Collision Investigation

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When two spheres collide, compression waves propagate throughout in every direction. To directly visualize the wave propagation through a sphere, a computer simulation is created. Particles arranged on an octagonal lattice are connected to each other with virtual springs, representing intermolecular forces. The simulation propels the ball toward a stationary object with some initial speed. During the collision with the obstacle, compression waves propagate through the sphere in a manner predicted by the theory. The compression wave propagation through the ball is very similar to what would be expected in the collision of an elastic spherical solid.

INTRODUCTION

Though there has not been a recognized quantitative analysis of the collision of solid spheres, there are qualitative theories, notably that of B. F. Bayman. In his 1976 article, "Model of the behavior of solid objects during collision", Bayman[1] investigates the propagation of acoustic waves during a collision between Hooke's Law springs, offering a quantitative explanation for the conservation of energy between the springs. Bayman also gives a proposed qualitative description of the collision of two hard spheres, which is investigated in this paper.

According to the proposed model, during the collision of two hard spheres, an acoustic wave is generated at the point of impact and propagates through both spheres in all directions. This compression wave reflects off the inner surface of the sphere and continues without loss of energy. These reflected waves will bring about some relaxation of the initial compression waves, eliminating the internal energy of the ball, and thereby conserving its kinetic energy. This paper attempts to give a qualitative explanation with quantitative assistance. The wave equations for a general sphere are found, and a possible explanation for the conservation of kinetic energy is given.

THEORY

In order to describe the compression waves propagating through a material, it is necessary to determine the equation of motion of a particle in a general elastic material. During and after a collision the internal forces of a solid material are not at equilibrium. Rather, each particle projects a force upon its neighbors. It is these "internal forces", or f_{int}, that creates a wave of compression. Feynman[2] and Landau[3] show that the equation of motion through an elastic material is

$$f = (\lambda + \mu)\nabla(\nabla \bullet \vec{u}) + \mu\nabla^2\vec{u} \quad (1),$$

where λ and μ are constants depending upon the molecular structure of the material, ***u*** is a vector describing the displacement of a particle located in the region, and $(\nabla \bullet \nabla) = \nabla^2$ is the Laplacian. Essentially the vector ***u*** describes a vector field, describing the displacements of all the particles contained in the area *A*. Therefore, the vector ***u*** can be equated to the sum of two vectors $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$, where

$$\nabla \bullet \vec{u}_1 = 0 \quad \text{and} \quad \nabla \times \vec{u}_2 = 0 \quad (2).$$

These expressions can be substituted back into equation (1).

$$\rho\frac{\partial^2(\vec{u}_1 + \vec{u}_2)}{\partial t^2} = (\lambda + \mu)\nabla(\nabla \bullet \vec{u}_2) + \mu\nabla^2(\vec{u}_1 + \vec{u}_2) \quad (3)$$

To eliminate $\boldsymbol{u_1}$, one can take the divergence of Equation 3. The divergence operators function only on ***u*** and therefore can pass through the constants and derivatives with respect to time.

$$\rho\frac{\partial^2(\nabla \bullet \vec{u}_2)}{\partial t^2} = (\lambda + \mu)\nabla^2(\nabla \bullet \vec{u}_2) + \mu\nabla \bullet \nabla^2\vec{u}_2 \quad (4)$$

Factoring out the divergence operator from the equation gives

$$\nabla \bullet \left[\rho\frac{\partial^2\vec{u}_2}{\partial t^2} - (\lambda + 2\mu)\nabla^2\vec{u}_2\right] = 0 \quad (5).$$

Based upon the definition of $\boldsymbol{u_2}$ (the curl of $\boldsymbol{u_2}$ = 0), so the curl of the bracketed term is zero. Since the divergence of the curl is zero, $\nabla \bullet \left(\nabla \times \vec{b}\right) = 0$, the bracketed term is zero. Therefore, Equation 5 can be rewritten:

$$\nabla^2 \vec{u}_2 = \frac{\rho}{(\lambda + 2\mu)} \frac{\partial^2 \vec{u}_2}{\partial t^2} \quad (6)$$

The term $\rho / (\lambda + 2\mu)$ is the inverse square of the speed of sound propagating through the elastic material, c_l. This is the longitudinal wave equation, or compression equation. Because the curl of $\boldsymbol{u_2}$ is zero, there are no shear motions. However, recall how this result arrived by taking the divergence of Equation 3. Taking the curl of Equation 3 produces a completely different expression without $\boldsymbol{u_2}$. The resulting expression is Equation 7.

$$\rho \frac{\partial^2 \left(\nabla \times \vec{u}_1\right)}{\partial t^2} = \mu\left(\nabla \times \nabla^2 (\vec{u}_1)\right) \quad (7)$$

