

the several times over a 3-y period. The uncertainty in the Th-228/Ra-228 ratio comes from the counting statistics and from differences in the geometry (assumed to be 3% for this counting situation). This uncertainty produces a corresponding uncertainty in the age of the mantle. As the Th-228/Ra-228 ratio approaches unity, a small uncertainty in the Th-228/Ra-228 ratio results in a large uncertainty in the age. The results in Table II show that the uncertainty in the age of the last count is considerably larger than the first count.

The results show that this method is accurate enough to determine the age of a lantern mantle to within two months if the age is less than 3 y. For ages beyond 3 y, the uncertain-

ty becomes too large to make an accurate age determination.

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Undergraduate experiment in critical phenomena: Light scattering in a binary fluid mixture

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(Received 17 March 1982; accepted for publication 14 July 1982)

A relatively simple experiment has been designed to allow advanced undergraduates to investigate light scattering phenomena above the critical consolute point in a binary fluid mixture: methanol-cyclohexane. A He-Ne laser beam penetrates a fluid mixture sealed in a cylindrical aluminum cell while a dc Wheatstone bridge allows the cell temperature to be controlled to 0.01 °K. Transmitted light intensity measurements taken from 0.1 to 10 °K above the critical consolute point allow the turbidity to be calculated. The critical exponent γ , is determined using a modified Ornstein-Zernike theory.

I. INTRODUCTION

Liquid-gas critical phenomena have been known and investigated for almost 100 years, yet this field has seen enormous growth in the past 20 years due to better theories and more sophisticated instrumentation. What spurred this interest was the realization, both theoretically and experimentally, of the universality of the description of physical processes near a critical point in a wide variety of systems. Thus essentially the same relationships hold for liquid-gas, binary fluid mixtures, superfluidity in He⁴, ferromagnetism, and protein solutions, to name just a few.¹ The diversity of critical systems that can be described by universal relations indicates that experimental measurements on one system should yield the same information as on another. Although several review articles have been written,¹ few experiments or demonstrations² have been described that would allow this important field to be experienced by undergraduates. In this paper, a simple experiment for junior or senior undergraduates is described.

In order to provide a physical feel for critical phenomena and motivate the universality observed, a simple droplet model will be described. This model has been used to correctly predict several phenomena³ and yet has not always been verified.⁴ Consider it in the same light as the Bohr model of the hydrogen atom: a model that provides a good physical feel for the phenomena as well as some correct

quantitative predictions. For simplicity, consider a gas loaded in a cell at the proper temperature, density, and pressure so that the gas has its phase transition at the critical point. When the gas is well above the critical point in temperature, it can be described by the ideal gas law. However, as its temperature is decreased, the molecules begin to "stick together" and their finite volume becomes significant. The molecules that stick together do not remain together long while the gas is above the critical point. Closer to the critical point, more molecules "stick" to each other forming droplets. These droplet sizes vary spatially and temporally. However, an average size, or correlation length ξ , of these droplets may be defined. Very close to the critical point, these droplets become several thousand angstroms across. At the critical point, the droplets become infinite in size—a liquid phase forms. This phenomenon is readily observed as a milkiness (called critical opalescence) and is readily measured by light scattering—the experiment described below.

Since the correlation length is very large near the critical point compared to the molecular size, the behavior of the system is not determined by the type of gas but by its critical properties. It is these critical properties that have universal descriptors. For example, the correlation length ξ diverges close to the critical point as a power law $\xi = \xi_0 t^{-\nu}$ where $t \equiv (T - T_c)/T_c$, ξ_0 is the correlation length far from the critical temperature T_c , and ν is a uni-

versal (i.e., the same for all gases) critical exponent ($\nu \approx 0.64$).¹ Thus physical phenomena describing a system close to its critical point have a universal form independent of the system involved; however, the location of the critical point (critical temperature, density, and pressure) depends on the specific substance.

Not only will the critical exponents be the same for all gases near their critical point but also for many other thermodynamic systems that have a critical point (second-order phase transition). One example is a binary fluid mixture—two components which are partially miscible below a certain temperature and completely miscible above. The analogy with the liquid–gas transition described above using the droplet model is complete. One associates the liquid phase with one component and the gas phase with the other component. Thus below the critical temperature in a binary fluid mixture, a lower phase and upper phase coexist in equilibrium where each phase is rich in one component. The amount of a component in one of the phases is determined by how close the system is to its critical temperature—there exists a coexistence curve for binary fluid mixtures just as there is for a single-component liquid–gas. In fact, the only important distinction between critical phenomena in a liquid–gas and a binary fluid mixture is the parameters used to describe the critical point. Because the binary fluid mixture has two components, only two thermodynamic variables are needed to determine the critical point: the critical temperature and the critical composition (the latter corresponding to the critical density in a liquid–gas transition). The critical point for a binary fluid mixture is often called a critical consolute point.

