SD

Using Double-Exposure Interferometry to Measure Thermal Expansion in Aluminum

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Abstract: In this experiment, a hologram is taken of a soda can at room temperature. Then 100 mL of hot water was poured into the can at room temperature and the same plate is exposed again. The interference pattern produced was to be used to measure how much the can had expanded due to the heat, but it was concluded that the thermal expansion is too minute for the interference pattern to be workable. It was concluded that double exposure interferometry using red coherent light is an ineffective way to measure thermal expansion.

Introduction:

In 1948 the first idea of constructing a three-dimensional photographic image was conceived by Dr. Dennis Gabor[1] of the Imperial College in London. In 1960, the laser was invented and the first hologram was created. Soon after, the data storage capabilities of holography were used in engineering. W.R. Bradford[2] published a paper on Holographic measuring techniques. Later on holography was used in the creation of movies and television[3]. In this experiment, double exposure interferometry will be used to measure thermal expansion in aluminum.

Theory:

Holography:

In order to understand holography, we must first seek to understand the principles behind constructive and destructive interference in waves. We know that a wave solution is any solution to the wave equation[4]:

$$\nabla^2 \mathbf{E} = \frac{1}{v^2}\frac{\partial}{\partial t}\mathbf{E} \quad (1)$$

Consider two transverse waves, which can be described by $\mathbf{E}_1 = E_{01}e^{-i(k_1 x - \omega t + \theta)}\hat{\mathbf{x}}$ and $\mathbf{E}_2 = E_{02}e^{-i(k_{21} x - \omega t + \phi)}\hat{\mathbf{x}}$ where E_{01} and E_{02} are the respective amplitudes, k is the wave number equivalent to $\frac{2\pi}{\lambda}$, and ω is the angular frequency equivalent to $2\pi f$ where f is the frequency of the wave. When E_1 and E_2 interfere they add vectorially by the principle of superposition to give $\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 = E_{01}e^{-i(k_1 x - \omega t + \theta)}\hat{\mathbf{x}} + E_{02}e^{-i(k_2 x - \omega t + \theta)}\hat{\mathbf{x}}$. In this case, we are concerned not with the resultant electric fields, but with the resultant irradiance given by[2]:

$$I = (E_1 + E_2)(E_1^* + E_2^*) \quad (2)$$ where E^* denotes the complex conjugate of E

Holograms are based on this notion of constructive and destructive interference. Figure 1 illustrates the construction of a hologram.