Once again the curl operator can be factored out of the equation:

$$\nabla \times \left(\rho \frac{\partial^2 \vec{u}_1}{\partial t^2} - \mu \nabla^2 (\vec{u}_1)\right) = 0 \quad (8),$$

Much like the previous situation, the bracketed term is zero.

$$\nabla^2 \vec{u}_1 = \frac{\rho}{\mu} \frac{\partial^2 \vec{u}_1}{\partial t^2} \quad (9)$$

Because the divergence of $\boldsymbol{u_1}$ is zero, $\boldsymbol{u_1}$ produces no change in linear density as it propagates, and thus describes a shear or transverse wave equation. The transverse wave speed is $c_t^2 = \mu/\rho$, which is generally smaller than c_l^2.

I will assume that compression waves, consisting of a longitudinal and transverse wave, travel outward from the point of impact on the sphere's surface. These wave pairs contain the exact same energy, and propagate with the same speed. If such an assumption is true, then it is possible to describe these compression waves in terms of their angle from a horizontal, in this case, the equator. Because of the equality of all the waves' energy, the magnitude of the wave vector can be assumed to be 1, which will reduce the computational details.

As the angle of incidence (θ) describing each wave vector increases, the resulting angle with respect to the base wave, b, changes from positive interference to negative interference. When the x (horizontal) component of the reflected compression wave vector is greater than zero, the interference with the base wave is positive, increasing the magnitude of the vector, which signifies an increase in energy. Similarly, the interference is negative when the x-component of the reflection wave vector is less than zero and the base wave loses energy. There is a very interesting geometry present in the sphere. For larger angles of incidence, the waves will reflect off of the sphere multiple times before reaching the base wave. Equation 10 displays the formula used to find the limit angles.

$$\theta_n = \left(1 - \frac{1}{n+1}\right)\frac{\pi}{2} \quad (10)$$

$$n = 1, 2, 3, 4, 5, \cdots,$$

n refers to the number of reflections a wave has before it reaches the base wave. θ_n refers to the angle limit for a certain number of reflections for a wave. As the incident angle increases, the number of reflections (always an integer) increases as well. For example, the base wave propagates with an incident angle of $\theta = 0$. This implies that $n = 0$, according to Equation 10. This is correct, as the base wave does not reflect.

The cosine of all the angles with respect to the base wave are added. Recall that the magnitudes of all the wave vectors are equal to 1, implying that $\cos(\theta) = X/1$, using simple trigonometry (where θ is an angle in a right triangle). Using the geometric angle limits derived from Equation 10, a new formula can be created that calculates the total x component of all waves.

$$X = \lim_{n \to \infty} \left[\int_{\left(1-\frac{1}{n}\right)\frac{\pi}{2}}^{\left(1-\frac{1}{n+1}\right)\frac{\pi}{2}} \cos\left(\theta\left(1 + \frac{2}{n}\right)\right) d\theta \right] = 0 \quad (11)$$

X is the total horizontal component of the average velocity vector. This implies that there is no x component after all the waves come in contact with the base and total relaxation has occurred.

SIMULATION

The simulation is a significantly modified version of a program written by Dr. John Lindner. Since the partial non-linear differential equations describing the wave nature of the elastic solid are probably impossible to solve analytically, the actual motion of the particles in a sphere is simulated, and the desired compression wave can hopefully be seen.

The simulation consists of an array of "particles" arranged on a rectangular grid. The particles are connected to each other by virtual springs, which represent the intermolecular forces present between the particles of a solid. It is best to consider these "particles" regions of multiple particles, which allow a circle with a small number of particles to demonstrate realistic behavior. Each particle is connected to its eight surrounding neighbors, and reacts to their displacement according to the intermolecular forces. There are eight discrete forces corresponding to each direction that are combined into the total spring force.

$$SpringForce = \frac{1}{2}\frac{k\,l}{\left(l^2 \square l_{\max}{}^2\right)^2} \quad (12)$$

Here, k is the virtual spring constant or *spring stiffness*, l refers to the displacement from the equilibrium length, and l_{max} refers to the maximum extended stretch of the spring. When two particles are extended, they attract each other; when two particles are compressed, they repel each other. Equation 12 was specially derived to ensure that when l is small, the spring force is linear resembling the Hooke's Law spring force, yet when l is near l_{Max} the force is infinite preventing the particles from "breaking apart". There is also a viscosity present in springs, linearly dependent upon velocity, which prevents the solid from freely oscillating after a displacement. There is a spring force and a spring viscosity for all eight directions. The force constant and viscosity constants for each virtual spring are arranged in a two-dimensional array and are user selected. However, this simulation is meant to capture the properties of a circular (and hence spherical) geometry. Therefore, a circular boundary is created, outside of which the spring force constants and the spring viscosities are set to zero, effectively removing their presence from the program.

