the speed of sound of the medium, and the logarithm of the trapping force, with starting displacement

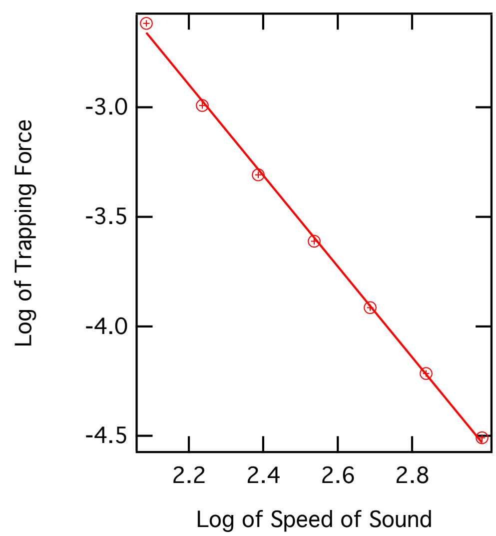

FIG. 12: The logarithm of the speed of sound of the medium, and the logarithm of the trapping force, with starting velocity.

TABLE I: The results for the displacement, velocity starts. "Pre." means predicted, "Disp." is initial displacement start, and "Vel." initial velocity start.

Disp.	Pre	Measured	Vel.	Pre	Measured
A	2	2.08 ± 0.03	A	1	1.06 ± 0.02
ω	2	2.017 ± 0.008	ω	1	1.04 ± 0.01
c	-4	-4.11 ± 0.04	c	-2	-2.07 ± 0.03

Assembling all 2048 data points, the actual data points were compared with the value of $A^2\omega^2/c^4$, or $A\omega/c^2$. The results are displayed in Fig. 13, and 14. Using that data, and factoring in initial values for $x\mid_{t=0}= 2^{-16}$ in the case of the displacement based experiment, and $\dot{x}\mid_{t=0}= 2^{-20}$ for the velocity based experiment, values for η and

ζ could be determined. As a result, $\eta = 0.04 \pm 0.01$ and $\zeta = 0.5 \pm 0.1$

V. ANALYSIS

The first major thing of note about the results of this are that there seem to be systematic patterns by which each of the values for the data vary from the fits. This is not entirely unexpected as, there were many approximations that went into equation 14. Additionally there seems to be a surprising result in that the system responds differently to a displacement from the node, as opposed to the velocity just in the initial conditions. During each run, the particles do oscillate about the trap, but when the displacement initial condition is selected, they oscillate much more slowly. It is important to notice that the period of oscillation for the waveform is not the oscillations about the trap, changing in both velocity and displacement. This is demonstrated in Fig. 15. This difference is likely what makes the initial conditions produce different results. In general, the term due to the velocity acts much more strongly on the particle due to the fact that ζ is about ten times larger than η. However, the velocities involved when the particle is let free from a displaced location are so low that the displacement term overpowers the other term. Note also that the dependence on each independent variable is slightly larger than that which would be expected. This could be due to the fact that there are in fact two terms always involved, but one dominates, due to the initial conditions.

VI. CONCLUSION

In conclusion, the most any of the predictions that the power of a particular value deviates from the expected value is 0.11. This experiment however lacks, in that it did not test an actual acoustic trap. This could possibly be an area of future research In addition other possible methods of further study on this topic include, more accurate approximation of the trapping force, as well as exploring how the size of the particle influences the trap. Moreover, there were a number of other factors that I did not examine closely. For nearly all of my computations, I assumed there was very little drag. This could be explored, and how drag influences the trap. Moreover the equation of motion may have a solution, in that it oscillates quickly and with very little amplitude with respect to the overarching motion, and also has a larger oscillation that mimics damped oscillations. In addition, one could look at adding gravity to the simulation, as some very strong acoustic traps can fight the force of gravity. Lastly the effect the curved potential, whose potential is Eq. 19 could be more closely examined. In summary, the influence that amplitude of a waveform, frequency, and speed of sound in the medium were verified to within 10 percent, but there are still many more factors that can be explored.

VII. ACKNOWLEDGEMENTS

I would like to thank my professor Dr. Lehman for providing advice on the project. I would like to also thank Dr. Lindner for advising me on the building of the computer simulation.