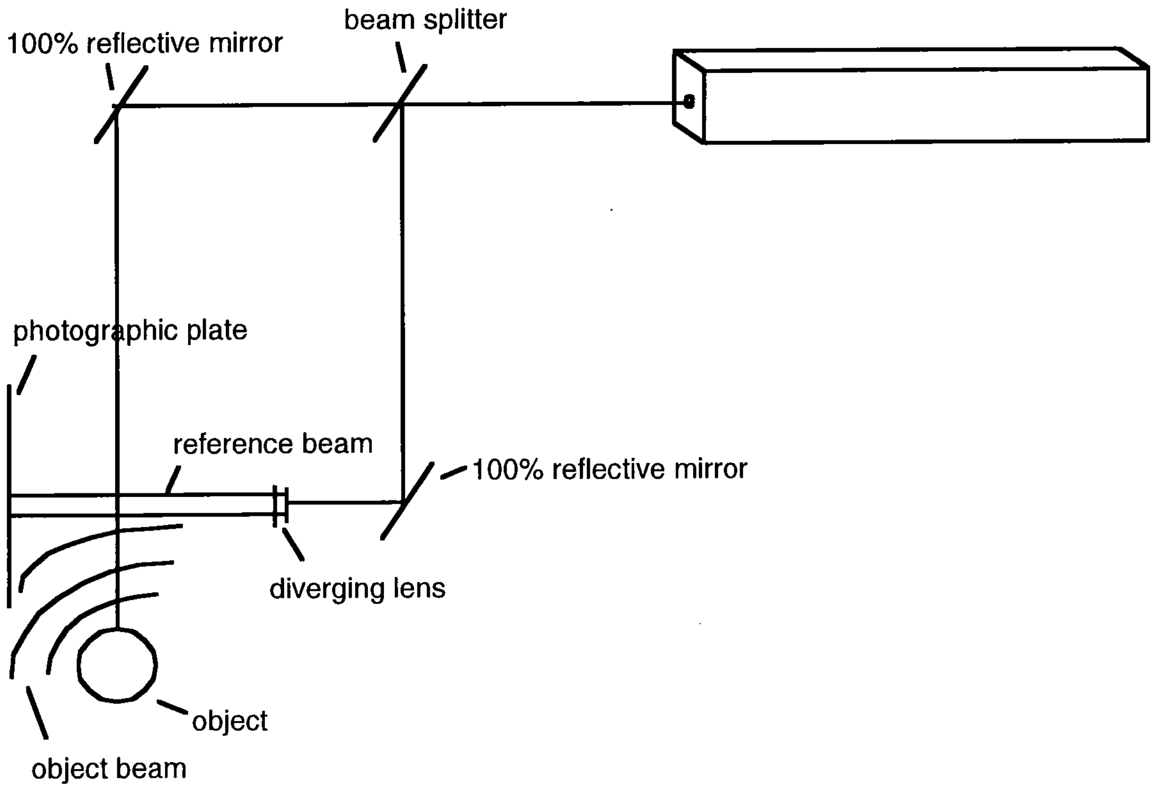

Figure 1. The construction of a hologram

In this example we will call the reference beam

$$E_R = re^{i(\omega t+\varphi)} \quad (3)$$

where φ includes both the spatial variable and the phase constant and r is the amplitude. We see in figure 2 a blown-up picture of the reference beam hitting the photographic plate.

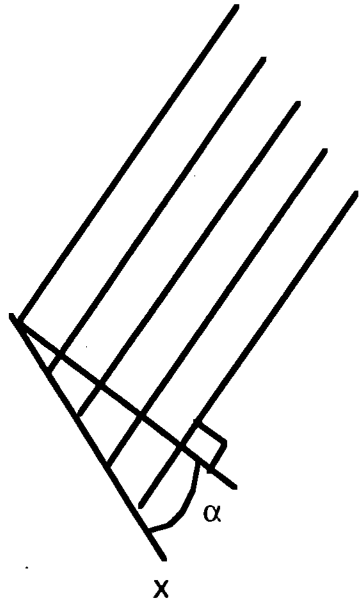

Figure 2. The reference beam hitting the photographic plate

The reference beam hits the plate at angle α, thus φ is a linear function $\varphi = \frac{2\pi}{\lambda} x \sin\alpha$. When we plug this value into equation 3, we modify the reference beam equation.

$$E_R = re^{i\omega t} e^{i\left(\frac{2\pi}{\lambda} x \sin\alpha\right)} \quad (4)$$

The object beam looks similar to the reference beam: $E_O = se^{i(\omega t+\theta)}$ where s is the amplitude, except in this case the phase term, θ is complicated and depends on the shape of the object. As the two beams hit the photographic plate, they will induce a resultant irradiance of:

$$I_T = (E_R + E_O)(E_R^* + E_O^*) \quad (5)$$

$$= r^2 + s^2 + rse^{i(\omega t+\theta)} e^{-i(\omega t+\varphi)} + rse^{i(\omega t+\varphi)} e^{-i(\omega t+\theta)}$$

$$= r^2 + s^2 + rse^{i(\theta-\varphi)} + rse^{-i(\theta-\varphi)}$$

This represents the transmittance function that will determine the hologram upon reconstruction. When the hologram is reconstructed, the hologram will appear according to:

$$E_H = I_F E_R = (r^2 + s^2)E_R + r^2 se^{i(\omega t+\theta)} + r^2 e^{i(2\varphi)} se^{i(\omega t-\theta)} \quad (6)$$

We interpret the first term in equation 5 as simply a magnified reference beam. The second term is essentially the subject beam modified in amplitude. Thus it diverges outside the plate, but will appear as a virtual image inside the plate at angle α. The third and final term represents the object beam with a phase reversal. The significance of this is that the beam will *converge* outside the plate and appear as a real image at an angle of 2α. In this experiment, we will use the real image to get our results.