There is also an obstacle placed in the path of the ball, which repels the ball via a version of the electrostatic force. The total spring, viscosity, and obstacle (if applicable) forces for each particle are added and that specific particle displaces according to the total force it experiences. A random number generator iterates through the array of spring force and spring viscosity constants and assigns new random constants. Because of these randomized spring forces and viscosities, the ball is not at equilibrium, and it shifts internally. It relaxes into a new and forms distinct regions based upon the assignment of the spring forces and viscosities. These regions can be thought of as grain patterns within an actual steel ball. As the program commences, the ball is given an initial speed in the positive x direction (there is no initial vertical velocity). With this initial velocity, it moves towards the obstacle. When each particle reaches the obstacle boundary the particles begin to slow and change direction. However, the particles on the other side of the ball are not experiencing the obstacle force and therefore do not slow down. This causes horizontal and vertical compression around the point of impact, as the springs between each particle compress. However, the springs then apply extension forces upon each other, pushing the atoms outward. This compression and extension propagates from the point of impact to the opposite side of the ball, simulating the compression waves discussed in the theory section.

RESULTS AND ANALYSIS

The simulation exhibits a direct representation of the compression waves propagating through the ball. At incredibly high speeds, the ball will deform significantly both horizontally and vertically upon impact with the obstacle as seen in Figure 2.

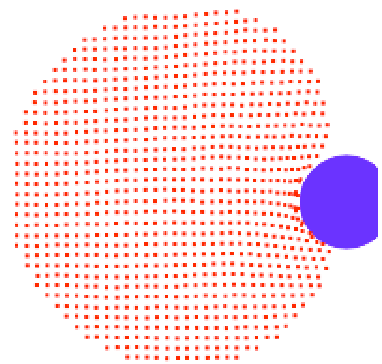

FIG. 2: Notice the significant deformation of the ball. This is due to unrealistic initial speeds.

The Collision simulation did show the propagation of a compression wave from the point of impact to the opposite side of the ball in a manner predicted by the theory. This can be seen in Figure 3. Therefore, the assumption that the numerous compression waves propagate outward in all directions from the impact point is most likely true. The reflection and relaxation of the compression waves is more difficult to perceive, though was moderately visible.

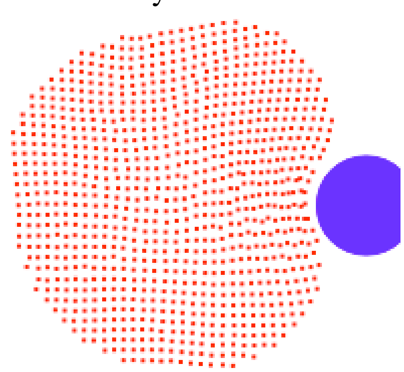

FIG. 3: The compression wave propagates outward from the point of impact with the obstacle.

The simulation does require some fine-tuning, especially in the speed of its operation. The most substantial hindrance of the Collision simulation is the presence of two time frames. Compression waves passing through an object move *much* faster than implied by the program. In steel, the speed of sound is 5100 m/s. Since the ball shifts toward the obstacle at a rate much faster than the compression waves propagate through it, the initial velocity of the ball is non-realistic. Irreversible effects such as plastic deformation and thermal breakdown would occur as such speeds. The limiting factor is once again the ball size. In a solid object, the distances between particles are incredibly small, and it takes very little time for a particle to collide into its neighbor. Solving the time reference frame and the ball size vs. speed issues would be an excellent area for future work.

CONCLUSION

The Collision investigation added some interesting insight into the conservation of kinetic energy of spheres. The theory has not been proven, though it certainly was not invalidated. There is a possibility that the assumptions made, notably the superposition of compression waves oversimplified the collision, producing a nonrealistic scenario.

With regard to the simulation, there are many possible features that could be added to improve the functionality. Firstly, it would be interesting to arrange the particles upon a different lattice, possibly a hexagonal or circular lattice. The collision would most likely follow a similar pattern, yet another grid may have added benefits: speed of operation, more distinct waves, etc. Also, the particle arrangement could be strictly assigned to mimic the grain patterns of different materials. Using a circular grid, the particles would be arranged much like a forged sphere, where the grains all converge at the center. The density of particles is a bit disappointing as well. It can be very difficult to see the compression wave propagation due to the small size of the ball. There are a few annoying bugs present in the program that cause some strange effects at times, though they are most likely attributed to the random assignment of spring forces and viscosities. The simulation could be modified to allow multiple spheres at one instance. Multiple sphere impacts could be simulated to determine whether they are a chain of successive two body collisions as the theory predicts.

REFERENCES

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