Building and Testing a Chaotic Jerk Circuit

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An electrical circuit was built using a jerk equation with no quadratic non-linearity. The attractor produced by the circuit was compared to a numerical simulation designed to plot the phase space and look for attractors. The attractor produced by the circuit was remarkably similar to that found by the simulation. The circuit was also used to estimate Feigenbaum's delta. The value obtained was 4.11 ± 0.40, which is within two standard deviations from the accepted value of 4.664.

INTRODUCTION

The qualitative solutions of ordinary differential equations become more interesting as the dimension of the flow increases. A flow is the entire pattern of trajectories in the phase space ((x, v) – plane).[1, 2]

In 3D, in addition to the fixed points of 1D and the saddles, nodes, spirals and limit cycles of 2D, we get attractors.[2] Attractors have a fractional dimension and have sensitive dependence to initial conditions. This leads to chaos. Lorenz found an example of a system of differential equations that leads to chaos in 1963.[3] The Lorenz system can be written as a single third order differential equation, but it would be too complicated.[4] To illustrate this, another example, will be used as shown in equation 1.

$$\dot{x} = y$$

$$\dot{y} = z$$

$$\dot{z} = -2.017\ddot{x} + \dot{x}^2 - x \qquad (1)$$

The three equations above can be written as

$$\dddot{x} = -2.017\ddot{x} + \dot{x}^2 - x \qquad (2)$$

The first derivative of position $\dot{x}$ is called velocity; the second derivative of position $\ddot{x}$ is called acceleration. The third derivative of position $\dddot{x}$ is called Jerk.[4, 5] Therefore the related equation, such as equation 2, will be called a Jerk equation.

THEORY

According to Sprott, after three decades of study, the sufficient conditions for chaos in a system of autonomous ordinary differential equations remain unknown.[6] Sprott, has been working on trying to find the simplest functional form of three-dimensional dynamical systems that exhibit chaos.[4, 6, 7] Working with Linz, Bai, and Lonngeren, he has also been studying Jerk equations that are explicit and autonomous. Some of the Jerk equations found have simple nonlinear functions that should permit easy electronic implementation.[4]

Sprott used a few of the jerk equations to build their related circuits. He compared the attractors produced by the circuits with those produced by a simulation he wrote.[4] The schematics of these circuits looked a lot simpler than those proposed by Chua and Elwakil.[8, 9]

A chaotic jerk equation was chosen from the list found by Sprott so that a circuit may be built and compared with a previous simulation. The equation chosen has the form

$$\dddot{x} + A\ddot{x} + \dot{x} - |x| + 1 = 0 \qquad (3)$$

Using a computer simulation the bifurcation diagram of this equation was studied by Linz and Sprott, for A between 0.5 and 0.8.[7] Using the initial conditions $\ddot{x} = \dot{x} = x = 0$, they observed period doubling route to chaos from the Feigenbaum diagram. The diagram was mapped for A between 0.8 and 0.64085.

An earlier simulation *Jerk Chaos*[10] was written by the author in C++ in the MetroWerks Code Warrior IDE, to plot the phase space of the Jerk equations. Version 1.1 of the program plots the $x \text{ vs } \dot{x}$ phase space of equation 3. It also allows the user to change the value of A while the program is running, change the initial conditions, change the integration time step dt, and also change the scale of the plot for better viewing. The integration method used to obtain the attractor was the fourth-order Runge-Kutta algorithm. The data obtained from the simulation will be analyzed and compared with the circuit later.

In order to implement equation 3 as a circuit we need three successive inverting integrators to generate $\ddot{x}, -\dot{x}$, and x from $-\dddot{x}$. [4, 6] The weighted sum of these three signals and a constant term generated with a dc voltage source are then fed back to the input of the first integrator. An op-amp configured as an integrator was used for this purpose.[4, 11, 12, 13] The absolute value part of equation 3 is implemented by two diodes acting as a full wave rectifier and an inverting unity gain amplifier.[4]

EXPERIMENT

Using the schematic from ref. 4, figure 2, and TL 082 op-amps[14], the circuit was built. The TL-082 was used to minimize output drift due to offset and bias current.