Heat Conduction:

In this experiment, the data stored in holograms will be used to quantify the thermal expansion of a soda can. As the heated water is placed into the can, the can will expand radially outward where the water is in contact with it immediately. The heat will then be conducted through the aluminum upwards according to the heat equation[5]:

$$\frac{\partial u}{\partial t} = \frac{K}{c\delta}\nabla^2 u \quad (7)$$

where u is a function with spatial and temporal dependencies and represents the temperature of the can. K is the thermal conductivity of aluminum, c is the specific heat of aluminum and δ is the density of the can. Equation 7 is derived from the empirical flux formula[5]:

$$\Phi(x,t) = -K\frac{\partial u}{\partial t} \qquad (8)$$

Because the conduction in this experiment will occur in only one direction, we can simplify the problem to one spatial dimension, x. To visualize the conduction, a small segment of a hollow cylinder should be considered.

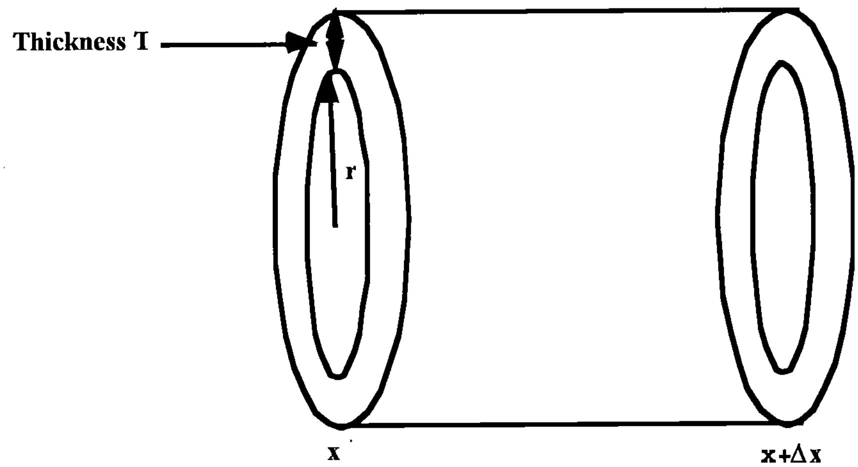

Figure 3. A hollow cylinder with thickness T

The area of the cross-section of the aluminum will be

$$\begin{aligned} A &= \pi\left[(r+T)^2 - r^2\right] \\ &= \pi\left(2rT + T^2\right) \end{aligned} \quad (8)$$

but for simplicity, we will call it A as we will find that it is unimportant to the final conductivity equation. Using equation 8, we will quantify the rate of flow into this segment of the cylinder:

$$\Phi(x,t) = \frac{R}{A}$$

$$\begin{aligned} R &= A\left[\Phi(x,t) - \Phi(x+\Delta x,t)\right] \\ &= KA\left[u_x(x+\Delta x,t) - u_x(x,t)\right] \end{aligned} \quad (9)$$

where u_x represents the first spatial derivative of u

The specific heat of a material, c is the amount of heat required to raise 1 cm^3 of the material one degree C. Therefore, $c\delta u$ calories are needed to raise one cubic centimeter of the material from zero to temperature u. A small segment of the can has length dx and volume Adx therefore $c\delta uAdx$ calories of heat are needed to raise the temperature of this short segment from 0 to u.