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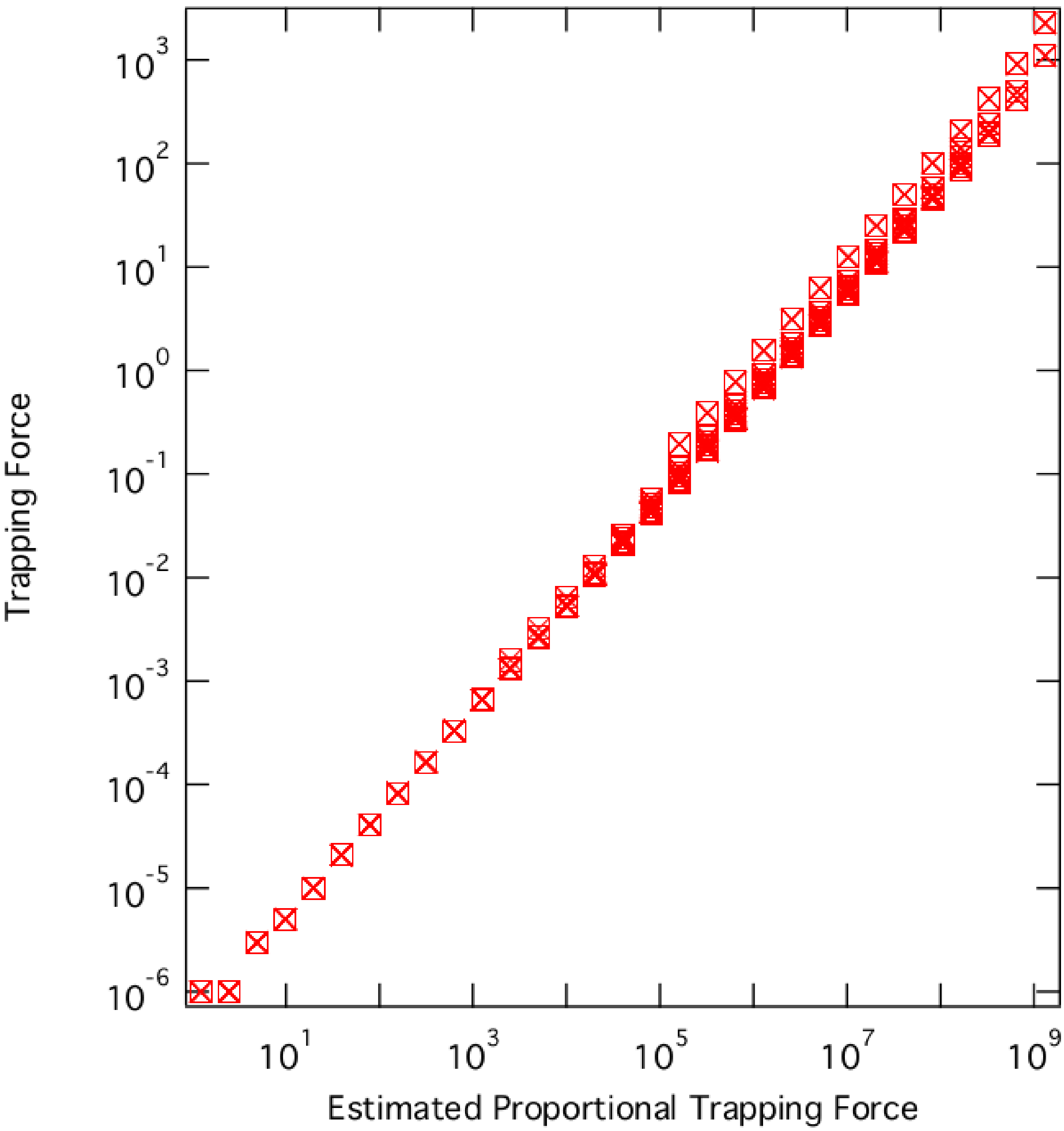

FIG. 13: The comparison of the of estimated proportional value for the trapping force, and the trapping force, with starting displacement. $\eta = 0.04 \pm 0.01$