This paper describes an experiment to measure the turbidity in a binary fluid mixture near its consolute point. A binary fluid mixture is an easy system to investigate because the fluids are loaded at atmospheric pressure and the critical temperature is usually close to room temperature. Temperature control is greatly simplified when T_c is slightly above room temperature since only a heater is needed and complicated thermostating is not essential to measure the effect. It is important to control the temperature of the cell so that the system may be brought close to the critical temperature in an equilibrium condition.

II. THEORY

The binary fluid mixture methanol–cyclohexane exhibits an upper consolute point, and when the mixture is well above its critical consolute temperature T_c , the fluids are one phase, homogeneous, and essentially clear. The constant transmitted light intensity at these temperatures is referred to as I_0 . As the temperature of the fluids approaches T_c , concentration fluctuations, “droplets,” begin to form and cause the transmitted light intensity I to be reduced. The total incremental intensity of light scattered per unit length is defined as the turbidity τ . The turbidity is dependent upon the transmitted light intensities I and I_0 by

$$\tau = -L^{-1} \ln(I/I_0),$$

where L is the optical path length.

The light scattering can be written in terms of two critical exponents, ν and η using the Ornstein–Zernike theory.⁵ The exponent η (≈ 0.11) has a small effect and is observable only with very precise measurements. If η is negligible, then the theory predicts an expression for the turbidity^{6,7}:

$$\tau = AK_T \{ (2a^2 + 2a + 1) [\ln(1 + 2a)] / a^3 - 2(1 + a) / a^2 \},$$

where $a = 2k_0^2 \xi^2$, $k_0 = 2\pi/\lambda$, ξ is the correlation length, A is a constant, K_T is the “static isothermal compressibility” which can be expressed as a power law expression $K_T = K_0 [(T - T_c)/T_c]^{-\gamma}$, and ν and γ are related critical exponents [$\gamma = (2 - \eta)\nu \approx 2\nu$]. We can approximate the behavior of the turbidity when very close to, but above, T_c (limit as $a \rightarrow \infty$) and when “far” above T_c (limit as $a \rightarrow 0$). In the latter case we find

$$\tau = BK_T = Ct^{-\gamma} \quad (T \gg T_c).$$

When the correlation length becomes large (as $T \rightarrow T_c$), the turbidity becomes

$$\tau = B' \ln(2a) \quad (T \gtrsim T_c),$$

where B , B' , and C are constants. Thus we expect the turbidity to have a power law dependence on the reduced temperature, $t \equiv (T - T_c)/T_c$, far above T_c , and the turbidity to vary slowly close to T_c . The slope of a \ln – \ln plot of turbidity versus reduced temperature will give a value for the critical exponent gamma (γ) when considering points sufficiently above T_c . Theoretical values have been calculated by series expansions; values for γ are 1.245 ± 0.003 by Guillou and Zinn-Justin⁸ and 1.238 ± 0.003 by Nickel.⁹

III. FLUIDS AND CELL

The binary fluid mixture methanol–cyclohexane is a good choice for this experiment. It exhibits an upper consolute point at 45 °C and both components are available in a nearly pure state.¹⁰ Their critical properties are known¹¹ and the critical opalescence is quite observable. Gravity effects are small for this mixture, but impurities can be a problem. To prevent water contamination, the cell should be filled and sealed in a dry box. The composition of the fluid mixture reported here is 29.0% by weight methanol which is very close to the critical composition of 28.94%.¹¹

The cell should be sealed securely, preferably with a teflon lined cap and not a stopper. The interior cell walls should be flat and parallel with an optical path length of a few centimeters. Several commercially available (“spectrophotometric”) cells satisfy these requirements. Owing to its convenience, a cylindrical aluminum cell (diameter of 6.5 cm and length of 8.2 cm) with an internal optical path length of 6.5 cm was used. The fluids were contained within two optically flat glass windows sealed with teflon *o*-rings. The inside of the cell was anodized black to prevent oxidation of the aluminum and to lessen the effect of multiple scattering from the He–Ne laser beam that passed directly through the center of the fluid mixture. The right circular cylinder had its base on a horizontal surface with the laser beam passing horizontally through the fluids in the cylinder.

IV. TEMPERATURE CONTROL

Since the turbidity is expected to have a power law dependence on the difference between the cell temperature and the fluid’s critical temperature, then precise temperature control is important. It is possible to use a well-mixed water bath if the temperature control is about ± 0.01 °K, and if turbulence in the water and imperfections on the tank do not significantly affect the transmitted light intensity. Since a temperature controlling bridge of our design was available, the cell was simply encased in styrofoam and placed on a laboratory bench.