The heat content of the segment is defined:

$$Q(t) = c\delta A\int_x^{x+\Delta x} u(x,t)dx \qquad (10)$$

We assume that heat enters only from the circular ends of the cylinder and not the sides. Thus we can use equation 9 to show:

$$R = Q'(t) = KA[u_x(x+\Delta x,t) - u_x(x,t)] \quad (11)$$

We can also differentiate (10) with respect to t and use the mean value theorem and see that:

$$Q'(t) = \int_x^{x+\Delta x} c\delta A u_t(x,t)dx = c\delta A u_t(\bar{x},t)\Delta x \quad (12)$$

for some $\bar{x}$ in the interval $(x, x+\Delta x)$. We now equate the values in 11 and 12 to arrive at:

$$\begin{aligned}
&c\delta A u_t(\bar{x},t)\Delta x = KA[u_x(x+\Delta x,t) - u_x(x,t)] \\
&u_t(\bar{x},t) = \frac{K}{c\delta}\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x} \\
&\lim_{\Delta x \to 0} u_t(\bar{x},t) = \frac{K}{c\delta}\lim_{\Delta x \to 0}\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x} \quad (13) \\
&\frac{\partial u}{\partial t} = \frac{K}{c\delta}\frac{\partial^2 u}{\partial t^2}
\end{aligned}$$

Using the separation of variables technique for partial differential equations and applying appropriate boundary conditions, we can see that the complete solution to equation 13 is:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 kt/L^2} \sin\frac{n\pi x}{L} \quad (14)$$ where L is the length of the can

The Fourier coefficients A_n can be calculated using the formula:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad (15)$$

where $f(x)$ represents the initial condition $f(x) = u(x,0)$. In this case, the can is at room temperature (25 degrees centigrade) for all x in the interval $[0, L]$ at time $t=0$ therefore $f(x) = 25$.

Thermal Expansion:

Heat will be applied to the can at the base, thus the can will expand at $x=0$ if we define the origin to be at the base of the can. This expansion will cause stress in the can as x increases.

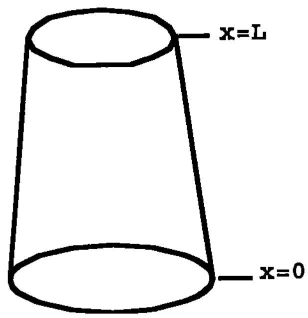

Figure 4. A greatly exaggerated model of the can expanding

Figure 4 is a greatly exaggerated model of how the can would expand if heat were applied to the base (*x=0*). If we take a hologram of the can before the heat is applied and after the heat is applied, the deflection will cause an interference pattern in the final exposure.

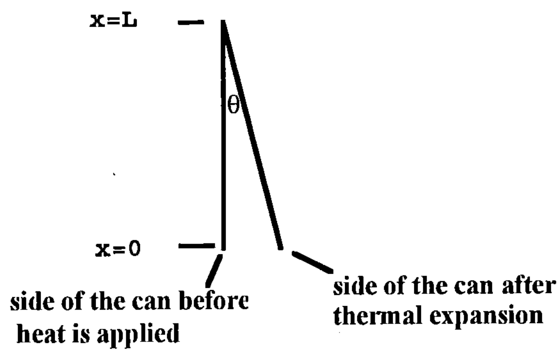

Figure 5. A diagram of the path difference

The path difference the laser light will travel from one exposure to the other will be determined by the thermal expansion of the base of the can. The following equation describes the thermal expansion[6]:

$\Delta C = n\lambda = \Delta T \alpha C_0$ (16) where α is the coefficient of thermal expansion and C is the circumference of the can

In different places along x, the path difference will be exactly $\frac{1}{2}\lambda$ which will cause a dark spot in the hologram. By counting the number of dark spots, we can deduce the total deflection due to the thermal expansion. If N represents the number of dark spots,

$$\Delta C = N\lambda = \alpha C_0 \Delta T \qquad (17)$$

From this, we can calculate ΔT and thus discover how much heat was transferred from the hot water to the can.

Equipment:

HeNe laser

Two diverging lenses

Beam splitter

Holographic plates (Holographic Recording Technologies) model BB-640

Soda can

Two 100% reflective mirrors

Hot Pot (Rival)

100 mL graduated cylinder

Alcohol thermometer

Thermos

Anchors for optical equipment

Apparatus and Setup:

The apparatus is shown in figure 1. In order to get good results, the optical equipment must be sufficiently anchored such that there is no wobble in the can or in the holographic plate. A shift in one of these objects so much as 1×10^{-8} meters can completely ruin the interference pattern.