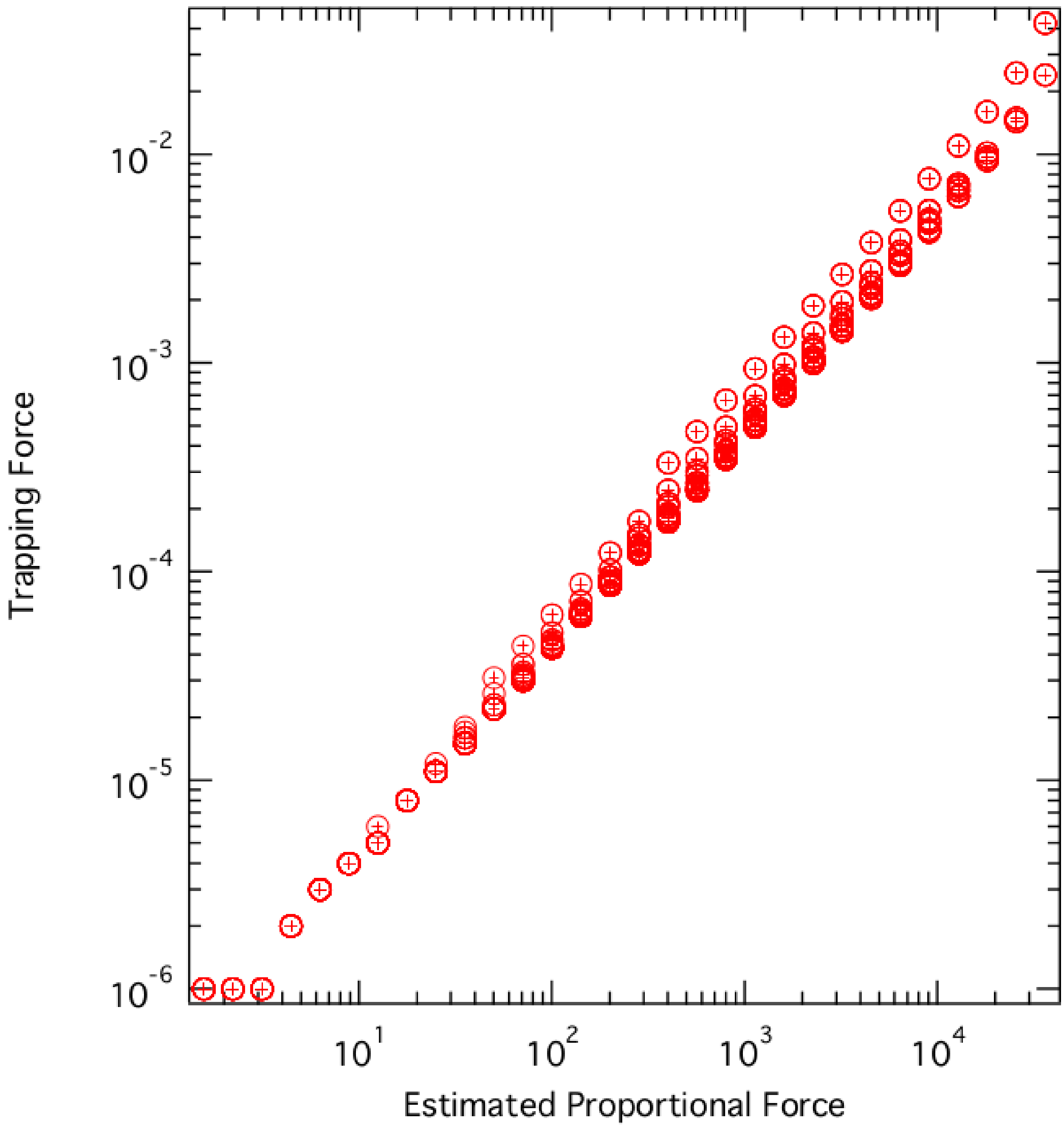

FIG. 14: The comparison of the of estimated proportional value for the trapping force, and the trapping force, with starting velocity. $\zeta = 0.5 \pm 0.1$

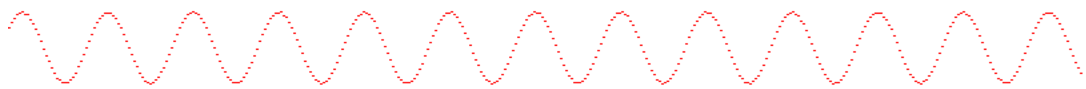

FIG. 15: Example of actual particle motion. Note this represents 1024 oscillations in the sound wave. The Y axis represents the displacement of the particle in arbitrary units