The following relation relates the variable resistor to the parameter A in equation 3

$$R = \frac{1}{A}(K\Omega) \qquad (4)$$

With the circuit built the voltage of the battery is set to 1 V, and the variable resistor is set so that A is below the region of chaos according to Linz and Sprott's Feigenbaum diagram for the system.[7] The ideal resistance is about 1.5 KΩ. The power is then turned on and off, this is done many times until the capacitors are charged with the appropriate initial conditions.[15, 16]

To observe the attractor, the horizontal input of the oscilloscope is connected to the x-output and the vertical input to the $-\dot{x}$- output of the circuit. When this is done the oscilloscope needs to be set to the x-y plot mode for the phase space to be observed.

DATA

By increasing the resistance of the variable resistor, period doubling, chaos, and windows of periodicity were observed in the circuit. The value of A for each range of periodicity or chaos was determined by measuring the value of the variable resistance and using equation 5. This value of A was entered in the simulation to check for agreement. If there was no agreement, then values of A close to the predicted value were tried until the desired attractor was observed in the simulation.

The sequence of periodicity, chaos, and windows of periodicity observed in the circuit were period-1, period-2, period-4, period-8, chaos, period-6, chaos, period-3, chaos, period-2, and chaos again. In the simulation the sequences that were obtained were period-1, period-2, period-4, chaos, period-6, and period-2.

The following are some of the printouts from the oscilloscopes and the related screenshots of the numerical simulation. They are presented in the sequence of their observation by decreasing the value of A from about 0.8 to about 0.5 as suggested by the Feigenbaum diagram. This is done by increasing the variable resistance. Note that the oscilloscope printouts are on the left and the simulation screen shot is on the right, unless otherwise noted. Also observe the remarkable similarity between the circuit and simulation.

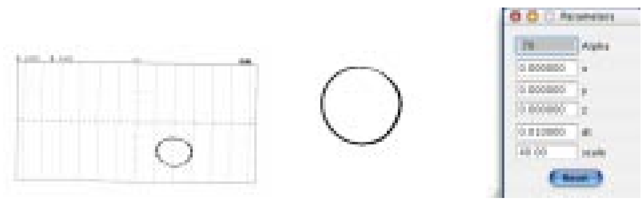

Figure 1: Oscilloscope printout of period-1, A= 0.578, R= 1.73 KΩ. The screenshot on the left is of the numerical simulation showing period 1. Note that A, which in the dialog box is labeled as alpha, is equal to 0.78.

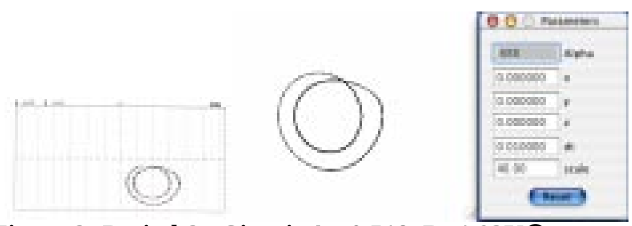

Figure 2: Period-2. Circuit A= 0.518, R=1.93KΩ. Simulation A= 0.688

Figure 3: Period-4. Circuit, A= 0.51, R=1.94KΩ. Simulation A=0.65

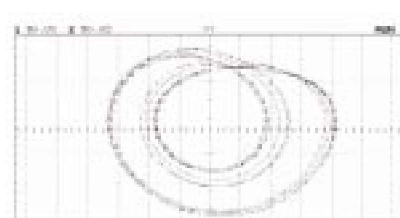

Figure 4: Period-8. Circuit A= 0.5025, R=1.99KΩ. Note that the simulation did not observe this.

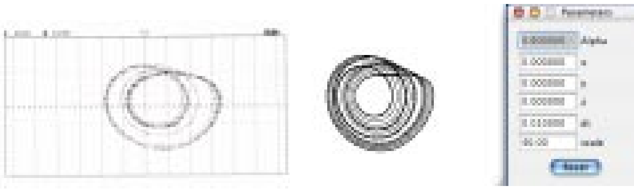

Figure 5: Chaos. Circuit, A= 0.4995, R=2.002 KΩ, simulation A= 0.6

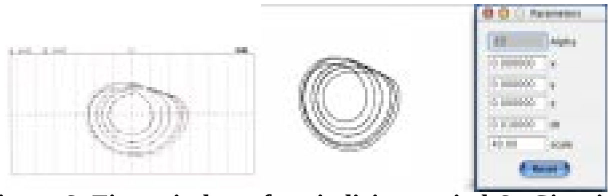

Figure 6: First window of periodicity, period-6. Circuit A= 0.4916, R=2.034KΩ, simulation A= 0.63