The diverging lenses must also be cleaned with acetone before the holograms are taken. In this experiment, many holograms did not develop well due to poor quality reference beams and object beams. It is also important to expose the plates for 20 seconds both before and after the heat is applied. This will ensure the quality of the interference pattern.

Procedure:

First, the optical equipment was arranged as shown in figure 1. Then the developing chemicals were prepared. The soda can was anchored well to the table such that it cannot be moved as was the stand for the holographic plate. The lights were then turned off and the holographic plate was placed in the stand. The unheated can was then exposed for 20 seconds to the holographic plate. Next, 100 mL of water (when 100 mL of water is poured into the can, the water lever ends at *x=3.4 cm*)* was heated to 30 degrees Celsius, and placed in the thermos to be transported to the darkroom with minimal heat loss to the water. The water was then placed into the can and immediately another 20-second exposure to the light was made. The plate was then placed into the developer for 2 minutes followed by 3 minutes of rinsing. The plate was then bleached and cleaned. When the plate was fully developed, the hologram was reconstructed and the number of dark spots in the interference pattern was counted. The procedure was then repeated for water temperatures of 35 degrees, 40 degrees and 50 degrees Celsius.

Data and Analysis:

For data analysis, the following measurements were taken:

Circumference of unheated can (C)	21.10±.05 cm
Height of the can (L)	12.21±.05 cm
Thermal diffusivity for aluminum (k)	0.85 cm^2/s
Coefficient of thermal expansion for Al. (α)	$2.3X10^{-7}$/degree C

Table 1. measurements and constants

Firstly we must calculate how the heat will be conducted in the aluminum can over the exposure time of 20 seconds. If it is significant, the interference pattern will be affected. We see from equation 14 that

$$u(x,t)=\sum_{n=0}^{\infty}\left(A_n \sin\frac{n\pi x}{L}+B_n \cos\frac{n\pi x}{L}\right)e^{-n^2\pi^2 kt/L^2} \qquad (18)$$

The next task is to determine the Fourier coefficients A_n. For small *t*, we know that *u(0,t)=35*, and *u(L,t)=25*. Plugging these values into equation 18, we see that $B_0 = 35$ and $B_n = 0$ for *n*>0. To find A_n, formula 15. Initially, the function $f(x)$ is a piecewise function, which looks like:

$$f(x)=\begin{cases}0 \ldots 0 \le x \le L/2 \\ -10 \ldots L/2 < x < L\end{cases}$$ (for simplicity, we assume in this case that the water level is halfway up the can)

This is assuming that the can is the same temperature as the water at and below the water level and at room temperature above the water level. Using equation 15, we can now calculate the Fourier coefficients:

$$A_n = -\frac{2}{L}\int_{L/2}^{L} 10\sin\left(\frac{n\pi x}{L}\right) = -\frac{20}{L}\int_{L/2}^{L}\sin\left(\frac{n\pi x}{L}\right)$$

From this we get:

$$A_1 = -\frac{20}{\pi}$$
$$A_2 = \frac{20}{\pi}$$
$$A_3 = -\frac{20}{3\pi}$$
$$A_4 = 0$$

and finally:

$$u(x,t) = 35 - \frac{20}{\pi}\sin\left(\frac{\pi x}{L}\right)e^{-\pi^2 kt/L^2} + \frac{20}{\pi}\sin\left(\frac{2\pi x}{L}\right)e^{-4\pi^2 kt/L^2} - \frac{20}{3\pi}\sin\left(\frac{3\pi x}{L}\right)e^{-9\pi^2 kt/L^2}$$

Using the first 4 terms of this series, we can calculate that the temperature .95 cm above the water level after 20 seconds is 33.01 degrees (see Appendix A). This is a significant temperature increase and if the theory holds, it may thwart our results.