The cell (surrounded by 4 in. of styrofoam) was heated using Minco thermofoil heaters placed on the curved cell surface. The heat loss of the cell was reasonably proportional to the temperature difference between the cell and room. Two Yellow Springs Instrument (No. 44004: 2250 Ω at 25 $^{\circ}\text{C}$) thermistors were embedded in the cell wall adjacent to the fluids and were used for temperature control and measurement. The thermistors were calibrated to a calibrated thermistor (traceable to the National Bureau of Standards). One thermistor was designated the control thermistor and was connected to a dc Wheatstone bridge; the other thermistor was used to monitor the cell temperature. The monitoring thermistor was connected in series with a 1.35-V mercury battery and a 50-k Ω Vishay standard resistor. By measuring the voltage across the thermistor and the standard resistor with a five-digit CIMRON voltmeter, the thermistor resistance could be precisely determined. An empirical equation¹² was used to determine the cell temperature from the thermistor resistance.

The temperature control was accomplished with the control thermistor and a 1-k Ω Vishay standard resistor forming one arm of a dc Wheatstone bridge. The other arm was a four-decade ESI dekapot (a precise voltage divider). The bridge error signal was amplified by an instrumentation amplifier and used with a Kepco programmable power supply to provide the necessary current to heat the cell. The steady state voltage necessary to maintain the cell temperature above ambient was set by an adjustable battery in series with the output of the dc bridge. The desired cell temperature was determined by the setting on the dekapot. The cell temperature was controlled to a precision of 0.01 $^{\circ}\text{K}$.

V. LIGHT INTENSITY MEASUREMENT

Since the transmitted light intensity decreases so dramatically close to the critical temperature, a fairly simple photometer may be used. In this experiment, the transmitted light intensity was measured with a Pasco photometer with a fiber-optic probe extension. This photometer measured intensity from 0.1 to 1000 lux full scale using a photoconductivity cell. The light source was an unpolarized He-Ne laser (Spectra Physics 155) that provided an intense monochromatic beam ($\lambda = 632.8$ nm). The laser operated continuously and had a constant output intensity.

VI. PROCEDURE

One first needs to find the critical temperature T_c and then measure the turbidity at temperatures higher than T_c . For a binary fluid mixture exhibiting an upper consolute point and loaded into the cell at the critical composition, the critical temperature is the highest temperature at which the two phases are indistinguishable. A meniscus forms when the temperature of the fluid mixture is lowered to T_c ; at this temperature, the fluids separate into two distinct phases. In this experiment, observations of this phase transition were difficult to observe due to convection which appeared when very close to T_c . The cell was submerged in an insulated water bath to reduce temperature gradients and T_c was determined to be 47.22 ± 0.05 $^{\circ}\text{C}$; this agrees with observations made when the cell was in the styrofoam. This value of T_c is higher than reported elsewhere¹¹ due to small amounts of impurity present. (The largest impurity present in these fluids is water which causes the critical temperature to increase 4 $^{\circ}\text{K}$ for every 0.1% by volume

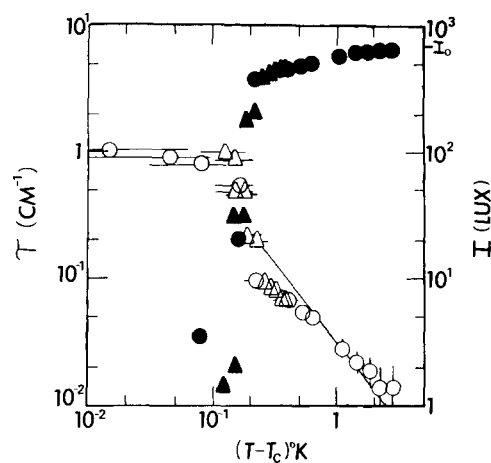


Fig. 1. Turbidity τ and transmitted light intensity I versus temperature from critical. The turbidity and the transmitted light intensity are represented by open and closed symbols, respectively. Circles are run 1 and triangles are run 2. The line drawn through the turbidity data far from T_c has a slope of 1.24 (theoretical value).

impurity.¹³ However, the critical exponents are not significantly affected by small levels of impurities.)