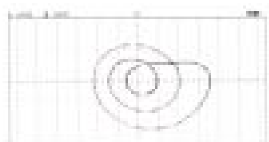

Figure 7: Period-3. Circuit A= 0.48, R= 2.043 KΩ, not observed in simulation.

Figure 8: Period-2 again. Circuit A= 0.45, R= 2.257 KΩ, simulation A= 0.55

ANALYSIS AND INTERPRETATION

Due to the many chaotic regimes in the bifurcation diagram, it was impossible to nail down all the periods observed in the circuit because, the value of A in the simulation was not exactly equal to the value calculated from the circuit. Looking at the attractors produced by both the simulation and the circuit, similarities can be seen in their pattern.

Looking at figure 1 and 2 we see that the attractor in the circuit is moved to the bottom right from the origin as compared to the simulation. This can be understood by taking a closer look at the electrical schematic (ref 4, fig.2), we see that the vertical input is obtained from $-\dot{x}$ and not $\dot{x}$. Thus all the $\dot{x}$ points become $-\dot{x}$, moving the attractor to the bottom right of the origin.

We can also compare the values of A from the simulation and from the circuit, by plotting them against each other.

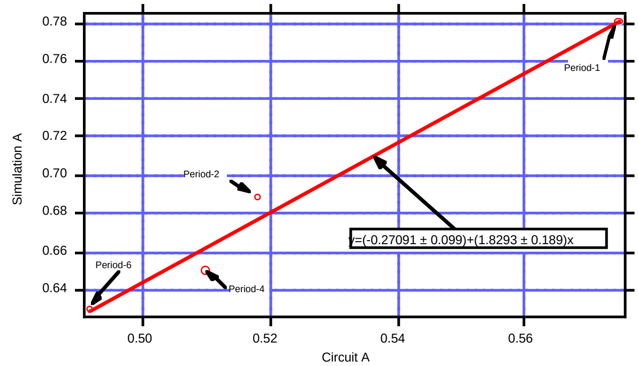

Figure 9: Plot of Simulation A vs Circuit A.

From figure 9, although only a small fraction of the values of A are plotted due to limitations in the search using the simulation, the relation seems to be pretty linear.

The resistance of the variable resistor was measured just before period doubling occurred. When this occurs we say a bifurcation has occured.[1, 2] This was done so that Feigenbaum's δ could be calculated. This could not be done in the simulation because there was no clear transition between periods.

	Period Transition	Resistance of Variable Resistor (KΩ)- Run #4	Resistance of Variable Resistor (KΩ)- Run #5	Mean	SD
R_1	1 to 2	1.73	1.7357	1.7329	0.0040
R_2	2 to 4	1.94	1.9367	1.9384	0.0023
R_3	4 to 8	1.99	1.9866	1.9883	0.0024

Table 1: data to calculate Feigenbaum's delta

Feigenbaum's delta can be calculated by using the following relation

$$\delta = \frac{R_2 - R_1}{R_3 - R_2} \qquad (11)$$

Therefore substituting the values from table 1 into equation 11 we get $\delta = 4.11 \pm 0.40$. This value is 12% off from the standard accepted value of 4.664.

CONCLUSION

The simulation was closer to what was predicted by the Feigenbaum diagram. Looking at figure 1 we see that period one exists until around A=0.71, and the simulation goes to period-2 around A=0.688, while the circuit goes to period-2 at A=0.518. Otherwise the circuit showed remarkable similarity to the numerical simulation in terms of the similarity of the attractors.

A future project might involve constructing the bifurcation diagram using the circuit; measuring the voltage of the loops could do this. Collecting the data from the voltage versus time plots would be too arduous. A way could be found to determine Feigenbaum's delta from the simulation.

An improvement on the simulation or if the scope signal were to be digitized, would be to find the fractional dimension of the attractor.

ACKNOWLEDGMENTS

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