Next we will calculate what the temperature of the can is at exposure time. From the trial when 40-degree water was used, the hologram showed a very uniform interference pattern. Upon reconstruction, 47 dark spots were counted. Using the accepted value for the expansion of aluminum[6], we can use equation 16 to calculate the temperature of the can:

$$(T_f - T_i)\alpha C_0 = n\lambda$$
$$T_f = \frac{(47)(632.8\times10^{-9} m)}{(2.3\times10^{7}\deg^{-1})(.211m)} + 25$$
$$= 637.84\deg$$

What this tells us is that the fringes must have been caused by something other than heat expansion since the can was obviously not 637.84 degrees. We calculate theoretically how many dark spots we expect to see if the water is 10 degrees warmer than room temperature:

$$\Delta T\alpha C_0 = n\lambda$$
$$n = \frac{(10)(2.3\times10^{-7}\deg^{-1})(.211m)}{632.8\times10^{-9} m}$$
$$= 0.76$$

The expansion is so small, we would not see even one dark spot!

We calculate the expected expansion of the base of the can theoretically. Assuming the can is 35 degrees where the water touches it and 25 degrees originally:

$$\Delta C = \Delta T\alpha C_0 = (10)(2.3\times10^{-7}\deg^{-1})(.211m) = 4.85\times10^{-7} m$$

Conclusions and Discussion:

Because the thermal expansion of the aluminum is so minute (relatively), holography using coherent light with wavelength 632.8 nm, is an ineffective way to measure it. If however lasers with smaller wavelengths are used, this may be an effective method. If wavelengths on the order of 10^{-7} meters are used, then results may be quite good. The interference fringes that were observed in the hologram for this experiment were more likely caused by the water pressure pushing the can outwards from the base. In the future, better ways of controlling temperature changes due to conductance should also be investigated.

Nice feature.

Appendix A

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} k$$

$$u = \sum_{n=0}^{\infty} \left(A_n \sin \frac{n\pi x}{L} + B_n \cos \frac{n\pi x}{L} \right) e^{-n^2\pi^2 kt/L^2}$$

$$A_n = -\frac{20}{L} \int_{-L/2}^{L} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= -\frac{20}{n\pi}\left[-\cos(n\pi) + \cos\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{20}{n\pi}\left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{20}{n\pi}\left[-1 - 0\right] = -\frac{20}{n\pi} \text{ when } n \text{ is odd}$$

$$= \frac{20}{n\pi}\left[1 + 1\right] = \frac{40}{n\pi}$$

$$A_1 = \frac{20}{\pi}(-1-0) = \frac{-20}{\pi} = -\frac{20}{\pi}$$

$$A_2 = \frac{20}{2\pi}(1+1) = \frac{40}{2\pi} = \frac{20}{\pi}$$

$$A_3 = \frac{20}{3\pi}(-1-0) = -\frac{20}{3\pi} = -\frac{20}{3\pi}$$

$$A_4 = \frac{20}{4\pi}(1-1) = 0 = 0$$

Therefore:

$$u(x,t) = 35 + \left(\frac{-20}{\pi}\right)\sin\left(\frac{\pi x}{L}\right)e^{-\pi^2 kt/L^2} + \frac{20}{\pi}\sin\left(\frac{2\pi x}{L}\right)e^{-4\pi^2 kt/L^2}$$
$$- \frac{20}{3\pi}\sin\left(\frac{3\pi x}{L}\right)e^{-9\pi^2 kt/L^2}$$

$L = 12.1$
$t = 20$
$x = 7$
$k = .85$

$$u(7,20) = 35 - (6.36)(.97)(.318) + (6.36)(-.474)(.01)$$
$$-(2.12)(-.738)(3.31\times10^{-5})$$

$$= 35 - 1.96 - .030 + 5.18\times10^{-5} = 33.01$$

References:

1. Fred Unterseher, Holography Handbook. (Ross Books 1982) pgs. 14-18

2. ER Robertson, The Engineering Uses of Holography. (Cambridge. 1970). Pg. 25

3. Caulfield. The Applications of Holography. (Wiley Interscience 1970). pg. 128

4. O'Shea, Callen, Rhodes. An Introduction to Lasers and Their Applications. (Addison-Wesley. 1978). Pgs185-217

5. C.H. Edwards. Elementary Differential Equations. (Prentice-Hall. New Jersey, 1993.) Pgs. 619-627

6. GWC Kaye. Tables of Phisical and Chemical Constants. (Longman London. 1973). Pg. 52