Transmitted light intensity data were collected in the temperature region 0.1 to 10 $^{\circ}\text{K}$ above the critical temperature. At temperatures higher than 10 $^{\circ}\text{K}$ above critical the transmitted light intensity I was observed to be a constant of 700 lux. The procedure used to take a data point started with setting the desired cell temperature with the dekapot, adjusting the steady-state voltage to the heaters, and waiting for temperature equilibrium. The cell was then shaken and left for 30–90 min for the fluids to come to equilibrium. The shorter equilibration times were used when further from T_c . The voltage across the standard resistor and monitor thermistor as well as the transmitted light intensity were then recorded. Within 0.3 $^{\circ}\text{K}$ of critical, the transmitted light intensity was observed to decline dramatically. At approximately 0.1 $^{\circ}\text{K}$ above critical, the measured intensities were very small (0.01–0.001 lux)—so small that they were not well resolved on the photometer used. Figure 1 shows two data runs in this experiment. The error associated with a transmitted light intensity measurement was designated as one-half of the smallest photometer scale division—an amount corresponding to the fluctuation of the photometer needle. The scatter in these data runs illustrates the reproducibility of the measurements.

VII. ANALYSIS AND RESULTS

The data shown in Fig. 1 is plotted in such a way as to allow one to observe the direct relationship between the transmitted light intensity and the turbidity over the temperature range 0.01 $^{\circ}\text{K} < T - T_c < 3$ $^{\circ}\text{K}$. The right and left ordinates are the transmitted light intensity I and the turbidity τ , respectively. The transmitted light intensity can be seen to decrease dramatically at 0.3 $^{\circ}\text{K}$ above critical. As the temperature difference $T - T_c$ decreases, the ratio of the intensities, I/I_0 , decreases significantly allowing the effect on the turbidity to be observable. The data is plotted on a ln–ln plot to illustrate the power law dependence of the turbidity far above T_c ($T - T_c > 0.2$ $^{\circ}\text{K}$) and the ln dependence “close” to T_c . The experimental data exhibits a “linear” region over the temperature range 0.2 $^{\circ}\text{K} < T - T_c$

$< 3^\circ\text{K}$; the slope of this region is the critical exponent γ . Within the temperature region $0.14^\circ\text{K} < T - T_c < 0.7^\circ\text{K}$, the data can be observed to be systematically lower than the straight line drawn through the points above $T - T_c = 0.2^\circ\text{K}$. Convection in the fluids became obvious over this temperature region, and it is thought that the transmitted light intensity was increased as a result, and hence, the calculated turbidities were smaller.

Data run 2 displays a plateau region close to, but above T_c ($0.01^\circ\text{K} < T - T_c < 0.05^\circ\text{K}$). The horizontal error bars are large close to T_c due to the 0.05°K uncertainty in T_c . Far above T_c , we observe large errors on the turbidities due to the small differences between I and I_0 . The effects of multiple scattering were small when far above T_c ; however, close to T_c , the effect was important though negligible compared to other errors.

All of the data collected within the temperature range $0.1^\circ\text{K} < T - T_c < 3^\circ\text{K}$ were fit using a $\ln\text{-}\ln$, weighted, linear least squares routine.¹⁴ The slope γ was determined to be 1.34 ± 0.13 . This value agrees within experimental error to other experimental results and to the predicted value; for example, the binary fluid mixture, polystyrene-diethyl malonate, was found to have γ equal 1.23 ± 0.03 using a similar technique.⁷

VIII. CONCLUSION

A simple experiment on the turbidity of a binary fluid mixture near its critical consolute point has been described. Quantitative results are consistent with theoretical predictions of universal critical exponents. Much more precise data may be obtained by using a photomultiplier tube for light intensity measurements, and by using a two stage thermostat to reduce temperature gradients across the cell and hence eliminate convection currents when close to T_c . It is felt that this is an appropriate experiment for junior or

senior undergraduates and provides them with an opportunity to experience the exciting field of critical phenomena.

ACKNOWLEDGMENTS

We would like to acknowledge Research Corporation for partial support of the research which led to this work. We are also indebted to S. C. Greer and R. H. Kulas for helpful suggestions on an early version of this manuscript.

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New method for calculating electric and magnetic fields and forces

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(Received 10 May 1982; accepted for publication 13 August 1982)

It is shown that for the purpose of electric field calculations, polarized dielectric bodies and real electric charge distributions can be replaced by fictitious "polarization current" distributions. Electric fields can then be calculated by any of the techniques available for calculating magnetic fields, and vice versa. Also electric forces acting on electric charges and dielectrics can be calculated by using the techniques available for calculating magnetic forces on currents. The method is illustrated by examples on calculating electric fields produced by electrets and by examples on calculating forces on dielectric bodies.

I. INTRODUCTION

An important method for calculating electrostatic fields in the presence of dielectric media, sometimes known as the

"Poisson transformation," is the representation of such media by "equivalent" space and surface charge distributions. This method allows one to reduce an electrostatic system in the presence of dielectric media to a system of